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An Application of the Legendre Polynomials Roots

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Abstract: In order to determine $A=\int_a^b f(x)dx,$, the function $f(x)$ can be tabulated in the points x_j specified by the roots of Legendre polynomials $P_n(\xi)$, thus $y_j = f(x_j)$, then the Gaussian quadrature consists in to approximate A with the area under the corresponding Lagrange interpolating polynomial. If the points x_j are twice, then it is also necessary to give the values of the first derivative y'_j and the respective polynomial is constructed via the Hermite interpolation. Here it is shown in both cases explicit relations to implement the Gauss technique.

Key words: Legendre polynomials · Gaussian quadrature · Lagrange and Hermite interpolations

INTRODUCTION

The various techniques of quadrature try to determine with the minimal error the integral:

$$A=\int_a^b f(x)dx, \tag{1}$$

and in this point, the Gauss method (1814) [1-3] is one of the most efficient because employs the roots ξ_k of the Legendre polynomials [2, 4, 5], leading us to the following data points in [a, b]:

$$x_j=\frac{b+a}{2}+\frac{b-a}{2}\xi_j, \quad \xi_j\in[-1,1],\ j=1,2,...,n, \tag{2}$$

where the ξ_k are the zeros of $P_n(\xi)$. Then it is constructed an interpolating polynomial $G(x)$ which approximates to $f(x)$ in [a, b] and allows to give a value close to (1):

$$\overline{A}=\int_a^b G(x)dx \overset{2}{=} \frac{b-a}{2}\int_{-1}^{1}[G(x)]_{x\to\xi}\, d\xi. \tag{3}$$

Now the question is how to construct $G(x)$: In Secs. 2 and 3 it is indicated the implementation of the corresponding interpolating polynomial when all the x_j are simple and double points, respectively, looking to preserve the efficiency of the Gaussian quadrature. Furthermore, there are shown the resulting expressions for (3) that give an excellent approximation of the area (1).

Lagrange Interpolation. Simple Points: According to the Lagrangian technique [2, 6], the following polynomials are introduced:

a). Fundamental:

$$F(x)=(x-x_1)(x-x_2)\,....\,(x-x_n) \tag{4}$$

b). Auxiliar – Complementary:

$$\Phi_j(x)=\frac{F(x)}{x-x_j},\ \ \Phi_j(x_k)=0,\ j\neq k \tag{5}$$

c). Canonical:

$$p_j(x)=\frac{\Phi_j(x)}{\Phi_j(x_j)}=\frac{F(x)}{(x-x_j)F'(x_j)},\ \ p_j(x_k)=\delta_{jk}, \tag{6}$$

and it is simple to prove that under the gauge transformation (2):

$$p_j(x)|_{x\to\xi}=\frac{\Phi_j(\xi)}{\Phi_j(\xi_j)},\ \ \Phi_j(\xi)=\prod_{\substack{r=1\\ r\neq j}}^{n}(\xi-\xi_r). \tag{7}$$

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So the corresponding interpolating polynomial $G(x)$, of n-1 degree, adopts the form:

$$G(x) = y_1 p_1(x) + y_2 p_2(x) + ... + y_n p_n(x), \quad (8)$$

with the basic property $G(x_j) = y_j = f(x_j)$, then (3) implies the Gaussian quadrature formula for simple points:

$$\overline{A} = \frac{b-a}{2}\sum_{k=1}^{n} y_k \omega_k, \quad \omega_k = \frac{1}{\Phi_k(\xi_k)}\int_{-1}^{1}\Phi_k(\xi)d\xi \quad (9)$$

noticing that there exist Tables for the weight factors ω_k [7].

As an example of (9), let us consider the calculation of $\int_0^4 e^x dx,$, that is, $a = 0$, $b = 4$,

$f(x) = e_x$, for the case n=5, then [2, 4, 7]:

$\xi_1 = -\xi_5 = -0.9061\ 798459$, $\xi_2 = -\xi_4 = -0.5384\ 693101$, $\xi_3 = 0$,

$\omega_1 = \omega_5 = 0.2369\ 2688\ 51$, $\omega_2 = \omega_4 = 0.4786\ 2867\ 05$, $\omega_3 = 0.5688\ 8888\ 89$,

(10)

with the corresponding values:
$x_1 = 0.1876\ 4031$, $y_1 = 1.2063\ 9950$,
$x_2 = 0.9230\ 6138$, $y_2 = 2.5196\ 8405$,
$x_3 = 2.0000\ 0000$, $y_3 = 7.3890\ 5610$,
$x_4 = 3.0769\ 3862$, $y_4 = 21.6918\ 9349$,
$x_5 = 3.8123\ 5969$, $y_5 = 45.2571\ 0562$, (11)

and (9) leads to the following approximate value:

$$\int_0^4 e^x d_x \approx \overline{A} = 2\sum_{k=1}^{5} y_k \omega_k = 53.5981\ 3663, \quad (12)$$

which can be compared with the exact value 53.5981 5003.

Hermite Interpolation. Double Points: In order to get (2), the roots (10) are used with 10 decimals, but may be in the laboratory the instruments do not allow to work with so many decimals, therefore forcing to the Legendre roots ξ_j [8] to be rounded, so for instance, instead of (10) it could be employed the values:

$$\xi_1 = -\xi_5 = -0.90,\ \xi_2 = -\xi_4 = 0.54,\ \xi_3 = 0.00,$$
$$x_1 = 0.20,\ x_2 = 0.92,\ x_3 = 2.00,\ x_4 = 3.08,\ x_5 = 3.80, \quad (13)$$

which together with (9) would give $\overline{A}$ with much more error than (12) and in this way the Gaussian quadrature efficiency is lost.

This situation is solved ingeniously [2] just assuming as double points the x_j and consequently the corresponding interpolating polynomial of degree $(2n - 1)$ is constructed by means of the Hermite technique [9]:

$$G(x) = \sum_{k=1}^{n}\left[y_k p_k^{(1)}(x) + y'_k\, p_k^{(2)}(x)\right], \quad (14)$$

where:

$$p_k^{(1)}(x) = [1 - 2p_k'(x_k)(x - x_k)]p_k^2(x), p_k^{(2)}(x) = (x - x_k)p_k^2(x), \quad (15)$$

and (9) is modified:

$$\overline{A} = \frac{b-a}{2}\sum_{k=1}^{n} y_k w_k + \left(\frac{b-a}{2}\right)^2 \sum_{k=1}^{n} y'_k \tilde{w}_k, \quad (16)$$

such that:

$$\tilde{w}_k = \frac{1}{\Phi_k^2(\xi_k)}\int_{-1}^{1}(\xi - \xi_k)\Phi_k^2(\xi)d\xi,$$

$$w_k = \frac{1}{\Phi_k^2(\xi_k}\int_{-1}^{1}\Phi_k^2(\xi)d\xi - 2\frac{\Phi'_k(\xi_k)}{\Phi_k(\xi_k)}\tilde{w}_k \quad (17)$$

Therefore, when (13) and (17) are used:

$$\begin{array}{lll} y_1 = y'_1 = 1.2214\ 0275 & w_1 = 0.2364\ 0530 & \tilde{w}_1 = -0.0015\ 5377 \\ y_2 = y'_2 = 2.5092\ 9039 & w_2 = 0.4789\ 9553 & \tilde{w}_2 = 0.0005\ 8042 \\ y_3 = y'_3 = 7.3890\ 5609 & w_3 = 0.5691\ 9830 & \tilde{w}_2 = 0 \\ y_4 = y'_4 = 21.7584\ 0240 & w_4 = w_2 & \tilde{w}_4 = -\tilde{w}_2 \\ y_5 = y'_5 = 44.7011\ 8449 & w_5 = w_1 & \tilde{w}_5 = -\tilde{w}_1 \end{array} \quad (18)$$

then the expression (16) for double points gives the value 53.5981 3516, with an error of the same order that in (12), so the Gaussian quadrature efficiency being restored.

CONCLUSIONS

When the Gauss quadrature is applied, two situations can be arised:

- It may be posible to manage the Legendre roots with enough significative numbers, so (9) is an excellent approximation for the corresponding area, that is, such roots do participate as simple points.
- The zeros ξ_j may be rounded in order to reduce its number of decimals, then in this case it must be used (16) for double points to get an error of the same order of magnitude that (9).

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