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# On the Wannemacker's Identity for Stirling Numbers of the Second Kind

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**Abstract:** We deduce a result involving the Stirling numbers of the first and second kind, which implies the Wannemacker's relation.

**Key words:** Stirling numbers · Wannemacker's identity

## INTRODUCTION

Here we obtain the result:

$$A \equiv \sum_{k=0}^{n} S_n^{(k)} S_{j+k}^{[N]} = \begin{cases} 0, \ N < n, & (1.a) \\ n^j, \ \ N = n, & (1.b) \\ \dfrac{1}{(Nn)!} \sum_{r=N-n}^{N} \dbinom{n}{N-r} r! S_j^{[N]}, N > n, & (1.c) \end{cases}$$

where $S_r^{(i)}$ and $S_j^{(t)}$ are the Stirling numbers of the first and second kind, respectively [1-7]. It is clear that (1.b) is consequence of (1.c) if $N = n$, For the case $j = 0$ the relations (1) imply the known property [1, 4]:

$$\sum_{k=0}^{n} S_n^{(k)} S_k^{[N]} = \delta_{nN}, \tag{2}$$

The application of $\sum_{n=0}^{t} S_t^{[n]}$ to (1.c) gives:

$$S_{j+t}^{[N]} = \sum_{r=0}^{N} \sum_{q=0}^{N} \binom{q}{r} \frac{(N-r)!}{(N-q)!} S_j^{[N-r]} S_t^{[q]}, \quad 0 \le N \le j+t, \tag{3}$$

where (2) was used. The identity (3) was first deduced by Wannemacker [8] while making calculations in the Witt ring. For $j = 0,1$ the expression (3) leads to $S_t^{[N]} = S_t^{[N]}$ and to the recurrence relation $S_{t+1}^{[N]} = S_t^{[N-1]} + N S_t^{[N]}$, respectively.

**An Identity for Stirling Numbers:** We have the Euler's formula [1, 4]:

$$S_{j+k}^{[N]} = \frac{(-1)^N}{N!} \sum_{t=0}^{N} (-1)^t \binom{N}{t} t^{j+k}, \tag{4}$$

thus:

$$A = \frac{(-1)^N n!}{N!} \sum_{t=0}^{N} (-1)^t \binom{N}{t} \binom{t}{n} t^j, \tag{5}$$

where it was applied the property [1, 4]:

$$\sum_{k=0}^{n} S_n^{(k)} t^k = n! \binom{t}{n}. \tag{6}$$

The result (1.a) is immediate from (5) for $N < n$. If $N = n$ then (5) gives $N > n$ in agreement with (1.b).

Now we accept that $N > n$ With the relation [4, 9, 10]:

$$\binom{N}{t}\binom{t}{n} = \binom{N}{n}\binom{N-n}{t-n}, \tag{7}$$

and the expression [1, 4]:

$$t^j = \sum_{r=0}^{j} r! \binom{t}{r} S_j^{[r]}, \tag{8}$$

we obtain from (5):

$$A = \frac{(-1)^N}{(N-n)!} \sum_{r=0}^{j} r! S_j^{[r]} \sum_{t=n}^{N} (-1)^t \binom{N-n}{t-n} \binom{t}{r},$$

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$$= \frac{(-1)^{N-n}}{(N-n)!} \sum_{r=0}^{j} [\sum_{k=0}^{N-n} (-1)^k \binom{N-n}{k} \binom{k+n}{r}] r! S_j^{[r]}, \tag{9}$$

but the identity (3.47) in [11] implies:

$$\sum_{k=0}^{N-n} (-1)^k \binom{N-n}{k} \binom{k+n}{r} = (-1)^{N-n} \binom{n}{N-r}, \tag{10}$$

whose application in (9) gives:

$$A = \frac{1}{(N-n)!} \sum_{r=0}^{j} \binom{n}{N-r} r! S_j^{[r]}, \tag{11}$$

where it is simple consider the situations $N > j$ and $N \leq j$ to prove that in (11) the range $[0, j]$ for $r$ can be replaced by $[N-n, N]$, in harmony with (1.c).

From (1) for $j = n - 1$:

$$\sum_{k=0}^{n} S_n^{(k)} S_{n+k-1}^{[N]} = n^{n-1} \delta_{nN}, \tag{12}$$

which is compatible with the Olson's identity [4, 12]:

$$\sum_{k=0}^{n} S_m^{(k)} S_{n+k-1}^{[N]} = \begin{cases} 0, & N < m, \\ m^{n-1} & N = m, \end{cases} \tag{13}$$

for the case $m = n$

Our study shows that the Wannemacker's relation indicated in (3) can be seen as a consequence of (1).

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