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Rota-Kahaner-Odlyzko's Identity for Stirling Numbers of the First Kind

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Abstract: We deduce the companion identity of the expression obtained by Rota-Kahaner-Odlyzko for Stirling numbers of the first kind.

Key words: Harmonic and Stirling numbers

INTRODUCTION

Rota-Kahaner-Odlyzko [1-3] deduced the following identity:

$$\binom{m+n}{n} S_k^{(m+n)} = \sum_{r=n}^{k-m} \binom{k}{r} S_r^{(n)}\, S_{k-r}^{(m)}\,,\ k \geq m+n, m, n \geq 0, \tag{1}$$

Involving Stirling numbers of the first kind [4, 5]. The inversion of (1) is given by:

$$\binom{k}{n} S_{k-n}^{(m)} = \sum_{r=n}^{k-m} \binom{m+r}{m} S_k^{(m+r)}\, S_r^{[n]}\,,\ m, n \geq 0, k \geq m+n, \tag{2}$$

whose application for $m = 1$ allows obtain the interesting relation:

$$\sum_{r=n}^{k-1} r\, S_k^{(r+1)}\, S_r^{[n]} = \frac{(-1)^{k+n+1}\,(k-1)!}{(n-1)!\,(k-n)}\,,\ k \geq n+1, \tag{3}$$

where were employed the expressions [3, 4]:

$$S_{r+1}^{(1)} = (-1)^r\, r!\,,\quad \sum_{r=n}^{k-1} S_k^{(r+1)}\, S_r^{[n]} = \frac{(-1)^{k+n+1}\,(k-1)!}{n!}. \tag{4}$$

From (2) with $m = 2$ results an identity between Stirling and harmonic numbers [5]:

$$H_{k-n-1} = \frac{(-1)^{k-n}\, n!\,(k-n)}{2\cdot k!} \sum_{r=n}^{k-2} (r+2)\,(r+1)\, S_k^{(r+2)}\, S_r^{[n]}\,,\ k \geq n+2\,, \tag{5}$$

where it was used the property [4]:

$$S_r^{(2)} = (-1)^r (r-1)!\ H_{r-1}\,. \tag{6}$$

If in [2] we make the changes $k \to -k$, $m \to -m$ and we apply the relations [4, 6, 7]:

$$\binom{-m}{j} = (-1)^j \binom{m+j-1}{j},\quad S_{(-j)}^{(-m)} = (-1)^{m-j}\, S_m^{[j]}\,, \tag{7}$$

Is generated the expression:

$$\binom{k+n-1}{n} S_m^{[k+n]} = \sum_{r=n}^{m-k} \binom{m-1}{r} S_r^{[n]}\, S_{m-r}^{[k]}\,,\ m \geq k+n, k, n \geq 0, \tag{8}$$

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For Stirling numbers of the second kind, which we can consider as companion of (1). The identity (8) for $k = 1$ implies [3]:

$$S_m^{[n+1]} = \sum_{r=n}^{m-1} \binom{m-1}{r} S_r^{[n]} \,, m \geq n+1,\ n \geq 0, \tag{9}$$

and for $k = 2$:

$$\sum_{r=n}^{m-2} \binom{m-1}{r} 2^{m-r} S_r^{[n]} = 2\,(n+1)\left(S_m^{[n+2]} + S_{m-1}^{[n+1]}\right), m \geq n+2, n \geq 0, \tag{10}$$

where it was applied the relation [4]:

$$S_q^{[2]} = 2^{q-1} - 1, q \geq 2. \tag{11}$$

If in the expression [8]:

$$\sum_{r=0}^{n} S_{m+r}^{[k]}\ S_n^{(r)} = \sum_{j=k-n}^{m} n^{m-j} \binom{m}{j} S_j^{[k-n]}\,,\ m,n \geq 0,\ 0 \leq k \leq m+n, \tag{12}$$

we put $n = 2$ and after we realize the changes $m \to m-1, j \to r, k \to n+2$ results (10) again.

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