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On the Remainder of Lagrangian Interpolation Formula

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Abstract: We give an elementary deduction of the remainder term in the Lagrangian interpolation, with applications via explicit Green functions. The corresponding error term for the finite Taylor series is deduced as a particular case.

Key words: Interpolation by polynomials, Green's function, Finite Taylor series.

INTRODUCTION

If we have the data points $(x_j, f_j = f(x_j)$, $j = 1,2,...,n$ then the function $f(x)$ can be approximated by a polynomial of degree $n - 1$ constructed via the Lagrangian interpolation [1, 2]:

$$P_n(x) = \sum_{k=1}^{n} p_k(x).f_x, \quad P_n(x_j) = f_j, j = 1,...,n, \tag{1}$$

with the fundamental polynomial:

$$F_n(x) = (x - x_1)(x - x_2)...(x - x_n), \quad F_n(x_j) = 0, \quad \forall j, \tag{2}$$

such that:

$$\varphi_k(x) = \frac{F_n(x)}{x - x_k}, \quad \varphi_k(x_k) = F_n^{'}(x_k), \quad \varphi_k(x_j) = 0, \quad j \neq k,$$
$$p_k(x) = \frac{\varphi_k(x)}{\varphi_k(x_k)}, \quad p_k(x_j) = \delta_{jk}. \tag{3}$$

The remainder term in this Lagrangian expansion is given by $\eta_n(x) = f(x) - P_n(x)$, satisfying a differential equation with boundary conditions:

$$\eta_n^{(n)}(x) = f^{(n)}(x), \quad \eta_n(x_j) = 0, \quad j = 1,...,n, \tag{4}$$

whose solution can be written in terms of the Green's function [2-7]:

$$\eta_n(x) = \int_{x_1}^{x_n} f^{(n)}(\xi).G(x,\xi)d\xi, \quad \frac{dn}{dx^n}G(x,\xi) = \delta(x - \xi),$$
$$G(x_j,\xi) = 0, \quad \forall j. \tag{5}$$

The Rolle theorem applied to (5) allows obtain the following estimation:

$$\eta_n(x) = f^{(n)}(\bar{x}).\int_{x_1}^{x_n} G(x,\xi)d\xi, \quad \bar{x}\varepsilon[x_1, x_n], \tag{6}$$

thus the goal is to determine the function:

$$g(x) = \int_{x_1}^{x_n} G(x,\xi)d\xi, \quad g(x_j) = 0, \quad j = 1,...,n. \tag{7}$$

In Sec. 2 we give an elementary process to obtain explicitly the integral (7), which in Sec. 3 is verified with the corresponding Green functions for two and three data points.

Remainder of Lagrange's Polynomial Expansion: From (5) we see that $G^{(n)} = 0$ for $x \neq \xi$, then G is a polynomial of degree $n - 1$ in x and we know that (7) is applied in the form $g(x) = \int_{x_1}^{x_n} G_1 \, d\xi + \int_{x}^{x_n} G_2 \, d\xi,$, therefore $g(x)$ is a polynomial of degree n in x and the x_j are its roots, hence (7) has the structure:

$$g(x) = c(x - x_1)(x - x_2)...(x - x_n), \tag{8}$$

and from (5), (7) and (8):

$$g^{(n)}(x) = \int_{x_1}^{x_n} G^{(n)}d\xi = \int_{x_1}^{x_n} \delta(x - \xi)d\xi = 1 = n!c, \; ,$$

that is:

$$\int_{x_1}^{x_n} G(x,\xi)d\xi = \frac{1}{n!}F_n(x), \tag{9}$$

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thus (6) and (9) imply the following expression for the remainder term of Lagrangian interpolation formula [2]:

$$\eta_n(x) = \frac{1}{n!} f^{(n)}(\bar{x}) F_n(x). \tag{10}$$

Green Functions for Two and Three Data Points: If $n = 2$ then the Green function verifying the properties (5) is given by:

$$G_+ = \frac{1}{x_2 - x_1} (\xi - x_1)(x - x_2), \quad x > \xi, \qquad G_- = \frac{1}{x_2 - x_1} (x - x_1)(\xi - x_2), \quad x < \xi, \tag{11}$$

therefore:

$$\int_{x_1}^{x_2} G(x,\xi)\, d\xi = \int_{x_1}^{x} G_+ \, d\xi + \int_{x}^{x_2} G_- \, d\xi = \frac{1}{2}(x - x_1)(x - x_2),$$

in according with (9).

For $n = 3$ we must consider two regions with their corresponding Green function:

$x_1 \le \xi \le x_2$:

$${}_1G_- = \frac{(x-x_1)\left[(x-x_3)(x_3-x_1)(x_2-\xi)^2 - (x-x_2)(x_2-x_1)(x_3-\xi)^2\right]}{2(x_2-x_1)(x_3-x_1)(x_3-x_2)}, \quad x \le \xi; \quad {}_1G_+ = \frac{1}{2}(x-\xi)^2 + {}_1G_-, \quad x \ge \xi, \tag{12}$$

$x_2 \le \xi \le x_3$:

$${}_2G_- = -\frac{(x-x_1)(x-x_2)(x_3-\xi)^2}{2(x_3-x_1)(x_3-x_2)}, \quad x \le \xi; \qquad {}_2G_+ = \frac{1}{2}(x-\xi)^2 + {}_2G_-, \quad x \ge \xi,$$

and the verification of $\int_{x_1}^{x_n} G d\xi$ also is in two regions:

$$x_1 \le x \le x_2: \quad \int_{x_1}^{x_3} G\, d\xi = \int_{x_1}^{x} {}_1G_+\, d\xi + \int_{x}^{x_2} {}_1G_-\, d\xi + \int_{x_2}^{x_3} {}_2G_-\, d\xi = \frac{1}{3!}(x-x_1)(x-x_2)(x-x_3),$$

$$x_2 \le x \le x_3: \quad \int_{x_1}^{x_3} G\, d\xi = \int_{x_1}^{x_2} {}_1G_+\, d\xi + \int_{x_2}^{x} {}_2G_+\, d\xi + \int_{x}^{x_3} {}_2G_-\, d\xi = \frac{1}{3!} F_3(x),$$

in harmony with (9).

For the general case the Green function is given by [2]:

$${}_jG_- = -\frac{1}{(n-1)!} \sum_{k=j+1}^{n} p_k(x)(x_k - \xi)^{n-1}, \quad x \le \xi; \qquad {}_jG_+ = \frac{1}{(n-1)!}(x-\xi)^{n-1} + {}_jG_-, \quad x \ge \xi, \tag{13}$$

for the intervals $x_j \le \xi \le x_{j+1}, \ j = 1,2,\ldots,n-1$.

If we consider that the data points are equidistant and that they all collapse into $x_1 = a$, then (10) implies the remainder term for the finite Taylor expansion:

$$\eta_n(x) = \frac{1}{n!} f^{(n)}(\bar{x})(x-a)^n, \quad \bar{x}\varepsilon[a,x], \tag{14}$$

thus the Lagrangian interpolation is transformed to Taylor extrapolation.

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