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Bernoulli, Stirling and Lah numbers

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Abstract: We prove an identity involving Stirling and Lah numbers, which allows deduce a relation for the Bernoulli numbers in terms of the Stirling numbers of the second kind

Key words: Stirling - Lah and Bernoulli numbers

INTRODUCTION

First we shall show the identity:

$$A(n,j) \equiv \sum_{k=j+1}^{n} (-1)^k \, S_{k-1}^{[j]} \, S_n^{(k)} = \frac{(-1)^n}{n}(j+1)L_n^{[j+1]} = \frac{(-1)^n\,(n-1)!}{j!}\binom{n-1}{j}, \quad n \geq 1, \;\; 0 \leq j < n, \tag{1}$$

Involving the Stirling numbers [1-3] and the Lah numbers [4-6]. In fact:

$$A(n,j) = \sum_{k=j+1}^{n} (-1)^k \left[S_k^{[j+1]} - (j+1)\, S_{k-1}^{[j+1]} \right] S_n^{(k)} = (-1)^n \, L_n^{[j+1]} - (j+1)\, A(n, j+1),$$

Which is a recurrence relation whose solution is given by (1), q.e.d.

We have the inversion formula [1, 5]:

$$\sum_{k=0}^{n} f(k)\; S_n^{(k)} = g(n) \qquad \therefore \qquad \sum_{k=0}^{n} g(k)\; S_n^{[k]} = f(k), \tag{2}$$

and its application to (1) implies the expression:

$$\sum_{k=j+1}^{n} (-1)^k \,(k-1)!\binom{k-1}{j} S_n^{[k]} = (-1)^n \, j! \; S_{n-1}^{[j]}\,, \qquad n \geq 1, \quad 0 \leq j < n. \tag{3}$$

From [1, 7, 8]:

$$\frac{(1-2^k)}{k}\, B_k = \sum_{j=0}^{k-1} \frac{(-1)^j \, j!}{2^{j+1}}\, S_{k-1}^{[j]}\,, \tag{4}$$

Then:

$$\sum_{k=0}^{n} \frac{(-1)^k\,(1-2^k)}{k}\, B_k \, S_n^{(k)} = \sum_{j=0}^{n} \frac{(-1)^j \, j!}{2^{j+1}} \sum_{k=j+1}^{n} (-1)^k \; S_{k-1}^{[j]}\, S_n^{(k)} \overset{(1)}{=} \frac{(-1)^n\,(n-1)!}{2^n}, \tag{5}$$

where we can apply (2) to obtain the following relation deduced in [9, 10]:

$$\sum_{k=0}^{n} \frac{(-1)^k\,(k-1)!}{2^k}\, S_n^{[k]} = \frac{(-1)^n}{n}\,(1-2^n)\, B_n\,, \qquad n \geq 1. \tag{6}$$

In [11] are studied the numbers:

$$b_n(0) \equiv \int_0^1 \binom{x}{n} dx = \frac{1}{n!}\sum_{k=0}^{n} \frac{1}{k+1}\, S_n^{(k)}\,, \qquad n \geq 0, \tag{7}$$

That is:

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$$b_0(0) = 1, \quad b_1(0) = \frac{1}{2}, \quad b_2(0) = -\frac{1}{12}, \quad b_3(0) = \frac{1}{24}, \quad b_4(0) = -\frac{19}{720}, \dots \tag{8}$$

Then the application of (2) to (7) gives the expression:

$$\sum_{k=0}^{n} k!\ b_k(0)\ S_n^{[k]} = \frac{1}{n+1}. \tag{9}$$

On the other hand, from [12]:

$$\sum_{k=0}^{n} (-1)^k B_k\, S_n^{(k)} = \frac{(-1)^{n-1}\,(n-1)!}{n+1}, \qquad n \geq 1, \tag{10}$$

Hence (9) and (10) imply the identity:

$$\sum_{k=1}^{n} (-1)^k B_k\, S_n^{(k)} = (-1)^{n-1}\,(n-1)!\ \sum_{k=1}^{n} k!\ b_k(0)\ S_n^{[k]}, \qquad n \geq 1; \tag{11}$$

Similarly:

$$\sum_{k=1}^{n} B_k\ S_n^{(k)} = (-1)^n\, n!\ \sum_{k=1}^{n} k!\ b_k(0)\, S_n^{[k]}, \qquad n \geq 1. \tag{12}$$

Besides:

$$\sum_{m=0}^{n} \sum_{j=0}^{m} \frac{(-1)^j\, j!}{2^j}\, S_n^{(m)}\, S_m^{[j]} = \sum_{j=0}^{n} \frac{(-1)^j\, j!}{2^j} \sum_{m=j}^{n} S_n^{(m)}\ S_m^{[j]} = \sum_{j=0}^{n} \frac{(-1)^j\, j!}{2^j}\, \delta_{jn} = \frac{(-1)^n\, n!}{2^n}; \tag{13}$$

The properties (12) and (13) are the Theorems 2.3 and 2.4 in [11], respectively.

Ghosh [13] showed the relation:

$$\sum_{k=0}^{n} \binom{2n+1}{2k}\, 2^{2k}\, B_{2k} = 2n+1, \qquad n \geq 0, \tag{14}$$

To determine the values of the Riemann zeta function in even integers; however, (14) was obtained by Euler [14, 15]. Namias identity is given by [16-19]:

$$B_n = \frac{1}{2(1-2^n)} \sum_{k=0}^{n-1} 2^k \binom{n}{k} B_k, \tag{15}$$

which for *n* even or odd generates (14) and the expression:

$$B_{2m} = \frac{1}{2(1-2^{2m})}\left[-1 + \sum_{k=1}^{m-1} (2^{2k}-2)\binom{2m}{2k} B_{2k}\right], \quad m = 1, 2, \dots, \tag{16}$$

This relation (16) was obtained in [18] without using the gamma function as in [16].

Lee-Kim [20] showed the following identities for $n \geq 0$:

$$Q \equiv \sum_{k=0}^{n} \sum_{r=0}^{k} \binom{n+1}{k}\binom{k}{r} B_{k-r}\, B_r = (n+1)\, B_n, \tag{17}$$

$$R \equiv \sum_{k=0}^{n} \sum_{r=0}^{k} \binom{n+1}{k}\binom{k}{r} (-1)^k\, B_{k-r}\, B_r = (n+1)\big(n + \delta_{n,1} + B_n\big), \tag{18}$$

Then we shall give elementary proofs of them. In fact:

$$Q = \sum_{r=0}^{n} \binom{n+1}{r} B_r \sum_{k=r}^{n} \binom{n-r+1}{k-r} B_{k-r} = \sum_{r=0}^{n} \binom{n+1}{r} B_r \sum_{j=0}^{n-r} \binom{n-r+1}{j} B_j,$$

$$= \sum_{r=0}^{n-1} \binom{n+1}{r} B_r \sum_{j=0}^{n-r} \binom{n-r+1}{j} B_j + \binom{n+1}{n} B_n = (n+1)\, B_n, \quad q.e.d.$$

where it was applied the relation [1]:

$$\sum_{j=0}^{m} \binom{m+1}{j} B_j = 0, \qquad m \geq 1. \tag{19}$$

Similarly:

$$R = \sum_{r=0}^{n} (-1)^r \binom{n+1}{r} B_r \sum_{j=0}^{n-r} (-1)^j \binom{n-r+1}{j} B_j = \sum_{r=0}^{n} (-1)^r \binom{n+1}{r} (n-r+1)\, B_r\,, \tag{20}$$

where it was used the identity [1]:

$$\sum_{j=0}^{m} (-1)^j \binom{m+1}{j} B_j = m+1, \qquad m \geq 0. \tag{21}$$

It is clear that $R = 1$ for $n = 0$, then now we shall accept that $n \geq 1$:

$$R = (n+1) \sum_{r=0}^{n} (-1)^r \binom{n+1}{r} B_r - \sum_{r=0}^{n} (-1)^r \binom{n+1}{r} r\, B_r,$$

$$\overset{(21)}{=} (n+1)^2 - (-1)^n (n+1)\, B_{n+1} - \sum_{r=0}^{n+1} (-1)^r \binom{n+1}{r} r\, B_r,$$

$$= (n+1)[\, n+1-(-1)^n B_n] - (n+1)[1-(-1)^n (B_{n+1}+B_n)\,] = (n+1)[\, n+(-1)^n B_n], \tag{22}$$

where it was employed the property:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} k\, B_k = n\,[1+(-1)^n\,(B_n + B_{n-1})], \qquad n \geq 1; \tag{23}$$

our expression (22) is equivalent to (18), *q.e.d.*

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