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The Elementary Arithmetic Operators of Continued Fraction

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Abstract: The finite and infinite simple continued fractions are considered. The addition, subtraction and equality of two simple continued fractions are presented. The applications of continued fractions are also studied.

Key words: Simple continued fraction · Negative rational number · Multiplicative inverse

INTRODUCTION

The continued fractions have been studied in mathematical (Diophantine and Pell's equations) and physical (gear ratio) [1-4]. Our simple continued fractions that we study are reminiscent of various other continued fractions problems. However they are very different from ours. One is analytic of continued fractions [3-8], which has been discussed for the real and complex values. Another study is the continued fractions for the integer values [2, 4, 6].

Currently, continued fractions have many practical uses in mathematics. Forinstance, we can express any number, rational or irrational, as a finite or infinitecontinued fraction expression. We can also solve any Diophantine congruence thatis any equivalence of the form $ax \equiv b\ (mod\ m)$. In other words, in most real-world applications of mathematics, continued fractions are rarely themost practical way to solve a given set of problems as decimal approximations are much more useful (and computers can work with decimals at a much faster rate). However, some interesting observations can still be made using continued fractions. Namely, in this paper, we will be exploring how continued fractions can be used to add and subtract the numbers $\sqrt{a} \mp \sqrt{b}$ and $\frac{a}{b} \mp \sqrt{c}$. We start with some definitions and theorems that we used to defined the addition and the subtraction of two simple continued fractions.

Definition 1: An expression of the form.

$$a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cdots}}}$$

is called a continued fraction. The values a_0, a_1, a_2 and $b_0, b_1 b_2, \ldots$ can be either real or complex and their numbers can be either finite or infinite.

Definition 2: A finite simple continued fraction is a continued fraction (definition 1) in which $b_n = 1$ for all n, that is:

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_n}}}$$

wherea_n is positive integer for all $n \geq 1$, a_0 can be any integer number. The above fraction is sometime represented by $[a_1;a_1,a_2,\ldots,a_n]$ for finite simple continued fraction and $[a_{00};a_1,a_2]$ for infinite simple continued fraction. In this paper we will use the symbol (S.C.F.) for the simple continued fraction.

- $1 + \cfrac{1}{2 + \cfrac{2}{3i}}$ is a finite complex valued continued fraction.
- $3 + \cfrac{1}{6 + \cfrac{1}{7 + \cfrac{1}{9 + \cfrac{1}{17}}}} = [3,6,7,9,17]$ is finite S.C.F..
- $4 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{3 + \cdots}}} = [4;2,3,3,\cdots]$ is an infinite S.C.F..

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Theorem 1: A number is rational if and only if it can be expressed as a finite S.C.F. [4].

For example

$$\frac{62}{23}=2+\frac{16}{23}=2+\frac{1}{\frac{23}{16}}=2+\frac{1}{1+\frac{7}{16}}=2+\frac{1}{1+\frac{1}{\frac{16}{7}}}=2+\frac{1}{1+\frac{1}{2+\frac{2}{7}}}=2+\frac{1}{1+\frac{1}{2+\frac{1}{3+\frac{1}{2}}}}=[2;1,2,3,2].$$

Remark 1:

- To expand a rational number $\frac{b}{a}$, $(a > b > 0)$ into S.C.F. we write

$$\frac{b}{a}=0+\frac{1}{\frac{a}{b}},$$

and then we use the same techniques as in theorem 1 for $\frac{a}{b}$.

- To expand a negative rational number $-\frac{b}{a}$ $(a, b, > 0)$ into S.C.F. we take the greatest integer number $\left[\!\left[-\frac{b}{a}\right]\!\right]$ for the first term of S.C.F.

that is,

$$\left[\!\left[-\frac{b}{a}\right]\!\right]=-a_0' \qquad \text{where} \qquad -a_0' \le -\frac{b}{a} < -a_0' + 1.$$

We write

$$-\frac{b}{a}=-a_0'+\frac{1}{\frac{b'}{a'}},$$

and then we use the same techniques as in theorem 1 to get the remaining terms for $\frac{b'}{a'}$. That is, if $\frac{b'}{a'}=[a_1';a_2',...,a_n']$ then $-\frac{b}{a}=[-a_0';a_1',...,a_n']$.

Example 1:

(1) $\frac{3}{7}=0+\frac{1}{2+\frac{1}{3}}$ = [0, 2, 3] (2) $-\frac{27}{5}=-6+\frac{1}{1+\frac{1}{1+\frac{1}{2}}}$ = [-6, 1, 1, 2].

Lemma 1: Any finite S.C.F. $[a_0; a_1,...,a_n]$ can be also represented by $[a_0;a_1,...,a_n - 1,1]$ [5].

Definition 3:

The S.C.F. $[a_0; a_1,...,a_n]$ can be defined by

$$[a_0;a_1,...,a_n]=a_0+\frac{K_{n-1}(a_2)}{K_n(a_1)}, \text{ or } [a_0;a_1,...,a_n]=\frac{K_{n+1}(a_0)}{K_n(a_1)}.$$

where,

$K_0(a_0) = 1$ $\quad$ $K_0(a_1) = 1$

$K_1(a_0) = a_0$ $\quad$ $K_1(a_1) = a_1$

$K_2(a_0) = a_0 a_1 + 1$ $\quad$ $K_2(a_1) = a_1\ a_2 + 1$

$K_3(a_0) = a_0\ a_1 a_2 + a_0 + a_2$ $\quad$ $K_3(a_1) = a_1\ a_2 a_3 + a_1 + a_3$

$\vdots$ $\quad$ $\vdots$

$K_i(a_0) = a_{i-1}\ K_{i-1}(a_0) + K_{i-2}(a_0)$ $\quad$ $K_i(a_1) = a_i\ K_{i-1}(a_1) + K_{i-2}(a_1)$

In general

$$K_i(a_j) = a_{i+(j-1)}\ K_{i-1}(a_j) + K_{i-2}(a_j), \quad i = 1,2,...,n, \quad j = 0,1,...,n$$
$$K_{-i}(a_j) = 0 \qquad K_0(a_j) = 1$$

Lemma 2: If $C_j = 0$ in, $[c_0; c_1,...,c_{j-1}, c_j, c_{j+1},...c_n]$ for some j with $0 < j < n$, then we can replace $c_{j-1}$by $cj_{-1} = c_{j-1} + c_{j+1}$ and delete c_j, c_{j+1} from the simple continued fraction expansion, without changing the value of S.C.F.

Lemma 3: If $c_j = c_{j+1}$ 0 in $[\, c_0;c_1,...,c_{j-1},c_j,c_{j+1},c_{j+2},...,c_n\,]$, for some j with $0 < j < n$ then we can delete c_j, c_{j+1} from the simple continued fraction expansion, without changing the value of S.C.F..

Definition 4:

Let $[a_0;a_1,...,a_n]$, $[b_0;b_1,...,b_n]$ be two simple continued fractions, then,

- $[a_0;a_1,...,a_n] = [b_0;b_1,...,b_n]$ if $a_i = b_i$ for $i = 0,1,2,...,n$.
- $[a_0;a_1,...,a_{n-1},a_n] = [a_0;a_1,...,a_{n-1},a_n - 1,1]$.
- $[a_0;...,a_{j-2},a_{j-1},0,a_{j+1},a_{j+2},...,a_n] = [a_0;...,a_{j-2},a_{j-1} + a_{j+1},a_{j+2},...,a_n]$.

Theorem 2: Let $[a_0;a_1,...,a_m]$ and $[b_0;b_1,...,b_n]$ be two S.C.F. with $a_i \neq 0$ for all i=1,2,...m and, $b_j \neq 0$ for all j = 1,2,...n and $[a_0;a_1,...,a_m] \neq [b_0;b_1,...,b_n]$ then

- If $a_0 \neq b_0$, let $a_0 > b_0$, then

  $[a_0;a_1,...,a_m] > [b_0;b_1,...,b_n]$.

- If $a_i = b_i$ for all i=0,1,2,...k, k< min (m, n)

  let $a_{k+1} > b_{k+1}$, with k +1 $\leq$ min (m, n)

  then $[a_0;a_1,...,a_m] > [b_0;b_1,...,b_n]$ if k is odd

  and $[a_0;a_1,...,a_m] < [b_0;b_1,...,b_n]$ if k is even.

- If $a_i = b_i$ for all i = 0,1,2,...k. k = min (m, n)

  let$k$ = m, then $[a_0;a_1,...,a_m] > [b_0;b_1,...,b_n]$ if k is odd

  and $[a_0;a_1,...,a_m] < [b_0;b_1,...,b_n]$ if k is even.

For examples

- [3;2, 2] > [2;1, 4], since $a_0 > b_0$.
- [3;1, 1, 3]> [3;7], since $a_0 = b_0$ and $b_1 > a_i$.
  [1;2, 3] < [1;2, 3, 4], since $a_i = b_i$ for all i = 0, 1, 2 and $k = m$ = 2 (even).
- [1;2, 3, 4] > [1;2, 3, 4, 5], since $a_i = b_i$ for all i= 0, 1, 2, 3 and $k = m$ = 3 (odd).

Definition 5: Let $[a_0;a_1,...,a_m]$ and $[b_0;b_1,...,b_m]$ be two S.C.F., we define *addition* by

(1) If m=n then

$$[a_0;a_1,...,a_n]+[b_0;b_1,...,b_n]=[c_0;c_1,...,c_n] \tag{5a}$$

where,

$$\begin{aligned}
c_0 &= a_0 + b_0 = K_1(a_0) + K_1(b_0)\\
c_1 &= \left[\!\left[\frac{a_1 b_1}{a_1 + b_1}\right]\!\right] = \left[\!\left[\frac{K_1(a_1)K_1(b_1)}{K_1(a_1)+K_1(b_1)}\right]\!\right]\\
c_2 &= \left[\!\left[\frac{a_2(b_1b_2+1)+b_2(a_1a_2+1)}{(a_1a_2+1)(b_1b_2+1)-c_1[a_2(b_1b_2+1)+b_2(a_1a_2+1)]}\right]\!\right]\\
&= \left[\!\left[\frac{K_1(a_2)K_2(b_1)+K_1(b_2)K_2(a_1)}{K_2(a_1)K_2(b_1)-c_1[K_1(a_2)K_2(b_1)+K_1(b_2)K_2(a_1)]}\right]\!\right]\\
c_3 &= \left[\!\left[\frac{K_3(a_1)K_3(b_1)K_0(c_2)-[K_2(a_2)K_3(b_1)+K_2(b_2)K_3(a_1)]K_1(c_1)}{[K_2(a_2)K_3(b_1)+K_2(b_2)K_3(a_1)]K_2(c_1)-K_3(a_1)K_3(b_1)K_1(c_2)}\right]\!\right]\\
&\vdots \qquad\qquad \vdots \qquad\qquad \vdots\\
c_i &= \begin{cases}
\left[\!\left[\dfrac{K_i(a_1)K_i(b_1)K_{i-3}(c_2)-[K_{i-1}(a_2)K_i(b_1)+K_{i-1}(b_2)K_i(a_1)]K_{i-2}(c_1)}{[K_{i-1}(a_2)K_i(b_1)+K_{i-1}(b_2)K_i(a_1)]K_{i-1}(c_1)-K_i(a_1)K_i(b_1)K_{i-2}(c_2)}\right]\!\right], & i \text{ odd}\\[2ex]
\left[\!\left[\dfrac{[K_{i-1}(a_2)K_i(b_1)+K_{i-1}(b_2)K_i(a_1)]K_{i-2}(c_1)-K_i(a_1)K_i(b_1)K_{i-3}(c_2)}{K_i(a_1)K_i(b_1)K_{i-2}(c_2)-[K_{i-1}(a_2)K_i(b_1)+K_{i-1}(b_2)K_i(a_1)]K_{i-1}(c_1)}\right]\!\right], & i \text{ even}
\end{cases}
\end{aligned}$$

for i = 2, …, n.

The last term c_n of the resulting S.C.F. is to be expanded again as a S.C.F. if necessary and not to be treated as the greatest integer number as the preceding terms have been treated.

(2) If $m \neq n$ then

Suppose that m<n then

$$[a_0;a_1,...,a_m]+[b_0;b_1,...,b_m,b_{m+1},...,b_n]=[c'_0;c'_1,...,c'_m,c'_{m+1},...,c'_n] \tag{5b}$$

where $c'_j = c_j$ for j = 0, 1, 2, …, m ,

and c_j as we did for case m=n, while $c'_j = c_{j,j-m}$ and

$$c_{j,j-m}=\begin{cases}\left[\!\left[\dfrac{K_m(a_1)K_j(b_1)K_{j-3}(c_2')-[K_{m-1}(a_2)K_j(b_1)+K_{j-1}(b_2)K_m(a_1)]K_{j-2}(c_1')}{[K_{m-1}(a_2)K_j(b_1)+K_{j-1}(b_2)K_m(a_1)]K_{j-1}(c_1')-K_m(a_1)K_j(b_1)K_{j-2}(c_2')}\right]\!\right], & j \text{ odd}\\[2ex] \left[\!\left[\dfrac{[K_{m-1}(a_2)K_j(b_1)+K_{j-1}(b_2)K_m(a_1)]K_{j-2}(c_1')-K_m(a_1)K_j(b_1)K_{j-3}(c_2')}{K_m(a_1)K_j(b_1)K_{j-2}(c_2')-[K_{m-1}(a_2)K_j(b_1)+K_{j-1}(b_2)K_m(a_1)]K_{j-1}(c_1')}\right]\!\right], & j \text{ even}\end{cases}$$

for$j = m+1, m+2, \ldots, n$.

Also, the last term of the resulting S.C.F. is to be treated as mentioned earlier.

Remark 2:

- If the number of the S.C.F. term's for $[a_0;a_1,...,a_n]$ and $[b_0;b_1,...,b_n]$ are n, then it is not necessary for the number of terms of the resulting S.C.F. $[a_0;a_1,...,a_n] + [b_0;b_1,...,b_n]$ to be n.
- If $[a_0]$ and $[b_0;b_1,...,b_n]$ are two S.C.F. then $[a_0] + [b_0;b_1,...,b_n] = [c_0 b_0;b_1,...,b_n]$ where c_0 is given in equation (5a).

Example 2:

Find [1;2,5] + [4;3,2].

Solution:

Let [1;2, 5] = $[a_0;a_1,a_2]$ and [4;3, 2] = $[b_0 b_1,b_2]$, we get m=n= 2.
From equation (5a) we have
[1;2,5] + [4;3,2] = $[a_0;a_1,a_2] + [b_0;b_1,b_2 = [c_0;c_1,c_2]$,
where c_2 is the last term and

$$c_0 = a_0 + b_0 = 1 + 4 = 5$$
$$c_1 = \left[\!\left[\frac{a_1 b_1}{a_1+b_1}\right]\!\right] = \left[\!\left[\frac{2\cdot 3}{2+3}\right]\!\right] = \left[\!\left[\frac{6}{5}\right]\!\right] = 1$$
$$c_2 = \frac{a_2(b_1b_2+1)+b_2(a_1a_2+1)}{(a_1a_2+1)(b_1b_2+1)-c_1[a_2(b_1b_2+1)+b_2(a_1a_2+1)]}$$
$$= \frac{5(3\cdot 2+1)+2(5\cdot 2+1)}{(3\cdot 2+1)(5\cdot 2+1)-1[5(3\cdot 2+1)+2(5\cdot 2+1)]}$$
$$= \frac{5(7)+2(11)}{(7)(11)-[35+22]} = \frac{57}{77-57} = \frac{57}{20} = [2;1,5,1,2],$$

then [1;2,5] + [4;3,2] = [5;1,2,1,5,1,2]

Example 3:

Find [17;7] + [18;1,1,3].

Solution:

Let [17;7] = $[a_0; a_1]$ and [18;1, 1, 3] = $[b_0, b_1,b_2,b_3]$, we get m= 1, n= 3 and $m < n$.
From equation (5b) we have,

$[17;7] + [18;1,1,3] = [a_0;a_1] + [b_0;b_1,b_2,b_3] = [c_0;c_1,c_2',c_3']$

where c_3' is the last term and

$$c_0 = c_0' = a_0 + b_0 = 17 + 18 = 35$$
$$c_1 = c_1' = \left[\!\left[\frac{a_1 b_1}{a_1 + b_1}\right]\!\right] = \left[\!\left[\frac{7\cdot 1}{7+1}\right]\!\right] = \left[\!\left[\frac{7}{8}\right]\!\right] = 0$$
$$c_2' = c_{2,1} = \left[\!\left[\frac{[K_0(a_2)K_2(b_1) + K_1(b_2)K_1(a_1)]K_0(c_1') - K_1(a_1)K_2(b_1)K_{-1}(c_2')}{K_1(a_1)K_2(b_1)K_0(c_2') - [K_0(a_2)K_2(b_1) + K_1(b_2)K_1(a_1)]K_1(c_1')}\right]\!\right]$$
$$= \left[\!\left[\frac{[1\cdot(b_1b_2+1) + b_2a_1]\cdot 1 - a_1\cdot(b_1b_2+1)\cdot 0}{a_1(b_1b_2+1) - [(b_1b_2+1) + b_2a_1]\cdot c_1'}\right]\!\right]$$
$$= \left[\!\left[\frac{(1\cdot 1+1) + 1\cdot 7}{7\cdot 2}\right]\!\right] = \left[\!\left[\frac{9}{14}\right]\!\right] = 0$$
$$c_3' = c_{3,2} = \frac{K_1(a_1)K_3(b_1)K_0(c_2') - [K_0(a_2)K_3(b_1) + K_2(b_2)K_1(a_1)]K_1(c_1')}{[K_0(a_2)K_3(b_1) + K_2(b_2)K_1(a_1)]K_2(c_1') - K_1(a_1)K_3(b_1)K_1(c_2')}$$
$$= \frac{a_1(b_1b_2b_3 + b_1 + b_3) - [(b_1b_2b_3 + b_1 + b_3) + a_1(b_2b_3+1)]c_1'}{[(b_1b_2b_3 + b_1 + b_3) + a_1(b_2b_3+1)](c_1'c_2'+1) - a_1(b_1b_2b_3 + b_1 + b_3)\cdot c_2'}$$
$$= \frac{7\cdot(1\cdot 1\cdot 3 + 1 + 3) - [(1\cdot 1\cdot 3 + 1 + 3) + 7\cdot(1\cdot 3+1)]\cdot 0}{[(1\cdot 1\cdot 3 + 1 + 3) + 7\cdot(1\cdot 3+1)] - 7\cdot(1\cdot 1\cdot 3 + 1 + 3)\cdot 0}$$
$$= \frac{7\cdot(1\cdot 1\cdot 3 + 1 + 3)}{[(1\cdot 1\cdot 3 + 1 + 3) + 7\cdot(1\cdot 3+1)]}$$
$$= \frac{7\cdot 7}{7 + 7\cdot 4} = \frac{7}{1+4} = \frac{7}{5} = [1;2,2,]$$

Therefore [17;7] + [18;1,1,3] = [35;0, 0, 1, 2, 2]= [35;1, 2, 2].

Example 4:
Find [4] + [1;1, 2].

Solution:

Let $[4] = [a_0]$ and $[1;1, 2] = [b_0;b_1,b_2]$ we get $m= 0$, $n= 2$, $m<n$.
From remark 2(ii) we have,
$[4] + [1;1, 2] = [a_0] + [b_0,b_1,b_2] = [c_0b_1,b_2]$ where
$c_0 = a_0 + b_0 = 4 + 1 = 5$.

Therefore [4] + [1, 1, 2] = [5, 1, 2].

Corollary 1: If c is a non-zero integer then $c[a_0, a_1, a_2, a_3, ...] = [ca_0, \frac{a_1}{c}, ca_2, \frac{a_3}{c}, ...]$ which is not necessary a S.C.F..

Example 5:
Find –2[1;2, 2].

Solution:
From corollary 1 we have
$$-2[1;2,2] = [-2(1); \frac{2}{-2}, -2(2)] = [-2;-1,-4].$$

Remark 3: Note that the multiplication of S.C.F. by a non-zero integer $\neq 1$ does not necessarily lead to S.C.F. as we have seen in example 5.

Definition 6: Let $[a_0;a_1,...,a_n]$ be a S.C.F., then we can define *additive inverses* by $-[a_0;a_1,...,a_n]=[b_0';b_n']$

where,

$$b_0' = -1 - a_0 = d_0$$
$$b_n' = \frac{K_n(a_1)}{K_n(a_1) - K_{n-1}(a_2)} \qquad \text{(to be treated as S.C.F.)}$$
$$= [d_1;d_2,...,d_l]$$

therefore,

$$-[a_0;a_1,...,a_n]=[b_0';b_n']=[d_0;d_1,d_2,...,d_l].$$

Definition 7: If $[a_0;a_1,...,a_m]$ and $[b_0;b_1,...,b_n]$ are two S.C.F. then we define *Subtraction* $[a_0;a_1,...,a_m]$ - $[b_0;b_1,...,b_n]$ by the addition, $[a_0;a_1,...,a_m]+[d_0;d_1,...,d_l]$ where $[d_0;d_1,...,d_l]$ given by definition 6.

That is;

- If $m=l$ then

$$[a_0;a_1,...,a_m]+[d_0;d_1,...,d_m]=[c_0;c_1,...,c_m] \tag{7a}$$

where $c_0,c_1,...,c_m$ as given in equation5a.

- If $m\neq l$, $l<m$ then

$$[a_0;a_1,...,a_l,a_{l+1},...,a_m]+[d_0;d_1,...,d_l]=[c_0';c_1',...,c_l',c_{l+1}',...,c_m'] \tag{7b}$$

where $c_0',c_1',...,c_m'$ as given in equation 5b.

The motivations of our definitions and the analytic prove are published in [5].

Example 6:

Find $[1;2,2,3,5,3,1,2,4,9]+[1;1,8,4,2,1,3,5,3,2,2]$ and $-[1;1,8,4,2,1,3,5,3,2,2]$.

Solution:

- To find $[1;2,2,3,5,3,1,2,4,9]+[1;1,8,4,2,1,3,5,3,2,2]$.

Let $[1;2,2,3,5,3,1,2,4,9]=[a_0;a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9]$, $m=9$

and $[1;1,8,4,2,1,3,5,3,2,2]=[b_0;b_1,b_2,b_3,b_4,b_5,b_6,b_7,b_8,b_9,b_{10}]$, $n=10$, $n\neq m$.

From equation (5b) we have

$$[a_0;a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9]+[b_0;b_1,b_2,b_3,b_4,b_5,b_6,b_7,b_8,b_9,b_{10}]$$
$$=[c_0;c_1,c_2,c_3,c_4,c_5,c_6,c_7,c_8,c_9,c_{10}']$$

where c'_{10} is the last term and

$$c_0 = c'_0 = a_0 + b_0 = 1 + 1 = 2$$

$$c_1 = c'_1 = \left[\!\left[\frac{a_1 b_1}{a_1 + b_1} \right]\!\right] = \left[\!\left[\frac{2}{3} \right]\!\right] = 0$$

$$c_2 = c'_2 = \left[\!\left[\frac{K_1(a_2)K_2(b_1) + K_1(b_2)K_2(a_1)}{K_2(a_1)K_2(b_1) - c_1[K_1(a_2)K_2(b_1) + K_1(b_2)K_2(a_1)]} \right]\!\right]$$

$$= \left[\!\left[\frac{2(9) + 8(5)}{(9)(5)} \right]\!\right] = \left[\!\left[\frac{58}{45} \right]\!\right] = 1$$

$$c_3 = c'_3 = \left[\!\left[\frac{K_3(a_1)K_3(b_1)K_0(c_2) - [K_2(a_2)K_3(b_1) + K_2(b_2)K_3(a_1)]K_1(c_1)}{[K_2(a_2)K_3(b_1) + K_2(b_2)K_3(a_1)]K_2(c_1) - K_3(a_1)K_3(b_1)K_1(c_2)} \right]\!\right]$$

$$= \left[\!\left[\frac{K_3(a_1)K_3(b_1)}{K_2(a_2)K_3(b_1) + K_2(b_2)K_3(a_1) - K_3(a_1)K_3(b_1)K_1(c_2)} \right]\!\right]$$

$$= \left[\!\left[\frac{17(37)}{(33)(17) + (7)(37) - (17)(37)} \right]\!\right] = \left[\!\left[\frac{629}{191} \right]\!\right] = 3$$

$$c_4 = c'_4 = \left[\!\left[\frac{[K_3(a_2)K_4(b_1) + K_3(b_2)K_4(a_1)]K_2(c_1) - K_4(a_1)K_4(b_1)K_1(c_2)}{K_4(a_1)K_4(b_1)K_2(c_2) - [K_3(a_2)K_4(b_1) + K_3(b_2)K_4(a_1)]K_3(c_1)} \right]\!\right]$$

$$= \left[\!\left[\frac{[(37)(83) + (47)(90)] - (83)(90)(1)}{(83)(90)(4) - [9731](3)} \right]\!\right] = \left[\!\left[\frac{2261}{687} \right]\!\right] = 3$$

$$c_5 = c'_5 = \left[\!\left[\frac{K_5(a_1)K_5(b_1)K_2(c_2) - [K_4(a_2)K_5(b_1) + K_4(b_2)K_5(a_1)]K_3(c_1)}{[K_4(a_2)K_5(b_1) + K_4(b_2)K_5(a_1)]K_4(c_1) - K_5(a_1)K_5(b_1)K_3(c_2)} \right]\!\right]$$

$$= \left[\!\left[\frac{(287)(120)(4) - [(118)(120) + (107)(287)](3)}{(44869)(100) - [34440](13)} \right]\!\right] = \left[\!\left[\frac{3153}{970} \right]\!\right] = 3$$

$$c_6 = c'_6 = \left[\!\left[\frac{[K_5(a_2)K_6(b_1) + K_5(b_2)K_6(a_1)]K_4(c_1) - K_6(a_1)K_6(b_1)K_3(c_2)}{K_6(a_1)K_6(b_1)K_4(c_2) - [K_5(a_2)K_6(b_1) + K_5(b_2)K_6(a_1)]K_5(c_1)} \right]\!\right]$$

$$= \left[\!\left[\frac{[(377)(395) + (443)(155)](10) - (377)(443)(13)}{(167011)(43) - [217580](33)} \right]\!\right]$$

$$= \left[\!\left[\frac{4657}{1333} \right]\!\right] = 3$$

$$c_7 = c'_7 = \left[\!\left[\frac{K_7(a_1)K_7(b_1)K_4(c_2) - [K_6(a_2)K_7(b_1) + K_6(b_2)K_7(a_1)]K_5(c_1)}{[K_6(a_2)K_7(b_1) + K_6(b_2)K_7(a_1)]K_6(c_1) - K_7(a_1)K_7(b_1)K_5(c_2)} \right]\!\right]$$

$$= \left[\!\left[\frac{(1041)(2335)(43) - [(1041)(2082) + (2335)(428)](33)}{(3166742)(109) - [2430735](142)} \right]\!\right]$$

$$= \left[\!\left[\frac{19119}{10508} \right]\!\right] = 1$$

$$c_8 = c'_8 = \left[\!\left[\frac{[K_7(a_2)K_8(b_1) + K_7(b_2)K_8(a_1)]K_6(c_1) - K_8(a_1)K_8(b_1)K_5(c_2)}{K_8(a_1)K_8(b_1)K_6(c_2) - [K_7(a_2)K_7(b_1) + K_7(b_2)K_8(a_1)]K_7(c_1)} \right]\!\right]$$

$$= \left[\!\left[\frac{[(4541)(6641) + (7448)(1857)](109) - (4541)(7448)(142)}{(33821368)(6106) - [66331717](142)} \right]\!\right]$$

$$= \left[\!\left[\frac{4227522897}{197094169194} \right]\!\right] = 0$$

$$c_9 = c'_9 = \left[\!\left[\frac{K_9(a_1)K_9(b_1)K_6(c_2) - [K_8(a_2)K_9(b_1) + K_8(b_2)K_9(a_1)]K_7(c_1)}{[K_7(a_2)K_9(b_1) + K_8(b_2)K_9(a_1)]K_8(c_1) - K_9(a_1)K_9(b_1)K_7(c_2)} \right]\!\right]$$

$$= \left[\!\left[\frac{(41910)(17231)(185) - [(41410)(15364) + (17231)(17231)](142)}{(940812601)(109) - [722151210](142)} \right]\!\right]$$

$$= \left[\!\left[\frac{2584508}{3101709} \right]\!\right] = 0$$

$$c'_{10} = \frac{[K_8(a_2)K_{10}(b_1) + K_9(b_2)K_9(a_1)]K_8(c'_1) - K_9(a_1)K_{10}(b_1)K_7(c'_2)}{K_9(a_1)K_{10}(b_1)K_8(c'_2) - [K_8(a_2)K_{10}(b_1) + K_9(b_2)K_9(a_1)]K_9(c'_1)}$$

$$= \frac{[(41910)(37369) + (41910)(17231)](109) - (41910)^2(142)}{(1756448100)(185) - [2288286000](142)}$$

$$= \frac{7543800}{6286500}$$

$$= \frac{6}{5} = [1;5]$$

therefore,

$$[1, 2, 2, 3, 5, 3, 1, 2, 4, 9] + [1;18, 4, 2, 1, 3, 5, 3, 2, 2] = [2;0, 1, 3, 3, 3, 3, 1, 0, 0, 1, 5]$$
$$= [3;3, 3, 3, 3, 1, 0, 0, 1, 5] = [3;3, 3, 3, 3, 1, 1, 5]$$

To check, we have

$$[1;2,2,3,5,3,1,2,4,9] + [1;18,4,2,1,3,5,3,2,2] = \frac{59141}{41910} + \frac{79279}{41910}$$
$$= \frac{138420}{41910} = \frac{4614}{1397}$$

and

$$[3;3, 3, 3, 3, 1, 1, 5] = 3 + \frac{K_6(a_2)}{K_7(a_1)}$$
$$= 3 + \frac{423}{1397} = \frac{4614}{1397}$$

which is true.

- To find –[1;1,8,4,2,1,3,5,3,2,2].

Let [1;18, 4, 2, 1, 3, 5, 3, 2, 2] = $[a_0;a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,a_{10}]$, we get: n= 10,

From definition 6 we have

$$-[1;1,8,4,2,1,3,5,3,2,2] = [b_0';b_{10}'],$$

where,

$$b_0' = -1 - a_0 = -1 - 1 = -2$$
$$b_{10}' = \frac{K_{10}(a_1)}{K_{10}(a_1) - K_9(a_2)}$$

Since $[1;1,8,4,2,1,3,5,3,2,2] = \frac{79279}{41910} = a_0 + \frac{K_9(a_2)}{K_{10}(a_1)}$, (by definition 3)

and $K_{10}(a_1) = 41910$,

$$K_9(a_2) = 79279 - a_0 K_{10}(a_1)$$
$$= 79279 - 1(41910) = 79279 - 41910 = 37369$$

Therefore,

$$b_{10}' = \frac{K_{10}(a_1)}{K_{10}(a_1) - K_9(a_2)} = \frac{41910}{41910 - 37369}$$
$$= \frac{41910}{4541}$$

= [9;4, 2, 1, 3, 5, 3, 2, 2].

Therefore –[1;1,8,4,2,1,3,5,3,2,2] = [-2;9, 4, 2, 1, 3, 5, 3, 2, 2].

Theorem 3: Let $\beta = \beta_0$ be an irrational number and define the sequence $a_0,a_1,a_2,...$ recursively by $a_k = [\![\beta_k]\!], \beta_{k+1} = \frac{1}{\beta_k - a_k}$ for k =0, 1, 2, … .Then β is the value of infinite S.C.F. [a_0;a_1,a_2].

For example $\sqrt{6} = [a_0; a_1, a_2, a_3, a_4, a_5, \ldots] = [2; 2, 4, 2, 4\ldots] = [2; \overline{2,4}]$.

We can use the same operations of finite S.C.F. for infinite S.C.F..

Example 7:

Find $[1; \bar{2}] + [1; \overline{1,2}]$

Solution:

Let $[1; \bar{2}] = [1; 2, 2, 2, 2, \ldots] = [a_0; a_1, a_2, a_3, a_4, \ldots]$

and $[1; \overline{1,2}] = [1; 1, 2, 1, 2, \ldots] = [b_0; b_1, b_2, b_3, b_4, \ldots]$.

From equation (5a) we have $[a_0; a_1, a_2, a_3, a_4, \ldots] + [b_0; b_1, b_2, b_3, b_4, \ldots] = [c_0; c_1, c_2, c_3, c_4, \ldots]$

where,

$$c_0 = a_0 + b_0 = 1 + 1 = 2$$

$$c_1 = \left[\!\left[\frac{a_1 b_1}{a_1 + b_1} \right]\!\right] = \left[\!\left[\frac{2 \cdot 1}{2+1} \right]\!\right] = 0$$

$$c_2 = \left[\!\left[\frac{a_2(b_1 b_2 + 1) + b_2(a_1 a_2 + 1)}{(a_1 a_2 + 1)(b_1 b_2 + 1) - c_1[a_2(b_1 b_2 + 1) + b_2(a_1 a_2 + 1)]} \right]\!\right]$$

$$= \left[\!\left[\frac{2(3) + 2(5)}{(3)(5)} \right]\!\right]$$

$$= \left[\!\left[\frac{16}{15} \right]\!\right] = 1$$

$$c_3 = \left[\!\left[\frac{K_3(a_1) K_3(b_1)}{K_2(a_2) K_3(b_1) + K_2(b_2) K_3(a_1) - K_3(a_1) K_3(b_1) K_1(c_2)} \right]\!\right]$$

$$= \left[\!\left[\frac{12(4)}{5(4) + 3(12) - 12(4)} \right]\!\right]$$

$$= \left[\!\left[\frac{48}{8} \right]\!\right] = 6$$

$$c_4 = \left[\!\left[\frac{[K_3(a_2) K_4(b_1) + K_3(b_2) K_4(a_1)] K_2(c_1) - K_4(a_1) K_4(b_1) K_1(c_2)}{K_4(a_1) K_4(b_1) K_2(c_2) - [K_3(a_2) K_4(b_1) + K_3(b_2) K_4(a_1)] K_3(c_1)} \right]\!\right]$$

$$= \left[\!\left[\frac{[(12)(11) + (8)(29)] - (29)(11)(1)}{(29)(11)(7) - [364](6)} \right]\!\right]$$

$$= \left[\!\left[\frac{45}{49} \right]\!\right] = 0$$

$$c_5 = \left[\!\left[\frac{K_5(a_1) K_5(b_1) K_2(c_2) - [K_4(a_2) K_5(b_1) + K_4(b_2) K_5(a_1)] K_3(c_1)}{[K_4(a_2) K_5(b_1) + K_4(b_2) K_5(a_1)] K_4(c_1) - K_5(a_1) K_5(b_1) K_3(c_2)} \right]\!\right]$$

$$= \left[\!\left[\frac{(70)(15)(7) - [(29)(15) + (11)(70)](6)}{(1205)(1) - [1050](1)} \right]\!\right]$$

$$= \left[\!\left[\frac{120}{155} \right]\!\right] = 0$$

$$c_6 = \left[\!\left[\frac{[K_5(a_2)K_6(b_1) + K_5(b_2)K_6(a_1)]K_4(c_1) - K_6(a_1)K_6(b_1)K_3(c_2)}{K_6(a_1)K_6(b_1)K_4(c_2) - [K_5(a_2)K_6(b_1) + K_5(b_2)K_6(a_1)]K_5(c_1)} \right]\!\right]$$

$$= \left[\!\left[\frac{[(70)(41) + (30)(169)](1) - (169)(41)(1)}{(6929)(7) - [7940](6)} \right]\!\right]$$

$$= \left[\!\left[\frac{1011}{863} \right]\!\right] = 1$$

$$c_7 = \left[\!\left[\frac{K_7(a_1)K_7(b_1)K_4(c_2) - [K_6(a_2)K_7(b_1) + K_6(b_2)K_7(a_1)]K_5(c_1)}{[K_6(a_2)K_7(b_1) + K_6(b_2)K_7(a_1)]K_6(c_1) - K_7(a_1)K_7(b_1)K_5(c_2)} \right]\!\right]$$

$$= \left[\!\left[\frac{(408)(56)(7) - [(169)(56) + (41)(408)](6)}{(26192)(7) - [22848](8)} \right]\!\right]$$

$$= \left[\!\left[\frac{2784}{560} \right]\!\right] = 4$$

$$c_8 = \left[\!\left[\frac{[K_7(a_2)K_8(b_1) + K_7(b_2)K_8(a_1)]K_6(c_1) - K_8(a_1)K_8(b_1)K_5(c_2)}{K_8(a_1)K_8(b_1)K_6(c_2) - [K_7(a_2)K_7(b_1) + K_7(b_2)K_8(a_1)]K_7(c_1)} \right]\!\right]$$

$$= \left[\!\left[\frac{[(408)(153) + (112)(985)](7) - (985)(153)(8)}{(150705)(39) - [172744](34)} \right]\!\right]$$

$$= \left[\!\left[\frac{3568}{4199} \right]\!\right] = 0$$

$\vdots$

Therefore $[1;\overline{2}] + [1;\overline{1,2}]$ = [2;0, 1, 6, 0, 0, 1, 4, 0, …] = [3;6, 0, 0, 1, 4, 0, …] = [3;6, 1, 4, 0, …]

Example 8:

Find $[1;2] + [2;\overline{1,1,1,4}]$

Solution:

Let $[1;2] = [a_0;a_1]$ and $[2;\overline{1,1,1,4}] = [2;1,1,1,4,1,1,1,4,...] = [b_0;b_1,b_2,b_3,b_4,...]$

From (5b) we have $[a_0;a_1] + [b_0;b_1,b_2,b_3,b_4,...] = [c_0;c_1,c_2',c_3',c_4',...]$ where

$c_0 = c_0' = a_0 + b_0 = 1 + 2 = 3$

$$c_1 = c_1' = \left[\!\left[\frac{a_1 b_1}{a_1 + b_1} \right]\!\right] = \left[\!\left[\frac{2 \cdot 1}{2+1} \right]\!\right] = 0$$

$$c_2' = \left[\!\left[\frac{K_2(b_1) + K_1(b_2)a_1}{a_1 K_2(b_1)} \right]\!\right] = \left[\!\left[\frac{2+2}{4} \right]\!\right] = 1$$

$$c_3' = \left[\!\left[\frac{a_1 K_3(b_1)c_2'}{[K_3(b_1) + K_2(b_2)a_1] - a_1 K_3(b_1)c_2'} \right]\!\right]$$

$$= \left[\!\left[\frac{2(3)}{3 + 2(2) - 2(3)} \right]\!\right] = \left[\!\left[\frac{6}{1} \right]\!\right] = 6$$

$$c_4' = \left[\!\left[\frac{[K_4(b_1) + K_3(b_2)a_1] - a_1 K_4(b_1)c_2'}{a_1 K_4(b_1)K_2(c_2') - [K_4(b_1) + K_3(b_2)a_1]K_3(c_1')} \right]\!\right]$$

$$= \left[\!\left[\frac{[(14) + (9)(2)] - (2)(14)}{(2)(14)(7) - [32](6)} \right]\!\right]$$

$$= \left[\!\left[\frac{4}{4} \right]\!\right] = 1$$

$$
\begin{aligned}
c_5' &= \left\llbracket \frac{a_1K_5(b_1)K_2(c_2') - [K_5(b_1)+K_4(b_2)a_1]K_3(c_1')}{[K_5(b_1)+K_4(b_2)a_1]K_4(c_1') - a_1K_5(b_1)K_3(c_2')} \right\rrbracket \\
&= \left\llbracket \frac{(2)(17)(7) - [(17)+(11)(2)](6)}{(39)(7)-(34)(8)} \right\rrbracket \\
&= \left\llbracket \frac{4}{1} \right\rrbracket = 4 \\
c_6' &= \left\llbracket \frac{[K_6(b_1)+K_5(b_2)a_1]K_4(c_1') - a_1K_6(b_1)K_3(c_2')}{a_1K_6(b_1)K_4(c_2') - [K_6(b_1)+K_5(b_2)a_1]K_5(c_1')} \right\rrbracket \\
&= \left\llbracket \frac{[(31)+(20)(2)](7) - (2)(31)(8)}{(62)(39)-(71)(34)} \right\rrbracket \\
&= \left\llbracket \frac{1}{4} \right\rrbracket = 0 \\
c_7' &= \left\llbracket \frac{a_1K_7(b_1)K_4(c_2') - [K_7(b_1)+K_6(b_2)a_1]K_5(c_1')}{[K_7(b_1)+K_6(b_2)a_1]K_6(c_1') - a_1K_7(b_1)K_5(c_2')} \right\rrbracket \\
&= \left\llbracket \frac{(2)(48)(39) - [(48)+(31)(2)](34)}{(110)(7)-(96)(8)} \right\rrbracket \\
&= \left\llbracket \frac{4}{2} \right\rrbracket = 2 \\
&\vdots
\end{aligned}
$$

Therefore [1;2] + $[2;\overline{1,1,1,4}]$ = [3;0, 1, 6, 1, 4, 0, 2, …] = [4;6, 1, 6, …].

There are many applications of continued fractions: combine continued fractions with the concepts of golden ratio and Fibonacci numbers, Pell equation and calculation of fundamental units in quadratic fields, reduction of quadratic forms and calculation of class numbers of imaginary quadratic field.There is a pleasant connection between Chebyshev polynomials, the Pell equation and continued fractions, the latter two being understood to take place in real quadratic function fields rather than the classical case of real quadratic number fields [1, 5].

The very nice elementary application of simple continued fractions is Gosper's batting average problem which is, if a baseball player's (3-digit rounded) batting average is 0.334, what's the smallest number of at-bats that player could have? (Batting average is computed as $(\frac{\text{number of hits}}{\text{at} - \text{bats}})$. The solution proceeds by noting that a rounded average of 0.334 corresponds to an actual number in the range (3335,3345), finding the continued fractions for these values yields $0.3335 = \frac{667}{2000} = [0;2,1,666]$ and $0.3345 = \frac{669}{2000} = .[0;2,1,94,1,1,3]$. This implies that the 'simplest' number within the range is $[0,2,1,95] = \frac{69}{287} \approx 0.334495$ [7].

This paper is the first part for the operations of the simple continued fractions. The second part will define the multiplication, multiplicative inverse and the powers of the simple continued fractions.

CONCLUSION

We can also solve any Diophantine congruence that is any equivalence of the form $ax \equiv b(mod\ m)$. In other words, in most real-world applications of mathematics, continued fractions are rarely the most practical way to solve a given set of problems as decimal approximations are much more useful (and computers can work with decimals at a much faster rate). However, some interesting observations can still be made using continued fractions.Namely, in this paper, we will be exploring how continued fractions can be used to add and subtract the numbers $\bar{a} \mp \bar{b}$ and $\frac{a}{b} \mp \bar{c}$.

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