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Fejér's Kernel and its Associated Polynomials

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Abstract: We show that $[1 - T_{n+1}(x)]/(1-x)$, where $T_m(x)$ is a first-kind Chebyshev polynomial, is a polynomial of degree n in x.

Key words: Dirichlet's kernel · Chebyshev's polynomials · Fejér's kernel

INTRODUCTION

Here we consider the first-kind Chebyshev polynomials $T_m(x)$, $x \in [-1,1]$ defined by [1-6]:

$$T_0(x)=1,\ T_1(x)=x,\ T_2(x)=2x^2-1,\ T_3(x)=4x^3-3x,$$
$$T_4(x)=8x^4-8x^2+1,\ T_5(x)=16x^5-20x^3+5x,\dots \tag{1}$$

then:

$$(1-T_1)/(1-x)=1,\ (1-T_2)/(1-x)=2(1+x),$$
$$(1-T_3)/(1-x)=(2x+1)^2,\ (1-T_4)/(1-x)=8x^2(1+x),$$
$$(1-T_5)/(1-x)=16x^4+16x^3-4x^2-4x+1,\dots \tag{2}$$

that is:

$$\tilde{W}_{n-1}(x)\equiv\left[1-T_n(x)\right]/(1-x),\quad n=1,2,\dots \tag{3}$$

are polynomials of degree $(n - 1)$ in x, in fact, (1) can be written [5] in terms of the Gauss hypergeometric function [7]:

$$T_m(x)={}_2F_1\left(-m,m;\frac{1}{2};\frac{1-x}{2}\right),$$
$$=1-m^2(1-x)+\frac{m^2\left(m^2-1\right)}{6}(1-x)^2-\frac{m^2\left(m^2-1\right)\left(m^2-4\right)}{90}(1-x)^3+\dots, \tag{4}$$

thus it is evident that $[1-T_m(x)$ accepts to $(1-x)$ as factor, which is showed in (2). From (3) and (4) we can obtain the expression:

$$\tilde{W}_{n-1}(x)=n\sum_{r=1}^{n}\frac{2^r}{n+r}\binom{n+r}{2r}(x-1)^{r-1},\quad n=1,2,\dots \tag{5}$$

where we have explicitly to $\tilde{W}_m$ as a polynomial of degree m in x:

$$\tilde{W}_0=1,\ \tilde{W}_1=2x+2,\quad \tilde{W}_2=4x^2+4x+1,$$
$$\tilde{W}_3=8x^3+8x^2,\quad \tilde{W}_4=16x^4+16x^3-4x^2-4x+1,\dots \tag{6}$$

On the other hand, we know that [8, 9]:

$$T_n(\cos\theta)=\cos(n\theta),\quad x=\cos\theta, \tag{7}$$

therefore:

$$T_n=1-2\sin^2\left(n\frac{\theta}{2}\right)=1-2\sin^2\left(\frac{\theta}{2}\right)\frac{\sin^2\left(n\frac{\theta}{2}\right)}{\sin^2\left(\frac{\theta}{2}\right)},$$
$$=1-(1-\cos\theta)2\pi n\underset{F}{K_n}(\theta)=1-(1-x)2\pi n\underset{F}{K_n}, \tag{8}$$

being $\underset{F}{K_n}$ the Fejér's kernel [5, 8-11]:

$$\underset{F}{K_n}(\theta)=\frac{1}{2\pi n}\frac{\sin^2\left(n\frac{\theta}{2}\right)}{\sin^2\left(\frac{\theta}{2}\right)},\quad n=0,1,2,\dots \tag{9}$$

of great importance in Fourier series. Hence (3) and (8) imply the interesting relationship:

$$\tilde{W}_{m-1}(x)=2\pi m\underset{F}{K_m}(\theta),\quad x=\cos\theta, \tag{10}$$

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that is, the Fejér's kernel generates to $\tilde{W}_m$ which were first obtained by Lanczos [12, 13].

In [2, 3, 14, 15] we find the Chebyshev polynomials of fourth-kind:

$$W_n(x) = \frac{\sin\left(n+\frac{1}{2}\right)\theta}{\sin\left(\frac{\theta}{2}\right)} = 2\pi \underset{D}{K_n}(\theta), \quad x = \cos\theta, \tag{11}$$

where appears the Dirichlet Kernel $\underset{D}{K_n}(\theta)$ [5, 8, 9]. Thus (11) makes very natural the existence of (10), and then each kernel has its corresponding Chebyshev- like polynomials.

We also have the Chebyshev polynomials of the second and third kinds [3]:

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad V_n(x) = \frac{\cos\left(n+\frac{1}{2}\right)\theta}{\cos\left(\frac{\theta}{2}\right)}, \tag{12}$$

however, it is important to note the fundamental character of these non-orthogonal Lanczos polynomials $\tilde{W}_m$ because they generate the four kinds:

$$\begin{aligned} &T_m(x) = 1 + (x-1)\tilde{W}_{m-1}(x), \quad W_m(x) = \tilde{W}_m(x) - \tilde{W}_{m-1}(x), \\ &2(1+x)U_m(x) = \tilde{W}_{m+1}(x) - \tilde{W}_{m-1}(x), \\ &(1+x)V_m(x) = 2 + (x-1)\left[\tilde{W}_m(x) + \tilde{W}_{m-1}(x)\right]. \end{aligned} \tag{13}$$

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