World Engineering & Applied Sciences Journal 6 (1): 45-58, 2015
ISSN 2079-2204

DOI: 10.5829/idosi.weasj.2015.6.1.22189

Bayesian Estimation Procedures for Three-parameter Exponentiated-Weibull Distribution under Squared-Error Loss Function and Type II Censoring

Ajit Chaturvedi and Anupam Pathak

Department of Statistics, University of Delhi-110007, India

Abstract: The three-parameter exponentiated-Weibull distribution has been widely used especially in the modelling of life time event data. It provides a statistical model which has a wide variety of application in many areas and the main advantage is its ability in the context of life time event among other distributions. Bayesian estimation procedures are considered for estimating the reliability function R(t)=P(X>t) and P=P(X>Y) for three-parameter exponentiated-Weibull distribution under type II censoring. Considerations are given to squared-error loss. *Approach*: A new technique of obtaining Bayes estimators of these parametric functions is introduced in which major role is played by the estimators of the powers of the parameter and the functional forms of the parametric functions to be estimated are not needed. Simulation studied is performed.

Key words: Three-parameter exponentiated-Weibull distribution · Type II Censoring · Bayes estimators · Squared-error loss function (SELF)

INTRODUCTION

Reliability theory is mainly concerned with the determination of the probability that a system, consisting possibly of several components, will operate adequately for a given period of time in its intended application. The reliability function R(t) is defined as the probability of failure-free operation until time t. Thus, if the random variable (rv) X denotes the lifetime of an item, then R(t)=P(X>t). Another measure of reliability under stress-strength set-up is the probability **P=P(X>Y)**, which represents the reliability of an item of random strength X subject to random stress Y[1-16]. Many researchers have considered the problems of estimation of R(t) and 'P' for various lifetime distributions and for a brief review, one may refer to and others. Some of these models (particularly, exponential and Weibull models) are widely discussed in the literature for the analysis of lifetime data. These models accommodate either constant or monotone type of the hazard rates. But non-monotonic hazard functions such as unimodal shaped and bathtub shaped also arises in practice. For example, data in reliability analysis specially life cycle of the product often involve high initial hazard rate (infant mortality) and eventual high hazard rates due to aging and wear-out in the end indicating a bathtub hazard rate. Non-monotone hazard rate is very common, for example, in the field of science and engineering, in the field of medical, in the field of ecological and space explorations. Due to these reasons, modeling lifetime data for non-monotonic hazard rates seems to be a growing interest. In this context, exponentiated-Weibull family can be considered as a suitable modal, which was initially introduced by [12], as a simple generalization of two-parameter Weibull family and is obtained by introducing one additional shape parameter. This family allows bathtub shaped as well as unimodal hazard rates. The probability density function (pdf), cumulative distribution function (cdf) and reliability function R(t) of exponentiated-Weibull family, respectively, are

$$f(x;\alpha,\lambda,\delta)=\alpha\lambda\delta x^{\delta-1}e^{-\lambda x^{\delta}}\left(1-e^{-\lambda x^{\delta}}\right)^{\alpha-1};\ x,\alpha,\lambda,\delta>0, \tag{1.1}$$

$$F(x;\alpha,\lambda,\delta)=\left(1-e^{-\lambda x^{\delta}}\right)^{\alpha};\ x,\alpha,\lambda,\delta>0 \tag{1.2}$$

and

$$R(t)=1-\left(1-e^{-\lambda x^{\delta}}\right)^{\alpha};\ x,\alpha,\lambda,\delta>0. \tag{1.3}$$

Corresponding Author: Anupam Pathak, Department of Statistics, University of Delhi-110007, India.

Bayesian ideas were introduced for the first time in reliability and life testing by [1], who considered the problems of estimating the parameter and reliability function of one-parameter exponential distribution under type II censoring. Consideration was given to SELF. [11] considered a family of lifetime distribution and proposed Bayesian estimation procedures for the parameter and reliability function under SELF and type II censoring. For a brief review one may refer to the book by [10]. [6] derived Bayes estimator of 'P' under SELF when X and Y were assumed to follow exponential distributions. [15] discussed the classical and Bayesian methods of parameter estimation for complete sample case. [16] have discussed the classical and Bayesian methods of parameter estimation under type II censoring. This paper is an attempt in the direction of Bayesian estimation of R(t) and 'P' for three-parameter exponentiated-Weibull distribution under type II censoring.

The purpose of the present paper is manifold. For the distribution (1.1), Bayes estimators are derived for the powers (positive as well as negative) of the parameter, reliability function R(t) and 'P' under SELF. Type II censoring is considered. Deviating from the conventional methods of obtaining Bayes estimators of R(t) and 'P', Bayes estimators of the powers of the parameter are utilized to obtain Bayes estimator of the pdf at a specified point. This estimator is subsequently used to obtain Bayes estimator of R(t) and 'P'. Thus, in all the estimation problems, the major role is played by the estimators of the powers of the parameter. We have established an interrelationship between various estimation problems. Expressions for the risks, posterior risks and Bayes risks of various estimators are provided.

In Section 2, we give the set-up of the estimation problems and introduce the notations and definitions. In section 3, we obtain Bayes estimators of the powers of α, R(t) and 'P'. In Section 4, simulation study is carried out to investigate the performance of estimators. In Section 5, discussion is made. Finally, in Section 6 conclusion are given.

Set-up of the Estimation Problems, Notations and Definitions: Let the random variable (rv) X follows the three-parameter exponentiated-Weibull distribution whose pdf is given at (1.1). Throughout we assume that α is unknown but λ and δ are known. Suppose n items are put on a test and the test is terminated after the first r ordered observations are recorded. Let $0 \le X_{(1)} \le .... \le X_{(r)};\ 0 < r < n,$ be the lifetimes of the first r ordered observations. Obviously, (n-r) items survived until $X_{(r)}$. Denoting by $\underline{x} = (x_1, x_2, ..., x_r)$, we have likelihood function as

$$L(\alpha|\underline{x}) = \alpha^r \lambda^r \delta^r \left[\prod_{i=1}^{r} x_{(i)}^{\delta} \left\{x_{(r)}^{\delta}\right\}^{n-r}\right] \exp\left\{-\lambda\left[\sum_{i=1}^{r} x_{(i)}^{\delta} + (n-r)x_{(r)}^{\delta}\right]\right\}$$
$$\cdot \exp\left\{(\alpha-1)\left[\sum_{i=1}^{r}\ln\left\{1-e^{-\lambda x_{(i)}^{\delta}}\right\} + (n-r)\ln\left\{1-e^{-\lambda x_{(r)}^{\delta}}\right\}\right]\right\}.$$

Therefore,

$$L(\alpha|\underline{x}) \propto \alpha^r \exp[-\alpha S_r]. \tag{2.1}$$

Where,

$$S_r = -\left[\sum_{i=1}^{r}\ln\left\{1-e^{-\lambda x_{(i)}^{\delta}}\right\} + (n-r)\ln\left\{1-e^{-\lambda x_{(r)}^{\delta}}\right\}\right]. \tag{2.2}$$

Looking (2.1), we consider the natural conjugate prior distribution for α to be gamma with pdf

$$\prod(\alpha) = \frac{\mu^{\nu}}{\Gamma\nu}\alpha^{\nu-1}\exp(-\alpha\mu)\ ;\ \alpha,\ \mu > 0 \text{ and } \nu \text{ is positive integer.} \tag{2.3}$$

Combining (2.1) and (2.3) via Bayes theorem, the posterior density of α comes out to

$$h(\alpha \mid s_r) = \frac{(s_r+\mu)^{r+\nu}}{\Gamma(r+\nu)} \alpha^{r+\nu-1} \exp\{-(s_r+\mu)\alpha\}. \tag{2.4}$$

In order to estimate 'P', let n items on X and m items on Y are put through a life test and r and s being their truncation numbers, respectively. The rv X has pdf $f(x;\alpha_1,\lambda_1,\delta_1)$ and rv Y has pdf $f(y;\alpha_2,\lambda_2,\delta_2)$. Let us denote by

$$S_r = -\left[\sum_{i=1}^{r} \ln\left\{1-e^{-\lambda_1 x_{(i)}^{\delta_1}}\right\} + (n-r)\ln\left\{1-e^{-\lambda_1 x_{(r)}^{\delta_1}}\right\}\right]$$

and

$$T_s = -\left[\sum_{j=1}^{s} \ln\left\{1-e^{-\lambda_2 y_{(j)}^{\delta_2}}\right\} + (m-s)\ln\left\{1-e^{-\lambda_2 y_{(s)}^{\delta_2}}\right\}\right].$$

Here, we assume that λ_1, δ_1, λ_2 and δ_2 are known but α_1 and α_2 are unknown. We consider the conjugate priors for α_1 and α_2 given at (2.3) with parameters (μ_1, ν_1) and (μ_2, ν_2) respectively.

Let us make the transformation $U = -\ln(1-e^{-\lambda X^{\delta}})$. It is easy to see that U follows exponential distribution with pdf

$$f(u;a) = a\exp(-au);\ u>0.$$

If we consider the transformation $Z_i = (n-i+1)[U_{(i)} - U_{(i-1)}];\ i=1, 2, ..., r,$ then Z_i's are independent and identically distributed rv's each having exponential distribution. Moreover, since $\sum_{i=1}^{r} Z_i = S_r$, from the additive property of exponential distribution the pdf of S_r is

$$h(s_r;\alpha) = \frac{\alpha^r}{\Gamma(r)} s_r^{r-1} \exp(-\alpha s_r); \quad s_r > 0. \tag{2.5}$$

From (2.3) and (2.5) the marginal pdf of S_r is

$$f(s_r;\alpha) = \frac{\mu^{\nu} s_r^{r-1}}{\Gamma(r)\,\Gamma(\nu)} \int_0^{\infty} \alpha^{r+\nu-1} \exp\{-\alpha(s_r+\mu)\} d\alpha,$$

or,

$$f(s_r;\alpha) = \frac{\mu^{\nu} s_r^{r-1}}{B(r,\nu)(s_r+\mu)^{r+\nu}};\ s_r > 0. \tag{2.6}$$

Denoting by $\hat{\theta}_{BS}$ and $L(\hat{\theta}_{BS}, \theta)$ the Bayes estimators of $\theta = \Psi(\alpha)$, under SELF and the loss resulting from estimating θ by $\hat{\theta}_{BS}$, respectively, the associated risk is defined by

$$R_S(\hat{\theta}_{BS}) = E_{S_r|\alpha}\left\{ L(\hat{\theta}_{BS}, \theta)\right\}. \tag{2.7}$$

The posterior risk for estimating θ by $\hat{\theta}_{BS}$ is

$$R_{PS}\left(\hat{\theta}_{BS}\right) = E_{\alpha|S_r}\left\{ L\left(\hat{\theta}_{BS}, \theta\right)\right\} \tag{2.8}$$

and Bayes risk for estimating θ by $\hat{\theta}_{BS}$ is

$$R_{BS}\left(\hat{\theta}_{BS}\right) = E_{S_r}\left[E_{\alpha|S_r}\left\{ L\left(\hat{\theta}_{BS}, \theta\right)\right\}\right]. \tag{2.9}$$

It should be noted here that the risk of Bayes estimator of θ is a function of the modal parameter α and is independent of the sample data, the posterior risk is a function of the sample data S_r and of the prior parameters and is independent of α and Bayes risk is a function of the prior parameters only. We also note the following relationships

$$R_{BS}\left(\hat{\theta}_{BS}\right) = E_{\alpha}\left[\left\{ R_S\left(\hat{\theta}_{BS}\right)\right\}\right]$$

and

$$R_{BS}\left(\hat{\theta}_{BS}\right) = E_{S_r}\left[\left\{ R_{PS}\left(\hat{\theta}_{BS}\right)\right\}\right].$$

(*)

In what follows we obtain $\hat{\theta}_{BS}$ and it's various measures of performance under SELF.

Bayes Estimators of the Powers of α, R(t) and P Under SELF: The following theorem provides Bayes estimators of powers of α.

Theorem 1: For a positive integer p, under SELF, Bayes estimators of α^{-p} and α^{p} are given, respectively, by $\hat{a}^{-p}_{BS}$ and $\hat{a}^{p}_{BS}$, where

$$\hat{\alpha}^{-p}_{BS} = \frac{\Gamma(r+\nu-p)}{\Gamma(r+\nu)}(S_r+\mu)^p; \quad (p<r+\nu) \tag{3.1}$$

and

$$\hat{\alpha}^{p}_{BS} = \frac{\Gamma(r+\nu+p)}{\Gamma(r+\nu)}(S_r+\mu)^{-p}. \tag{3.2}$$

Proof: From (2.4) and using the result that Bayes estimator of any function of α is it's posterior mean, we have

$$\hat{\alpha}^{-p}_{BS} = \int_{\alpha} \alpha^{-p} h(\alpha | S_r) d\alpha$$

$$= \frac{\Gamma(r+\nu-p)}{\Gamma(r+\nu)}(S_r+\mu)^p; \; (p<r+\nu)$$

and the result (3.1) follows.

Similarly, we can prove result (3.2).

In the following theorem we derive expression for the risks, posterior risks and Bayes risks of Bayes estimators of the powers of α.

Theorem 2:

$$R_S\left(\hat{\alpha}_{BS}^{-p}\right)=\left\{\frac{\Gamma(r+\nu-p)}{\Gamma(r+\nu)}\right\}^2\sum_{i=0}^{2p}\binom{2p}{i}\mu^{2p-1}\left\{\frac{\Gamma(i+r)}{\Gamma(r)}\right\}\alpha^{-i}$$
$$-2\alpha^{-p}\left\{\frac{\Gamma(r+\nu-p)}{\Gamma(r+\nu)}\right\}\sum_{i=0}^{p}\binom{p}{i}\mu^{p-1}\left\{\frac{\Gamma(i+r)}{\Gamma(r)}\right\}\alpha^{-i}+\alpha^{-2p};\ (p<r+\nu), \tag{3.3}$$

$$R_{PS}\left(\hat{\alpha}_{BS}^{-p}\right)=\left[\frac{\Gamma(r+\nu-2p)}{\Gamma(r+\nu)}-\left\{\frac{\Gamma(r+\nu-p)}{\Gamma(r+\nu)}\right\}^2\right](S_r+\mu)^{2p};\ (2p<r+\nu), \tag{3.4}$$

$$R_{BS}\left(\hat{\alpha}_{BS}^{-p}\right)=\left[1-\frac{\{\Gamma(r+\nu-p)\}^2}{\Gamma(r+\nu)\Gamma(r+\nu-2p)}\right]\frac{\Gamma(\nu-2p)}{\Gamma(\nu)}\mu^{2p};\ (p<\nu/2), \tag{3.5}$$

$$R_S\left(\hat{\alpha}_{BS}^{p}\right)=\alpha^{2p}\left[\left\{\frac{\Gamma(r+\nu+p)}{\Gamma(r+\nu)}\right\}^2\frac{1}{\Gamma(r)}\int_0^\infty\frac{z^{r-1}e^{-z}}{(z+\mu\alpha)^{2p}}dz\right.$$
$$\left.-2\left\{\frac{\Gamma(r+\nu+p)}{\Gamma(r+\nu)}\right\}\frac{1}{\Gamma(r)}\int_0^\infty\frac{z^{r-1}e^{-z}}{(z+\mu\alpha)^{p}}+1\right]dz, \tag{3.6}$$

$$R_{PS}\left(\hat{\alpha}_{BS}^{p}\right)=\left[\left\{\frac{\Gamma(r+\nu+2p)}{\Gamma(r+\nu)}\right\}-\left\{\frac{\Gamma(r+\nu+p)}{\Gamma(r+\nu)}\right\}^2\right](S_r+\mu)^{-2p} \tag{3.7}$$

and

$$R_{BS}\left(\hat{\alpha}_{BS}^{p}\right)=\left[1-\frac{\{\Gamma(r+\nu+p)\}^2}{\Gamma(r+\nu)\Gamma(r+\nu+2p)}\right]\frac{\Gamma(\nu+2p)}{\Gamma\nu}\mu^{-2p}. \tag{3.8}$$

Proof: From (3.1) the risk corresponding to $\hat{a}_{BS}^{-p}$ is

$$R_S\left(\hat{\alpha}_{BS}^{-p}\right)=E_{S_r|\alpha}\left[\hat{\alpha}_{BS}^{-p}-\alpha^{-p}\right]^2$$

$$=\left\{\frac{\Gamma(r+\nu-p)}{\Gamma(r+\nu)}\right\}^2E_{S_r|\alpha}(S_r+\mu)^{2p}-2\alpha^{-p}\left\{\frac{\Gamma(r+\nu+p)}{\Gamma(r+\nu)}\right\}E_{S_r|\alpha}(S_r+\mu)^{p}+\alpha^{-2p}$$

$$=\left\{\frac{\Gamma(r+\nu-p)}{\Gamma(r+\nu)}\right\}^2E_{S_r|\alpha}\left[\sum_{i=0}^{2p}\binom{2p}{i}\mu^{2p-i}S_r^i\right]$$
$$-2\alpha^{-p}\left\{\frac{\Gamma(r+\nu+p)}{\Gamma(r+\nu)}\right\}E_{S_r|\alpha}\left[\sum_{i=0}^{p}\binom{p}{i}\mu^{p-i}S_r^i\right]+\alpha^{-2p}. \tag{3.9}$$

Now from (2.5) for q to be positive integer, we have

$$E\left(S_r^q\right)=\int_0^\infty \frac{\alpha^r S_r^{q+r-1}}{\Gamma(r)} e^{-\alpha S_r} dS_r$$

$$=\frac{G(q+r)}{G(r)} a^{-q}. \tag{3.10}$$

Hence result (3.3) follows on combining (3.9) and (3.10).

Now by definition

$$R_{PS}\left(\hat{\alpha}_{BS}^{-p}\right)=E_{\alpha|S_r}\left[L\left(\hat{\alpha}_{BS}^{-p},\alpha^{-p}\right)\right]$$

$$= E_{\alpha|S_r}\left[\alpha^{-2p}-2\alpha^{-p}\hat{\alpha}_{BS}^{-p}+\left(\hat{\alpha}_{BS}^{-p}\right)^2\right]$$

$$=E_{\alpha|S_r}\left(\alpha^{-2p}\right)-2\left[E_{\alpha|S_r}\left(\alpha^{-p}\right)\right]\hat{\alpha}_{BS}^{-p}+\left(\hat{\alpha}_{BS}^{-p}\right)^2. \tag{3.11}$$

Hence result (3.4) follows from (2.4), (3.1) and (3.11).

Using definition (*), (2.6) and (3.4), we get

$$R_{BS}\left(\hat{\alpha}_{BS}^{-p}\right)=E_{S_r}\left[R_{PS}\left(\hat{\alpha}_{BS}^{-p}\right)\right]$$

$$=\left[\frac{\Gamma(r+\nu-2p)}{\Gamma(r+\nu)}-\left\{\frac{\Gamma(r+\nu-p)}{\Gamma(r+\nu)}\right\}^2 E\left(S_r+\mu\right)^{2p}\right].$$

And hence (3.5) follows on using

$$E_{S_r}\left(S_r+\mu\right)^{2p}=\int_0^\infty \frac{\left(S_r+\mu\right)^{2p} S_r^{r-1}\mu^\nu}{B(r,\nu)\left(S_r+\mu\right)^{r+\nu}} dS_r$$
$$=\mu^{2p}\frac{\Gamma(r+\nu)\Gamma(\nu-2p)}{\Gamma(\nu)\Gamma(r+\nu-2p)} \ ;\ (2p<r+\nu).$$

The proofs of the results (3.6), (3.7) and (3.8) are similar to those of (3.3), (3.4) and (3.5), respectively.

In the following lemma we obtain Bayes estimator of the pdf (1.1) at a specified point 'x' with the help of Bayes estimators of powers of α.

Lemma 1: For $f(x;\alpha,\lambda,\delta)$ defined at (1.1)

$$\hat{f}_{BS}\left(x;\alpha,\lambda,\delta\right)=\frac{\lambda\delta(r+\nu)x^{\delta-1}e^{-\lambda x^\delta}}{(S_r+\mu)(1-e^{-\lambda x^\delta})}\left[1-\frac{\ln(1-e^{-\lambda x^\delta})}{(S_r+\mu)}\right]^{-(r+\nu+1)}.$$

Proof: We can write (1.1) as

$$f(x;\alpha,\lambda,\delta)=\frac{\lambda\delta x^{\delta-1}e^{-\lambda x^{\delta}}}{(1-e^{-\lambda x^{\delta}})}\alpha(1-e^{-\lambda x^{\delta}})^{\alpha}$$

$$=\frac{\lambda\delta x^{\delta-1}e^{-\lambda x^{\delta}}}{(1-e^{-\lambda x^{\delta}})}\sum_{i=0}^{\infty}\frac{[\ln(1-e^{-\lambda x^{\delta}})]^{i}}{i!}\alpha^{i+1}$$

Utilizing lemma 1 of [4] and (3.2), we get

$$\hat{f}_{BS}(x;\alpha,\lambda,\delta)=\frac{\lambda\delta x^{\delta-1}e^{-\lambda x^{\delta}}}{\left(1-e^{-\lambda x^{\delta}}\right)}\sum_{i=0}^{\infty}\frac{\left[\ln\left(1-e^{-\lambda x^{\delta}}\right)\right]^{i}}{i!}\hat{\alpha}_{BS}^{i+1}$$

$$=\frac{\lambda\delta x^{\delta-1}e^{-\lambda x^{\delta}}}{(S_r+\mu)\left(1-e^{-\lambda x^{\delta}}\right)}\sum_{i=0}^{\infty}\binom{r+\nu+i}{i}\left[\frac{\ln\left(1-e^{-\lambda x^{\delta}}\right)}{(S_r+\mu)}\right]^{i}$$

$$=\frac{\lambda\delta(r+\nu)x^{\delta-1}e^{-\lambda x^{\delta}}}{(S_r+\mu)(1-e^{-\lambda x^{\delta}})}\left[1-\frac{\ln(1-e^{-\lambda x^{\delta}})}{(S_r+\mu)}\right]^{-(r+\nu+1)}$$

which is the required result

In the following theorem, we obtain Bayes estimator of reliability function.

Theorem 3: For the reliability function given at (1.3)

$$\hat{R}_{BS}(t)=\left[1-\left\{1-\frac{\ln\left(1-e^{-\lambda t^{\delta}}\right)}{(S_r+\mu)}\right\}^{-(r+\nu)}\right].$$

Proof: We have,

$$\int_t^{\infty}\hat{f}_{BS}(x;\alpha,\lambda,\delta)dx=\int_t^{\infty}E_{\alpha|S_r}\{f(x;\alpha,\lambda,\delta)\}dx$$

$$=E_{\alpha|S_r}\int_t^{\infty}f(x;\alpha,\lambda,\delta)dx$$

$$=E_{\alpha|S_r}\left[R(t)\right]$$

$$=\hat{R}_{BS}(t) \tag{3.12}$$

From above, we conclude that Bayes estimator of R(t) can be obtained with the help of Bayes estimator of f(x;a,?,d). Thus from Lemma 1

$$\hat{R}_{BS}(t)=\frac{\lambda\delta(r+\nu)}{(S_r+\mu)}\int_t^{\infty}\frac{x^{\delta-1}e^{-\lambda x^{\delta}}}{\left(1-e^{-\lambda x^{\delta}}\right)}\left[1-\frac{\ln(1-e^{-\lambda x^{\delta}})}{(S_r+\mu)}\right]^{-(r+\nu+1)}dx$$

$$=(r+\nu)\int_1^{1-\frac{\ln(1-e^{-\lambda x^{\delta}})}{(S_r+\mu)}}z^{-(r+\nu+1)}dz$$

and the theorem follows.

The following theorem gives the expression for risk, posterior risk and Bayes risk of $\hat{R}_{BS}(t)$.

Theorem 4: For $\hat{R}_{BS}(t)$ obtained in Theorem 3,

$$R_S\left(\hat{R}_{BS}(t)\right)=\frac{1}{\Gamma(r)}\int_0^\infty \frac{\left[z+\alpha\left\{\mu-\ln\left(1-e^{-\lambda t^\delta}\right)\right\}\right]^{-2(r+\nu)} z^{r-1}e^{-z}}{(z+\alpha\mu)^{-2(r+\nu)}}dz$$
$$-2\frac{\left(1-e^{-\lambda t^\delta}\right)^{2\alpha}}{\Gamma(r)}\int_0^\infty \frac{\left[z+\alpha\left\{\mu-\ln\left(1-e^{-\lambda t^\delta}\right)\right\}\right]^{-(r+\nu)} z^{r-1}e^{-z}}{(z+\alpha\mu)^{-(r+\nu)}}dz+\left\{1-e^{-\lambda t\delta}\right\}^{2\alpha}, \tag{3.13}$$

$$R_{PS}\left(\hat{R}_{BS}(t)\right)=\left[1-\frac{2\ln\left(1-e^{-\lambda t^\delta}\right)}{(S_r+\mu)}\right]^{-(r+\nu)}-\left[1-\frac{\ln\left(1-e^{-\lambda t^\delta}\right)}{(S_r+\mu)}\right]^{-2(r+\nu)} \tag{3.14}$$

and

$$R_{BS}\left(\hat{R}_{BS}(t)\right)=\left[\frac{\mu}{\mu-2\ln\left(1-e^{-\lambda t^\delta}\right)}\right]^{\nu}-\frac{1}{B(r,\nu)}\sum_{i=0}^{r+\nu}\binom{r+\nu}{i}\left[\frac{\mu}{\mu-\ln\left(1-e^{-\lambda t^\delta}\right)}\right]^{r+2\nu-i}$$
$$\cdot B(r+i,r+2\nu-i). \tag{3.15}$$

Proof: By definition

$$R_S\left(\hat{R}_{BS}(t)\right)=E_{S_r|\alpha}\left[\hat{R}_{BS}(t)-R(t)\right]^2$$
$$=E_{S_r|\alpha}\left[1-\frac{\ln\left(1-e^{-\lambda t^\delta}\right)}{(S_r+\mu)}\right]^{-2(r+\nu)}-2\left(1-e^{-\lambda t^\delta}\right)^{\alpha}$$
$$\cdot E_{S_r|\alpha}\left[1-\frac{\ln\left(1-e^{-\lambda t^\delta}\right)}{(S_r+\mu)}\right]^{-(r+\nu)}+\left(1-e^{-\lambda t^\delta}\right)^{2\alpha}. \tag{3.16}$$

For q to be positive integer, from (2.5),

$$E_{S_r|\alpha}\left[\frac{S_r+\mu}{(S_r+\mu)-\ln\left(1-e^{-\lambda t^\delta}\right)}\right]^q=\frac{\alpha^r}{\Gamma(r)}\int_0^\infty\frac{(S_r+\mu)^q S_r^{r-1}e^{-\alpha S_r}}{\left[(S_r+\mu)-\ln\left(1-e^{-\lambda t^\delta}\right)\right]^q}dS_r$$
$$=\frac{1}{\Gamma(r)}\int_0^\infty\frac{(z+\alpha\mu)^q z^{r-1}e^{-z}dz}{\left[z+\alpha\left\{\mu-\ln\left(1-e^{-\lambda t^\delta}\right)\right\}\right]^q}. \tag{3.17}$$

Result (3.13) follows from (3.16) and (3.17).
Again by definition

$$R_{PS}\left(\hat{R}_{BS}(t)\right) = E_{\alpha|S_r}\left[\hat{R}_{BS}(t) - R(t)\right]^2$$
$$= \left[1 - \frac{\ln(1-e^{-\lambda t^{\delta}})}{(S_r+\mu)}\right]^{-2(r+\nu)} - 2\left[1 - \frac{\ln(1-e^{-\lambda t^{\delta}})}{(S_r+\mu)}\right]^{-(r+\nu)}$$
$$\cdot E_{\alpha|S_r}\left(1-e^{-\lambda t^{\delta}}\right)^{\alpha} + E_{\alpha|S_r}\left(1-e^{-\lambda t^{\delta}}\right)^{2\alpha}. \tag{3.18}$$

Using (2.4), we have

$$E_{\alpha|S_r}\left(1-e^{-\lambda t^{\delta}}\right)^{\alpha} = \int_0^{\infty} \frac{(S_r+\mu)^{r+\nu}}{\Gamma(r+\nu)} \alpha^{r+\nu-1} \exp\left\{-\alpha(S_r+\mu)\right\}\left(1-e^{-\lambda t^{\delta}}\right)^{\alpha} d\alpha$$
$$= \left[1 - \frac{\ln\left(1-e^{-\lambda t^{\delta}}\right)}{(S_r+\mu)}\right]^{-(r+\nu)}. \tag{3.19}$$

Similarly,

$$E_{\alpha|S_r}\left(1-e^{-\lambda t^{\delta}}\right)^{2\alpha} = \left[1 - \frac{2\ln\left(1-e^{-\lambda t^{\delta}}\right)}{(S_r+\mu)}\right]^{-(r+\nu)}. \tag{3.20}$$

Result (3.14) follows from (3.18), (3.19) and (3.20).

From (3.14),

$$R_{BS}\left(\hat{R}_{BS}(t)\right) = E_{S_r}\left[\left\{1 - \frac{2\ln\left(1-e^{-\lambda t^{t}}\right)}{(S_r+\mu)}\right\}^{-(r+\nu)} - \left\{1 - \frac{\ln\left(1-e^{-\lambda t^{t}}\right)}{(S_r+\mu)}\right\}^{-2(r+\nu)}\right]. \tag{3.21}$$

Using (2.6), we have

$$E_{S_r}\left[1 - \frac{2\ln\left(1-e^{-\lambda t^{t}}\right)}{(S_r+\mu)}\right]^{-(r+\nu)} = \left[\frac{\mu}{\mu - 2\ln\left(1-e^{-\lambda t^{\delta}}\right)}\right]^{\nu} \tag{3.22}$$

and

$$E_{S_r}\left[1 - \frac{\ln\left(1-e^{-\lambda t^{t}}\right)}{(S_r+\mu)}\right]^{-2(r+\nu)} = \frac{1}{B(r,\nu)}\sum_{i=0}^{r+\nu}\binom{r+\nu}{i}\left[\frac{\mu}{\mu - \ln\left(1-e^{-\lambda t^{\delta}}\right)}\right]^{r+2\nu-i}$$
$$\cdot B\left(r+i,\ r+2\nu-i\right). \tag{3.23}$$

Result (3.15) follows from (3.21), (3.22) and (3.23).

In what follows, we obtain Bayes estimator of P.

Theorem 5: For $C=1-\frac{(S_r+\mu_1)}{(T_s+\mu_2)}$ and $\lambda_1=\lambda_2=\lambda$ say and $\delta_1=\delta_2=\delta$, say,

$$\hat{P}_{BS}=\begin{cases}\frac{(r+\nu_1)}{(r+\nu_1+s+\nu_2)}(1-C)^{(r+\nu_1)}\,{}_2F_1(r+\nu_1+s+\nu_2,r+\nu_1+1;r+\nu_1+s+\nu_2+1;C);|C|<1\\ \frac{(r+\nu_1)}{(r+\nu_1+s+\nu_2)}(1-C)^{-(s+\nu_2)}\,{}_2F_1(r+\nu_1+s+\nu_2,s+\nu_2;r+\nu_1+s+\nu_2+1;\frac{C}{C-1});|C|\le -1,\end{cases} \tag{3.24}$$

where ${}_2F_1(a,b,c;z)=\sum_{j=0}^{\infty}\frac{a_{(j)}b_{(j)}}{c_{(j)}}.\frac{z^j}{(j)!},c\neq 0,-1,-2,....$, is Gauss hyper geometric series and $l_{(k)}=\frac{\Gamma(l+k)}{\Gamma(l)}$; $k=1,\ 2,\ .....$ Moreover, for $\alpha_1\neq\alpha_2,\ \lambda_1\neq\lambda_2$ and $\delta_1\neq\delta_2$

$$\hat{P}_{BS}=1-\frac{\lambda_2\delta_2(s+\nu_2)}{(T_s+\mu_2)}\int_{y=0}^{\infty}\left[1-\frac{\ln(1-e^{-\lambda_1 y^{\delta_1}})}{(S_r+\mu_1)}\right]^{-(r+\nu_1)}\cdot\frac{y^{\delta_2-1}e^{-\lambda_2 y^{\delta_2}}}{(1-e^{-\lambda_2 y^{\delta_2}})}\left[1-\frac{\ln(1-e^{-\lambda_2 y^{\delta_2}})}{(T_s+\mu_2)}\right]^{-(s+\nu_2+1)}dy. \tag{3.25}$$

Proof: For $\lambda_1=\lambda_2=\lambda$, say and $\delta_1=\delta_2=\delta$, say,

$$P=P(X>Y)=\int_{y=0}^{\infty}R(y;\alpha_1,\lambda,\delta)f(y;\alpha_2,\lambda,\delta)dy.$$

Putting $1-e^{-\lambda Y^{\delta}}=Z$, we get $P=\frac{\alpha_1}{\alpha_1+\alpha_2}$. From (2.6), the joint posterior density of (α_1,α_2) is

$$h(\alpha_1,\ \alpha_2)=\frac{(S_r+\mu_1)^{r+\nu_1}(T_s+\mu_2)^{s+\nu_2}}{\Gamma(r+\nu_1)\Gamma(s+\nu_2)}\alpha_1^{r+\nu_1-1}\alpha_2^{s+\nu_2-1}\cdot\exp\{-(S_r+\mu_1)\alpha_1-(T_s+\mu_2)\alpha_2\}. \tag{3.26}$$

In (3.26), let us make the transformation $P=\frac{\alpha_1}{\alpha_1+\alpha_2}$ and $W=\alpha_1+\alpha_2$. The Jacobian of transformation is w. From (3.26), the joint pdf of P and W is thus

$$h(P,\ w)=\frac{(S_r+\mu_1)^{r+\nu_1}(T_s+\mu_2)^{s+\nu_2}}{\Gamma(r+\nu_1)\Gamma(s+\nu_2)}w^{r+s+\nu_1+\nu_2-1}P^{r+\nu_1-1}(1-P)^{s+\nu_2-1}\times\exp\left[-w\{(S_r+\mu_1)P+(T_s+\mu_2)(1-P)\}\right]. \tag{3.27}$$

Integrating out w from (3.27), the marginal posterior pdf of ‘P’ comes out to be

$$h(P)=\frac{(S_r+\mu_1)^{r+\nu_1}(T_s+\mu_2)^{s+\nu_2}}{\Gamma(r+\nu_1)\Gamma(s+\nu_2)}\int_0^\infty w^{r+s+\nu_1+\nu_2-1}P^{r+\nu_1-1}(1-P)^{s+\nu_2-1}$$

$$\times\exp\left[-w\left\{(S_r+\mu_1)P+(T_s+\mu_2)(1-P)\right\}\right]dw$$

$$=\frac{(S_r+\mu_1)^{r+\nu_1}(T_s+\mu_2)^{s+\nu_2}}{B(n+\nu_1,\ m+\nu_2)}P^{r+\nu_1-1}(1-P)^{s+\nu_2-1}$$

$$\cdot\left\{(S_r+\mu_1)P+(T_s+\mu_2)(1-P)\right\}^{r+s+\nu_1+\nu_2}. \tag{3.28}$$

Denoting by $C=1-\frac{(S_r+\mu_1)}{(T_s+\mu_2)}$, we can write (3.28) as

$$h(P)=\frac{(1-C)^{r+\nu_1}}{B(r+\nu_1,s+\nu_2)}P^{r+\nu_1-1}(1-P)^{s+\nu_2-1}(1-CP)^{-(r+s+\nu_1+\nu_2)}.$$

Result (3.24) now fallows on applying a result of [7].

From the argument similar to used in Theorem 3, for $\alpha_1\neq\alpha_2,\ \lambda_1\neq\lambda_2$ and $\delta_1\neq\delta_2$

$$\hat{P}_{BS}=\int_{y=0}^{\infty}\int_{x=y}^{\infty}\hat{f}_{BS}(x;\alpha_1,\lambda_1,\delta_1)\hat{f}_{BS}(y;\alpha_2,\lambda_2,\delta_2)dxdy$$

$$=\int_{y=0}^{\infty}\hat{R}_{BS}(y;\alpha_1,\lambda_1,\delta_1)\hat{f}_{BS}(y;\alpha_2,\lambda_2,\delta_2)dy$$

$$=1-\frac{\lambda_2\delta_2(s+\nu_2)}{(T_s+\mu_2)}\int_{y=0}^{\infty}\left[1-\frac{\ln(1-e^{-\lambda_1 y^{\delta_1}})}{(S_r+\mu_1)}\right]^{-(r+\nu_1)}$$

$$\cdot\frac{y^{\delta_2-1}e^{-\lambda_2 y^{\delta_2}}}{(1-e^{-\lambda_2 y^{\delta_2}})}\left[1-\frac{\ln(1-e^{-\lambda_2 y^{\delta_2}})}{(T_s+\mu_2)}\right]^{-(s+\nu_2+1)}dy$$

and result (3.24) follows.

Corollary 1: For $\delta_1=\delta_2=\delta$, say

$$\hat{P}_{BS}=1-(s+\nu_2)\int_0^1 z^{(s+\nu_2-1)}\left[1-\frac{\left\{1-e^{(T_s+\mu_2)(1-1/z)}\right\}^{\lambda_1/\lambda_2}}{(S_r+\mu_1)}\right]^{-(r+\nu_1)}dz.$$

Proof: Corollary directly follows from (3.25), For $\delta_1=\delta_2=\delta$.

Remarks 1:

- The complete sample case results can be obtained on putting r = n and s = m.

- From (3.1),

$$\hat{\alpha}_{BS}^{-1} = \frac{\Gamma(r+\nu-1)}{\Gamma(r+\nu)}(S_r+\mu) = \frac{S_r+\mu}{r+\nu-1}$$

Now

$$E(\hat{\alpha}_{BS}^{-1}) = \frac{E(S_r)+\mu}{r+\nu-1} = \frac{\frac{1}{2\alpha}E(\chi_{2r}^2)+\mu}{r+\nu-1} = \frac{\frac{r}{\alpha}+\mu}{r+\nu-1}$$

$\rightarrow a^{-1}$ as $r \rightarrow \infty$.

Moreover,

$$V(\hat{a}_{BS}^{-1}) = \frac{V(S_r)}{4a^2(r+?-1)^2} = \frac{r}{a^2(r+?-1)^2} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

We can write (3.12) as

$$\hat{f}_{BS}(x;\alpha,\lambda,\delta) = \frac{\lambda\delta(r+\nu)x^{\delta-1}e^{-\lambda x^\delta}}{(S_r+\mu)(1-e^{-\lambda x^\delta})}\left[1-\frac{\ln(1-e^{-\lambda x^\delta})}{(S_r+\mu)}\right]^{-(r+\nu+1)}$$

$$\hat{f}_{BS}(x;\alpha,\lambda,\delta) = \frac{\lambda\delta(r+\nu)x^{\delta-1}e^{-\lambda x^\delta}}{(r+\nu-1)\alpha_{BS}^{-1}(1-e^{-\lambda x^\delta})}\left[1-\frac{\ln(1-e^{-\lambda x^\delta})}{(r+\nu-1)\alpha_{BS}^{-1}}\right]^{-(r+\nu+1)}$$

$$\rightarrow \frac{\alpha\lambda\delta x^{\delta-1}e^{-\lambda x^\delta}}{(1-e^{-\lambda x^\delta})}e^{\alpha\ln(1-e^{-\lambda x^\delta})} = f(x;\alpha,\lambda,\delta)$$

$\Rightarrow \hat{f}(x;\alpha,\lambda,\delta)$ is a consistent estimator of $f(x;\alpha,\lambda,\delta)$.

Hence, $\hat{R}_{BS(t)}$ and $\hat{P}_{BS}$ are also consistent estimators of R(t)and 'P', respectively.

- If we look at remark (2), we observe that the estimators of negative powers of α are used to prove the consistency of $\hat{f}(x;\alpha,\lambda,\delta)$. This justifies the estimation of negative powers of α.
- In order to obtain Bayes estimators of R(t) and 'P', in literature, authors first obtain the expression of these quantities and then their Bayes estimators, say, posterior mean under SELF. In the present approach to obtain Bayes estimators of R(t) and 'P' we made use of Bayes estimator of the pdf and one does not need their expression. Moreover, we have established an interrelationship between these two estimation problems.

Simulation Studies: In order to validate the performance of the estimators under SELF with respect to the actual reliability estimates obtained through numerical integration, we have simulated a sample of size r = n = 50 from (1.1) with $\alpha = 2$, $\delta = 2$ and $\lambda = 1$.

0.4173003, 0.9460239, 0.7186531, 0.9736477, 1.6545909, 1.1489930, 1.0387765, 0.8689265, 1.1750468, 0.9411219, 1.8013401, 1.1333229, 2.2251164, 0.8826638, 1.8679742, 1.2884515, 1.0713660, 1.4829736, 1.5310753, 0.7714249, 0.7771927, 1.0573936, 1.5314517, 0.8741688, 2.1261284, 1.1641412, 0.9074558, 1.3315983, 0.6891152, 1.5015916, 1.9824928, 1.2762304, 1.5224234, 0.7976915, 0.7174940, 1.0339409, 1.2862068, 0.9775802, 0.8620062, 1.1182395, 1.0568100, 0.9023865, 0.7227187, 1.9491991, 1.1022730, 1.6380632, 1.1811405, 1.7408185, 1.6126990, 0.9343715.

Table 1:

p = 1		p = 2	
$\hat{a}_{BS}$ 2.231829	$\hat{a}^{-1}_{BS}$ 0.4562096	$\hat{a}^{2}_{BS}$ 5.070009	$\hat{a}^{-2}_{BS}$ 0.2119814
$R_{PS}(\hat{a}_{BS})$ 0.08894753	$R_{PS}(\hat{a}^{-1}_{BS})$ 0.003854207	$R_{PS}(\hat{a}^{2}_{BS})$ 1.852177	$R_{BS}(\hat{a}^{-2}_{BS})$ 0.003489233
$R_{BS}(\hat{a}_{BS})$ 0.02947368	$R_{BS}(\hat{a}^{-1}_{BS})$ 0.02272727	$R_{BS}(\hat{a}^{2}_{BS})$ 0.3251993	$R_{BS}(\hat{a}^{-2}_{BS})$ 0.3752806

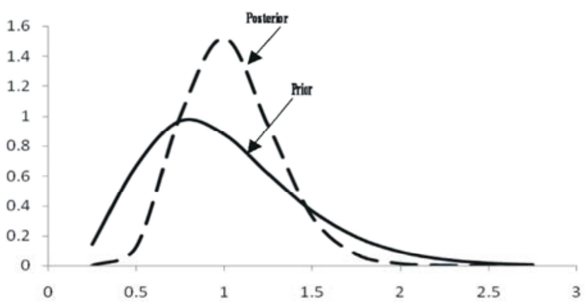

Fig. 1: Prior and Posterior densities

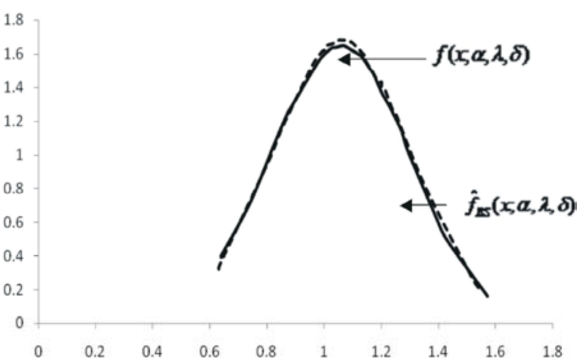

Fig. 2: Graph of f(x; α, λ, δ) and $\hat{f}_{BS}(x;\alpha,\lambda,\delta)$.

Table 2:

t	R(t)	$\hat{R}_{BS}(t)$
0.30	0.993	0.995
0.40	0.978	0.984
0.50	0.951	0.962
0.60	0.909	0.926
0.70	0.850	0.875
0.80	0.777	0.808
0.90	0.692	0.727
1.00	0.600	0.637
1.10	0.507	0.544

Here, we have S_r = 20.09153. For the case when prior parameters are μ = 5 and ν = 6, we have plotted prior and posterior densities in Fig. 1 and constructed Table 1.

In order to compute values of R(t) and $\hat{R}_{BS}(t)$, we assume that the data represents life spans of items in hours. With respect to above sample and for same prior parameters as above, computed values of R(t) and $\hat{R}_{BS}(t)$, over t=0.30(0.10)1.10 are given in Table 2.

For computing the value of $\hat{P}_{BS}$ we have simulated two samples from (1.1) of sizes r = n = 30 and s = m = 35, respectively. First is with parameters $(\alpha_1=3, \delta_1=2, \lambda_1=1)$ and second is with $(\alpha_2=2.5, \lambda_2=1, \delta_2=2)$, respectively.

X-Population:

1.1725308, 1.8200127, 1.7052567, 1.4018941, 1.2583337, 1.6616181, 0.9813064, 1.5846732, 0.5500817, 0.8015553, 1.4781880, 1.3155080, 0.9924352, 0.4397273, 1.0000568, 1.1561893, 0.7403056, 0.9018083, 0.5021047, 1.9817868, 1.4721579, 1.3098427, 1.4935193, 2.1591953, 0.8969304, 0.8221974, 1.5739064, 1.0102813, 0.5673100, 1.3246853.

Y-Population

1.1403903, 0.7555690, 1.2309668, 2.3932917, 1.2066033, 1.8399822, 1.3407853, 0.6030329, 1.1180712, 1.5327907, 1.5622690, 1.4313309, 1.6976184, 2.2411743, 0.7800085, 1.3023954, 1.0793686, 0.9517087, 1.4332955, 0.8087409, 0.7456061, 0.4338839, 1.3554415, 0.8024363, 1.0191814, 0.5005716, 0.6418640, 0.7757467, 0.6331233, 1.4079004, 1.1508947, 1.3903010, 1.4176388, 0.9870763, 1.2192468.

In the case when, prior parameters are $(\mu_1=5,\ ?_1=6)$ and $(\mu_2=7,\ \nu_2=5)$, respectively. For q=1, we get S_r =13.50066, T_s=16.5051, $P=0.5$ and $\hat{P}_{BS}=0.5326872$.

We have shown under Remark (1), that $\hat{f}(x;\alpha,\lambda,\delta)$ is a consistent estimator of $f(x;\alpha,\lambda,\delta)$. In order to verify these results, we have drawn a sample of sizes n = r = 30 from (1.1), with $\alpha=2, \lambda=1, \delta=3.5$ and $\mu=\nu=5$. In Fig. 2, we have plotted $\hat{f}(x;\alpha,\lambda,\delta)$ and $f(x;\alpha,\lambda,\delta)$, simultaneously. It is clear from Fig. 2 that both the curves overlap. This justifies the consistency property of the estimator.

DISCUSSION

[15] obtained the classical and Bayesian methods of parameter estimation for complete sample case. [16] obtained the classical and Bayesian methods of parameter

estimation under type II censoring. Here, we obtained Bayes estimators of the powers of the α, R(t) and 'P' under type II censoring. Consideration was given to SELF. We also obtained expressions for the risks, posterior risks and Bayes risks of the powers of α and R(t). Therefore we have extended previous research.

In the literature, the researchers have dealt with the estimation of R(t) and 'P', separately. If we look at the proofs of Theorems 3 and 5, we observe that the Bayes estimator of the sampled pdf is used to obtain the Bayes estimators of R(t) and 'P', respectively.

With the help of Fig. 2, we justified the consistency property of the estimators. Table 1 and 2 Shows that estimated values are very close to actual values.

CONCLUSION

Bayes estimators of R(t) and 'P' are derived under SELF and Type II censoring scheme. We have established interrelationship between the two estimation problems and extended the previous research. Moreover, in the present approach, one does not require the expressions of R(t) and 'P'.

REFERENCES

1. Bhattacharya, S.K., 1967. Bayesian approach to life testing and reliability estimation. Jour. Amer. Statist. Assoc., 62: 48-62.
2. Chao, A., 1982. On comparing estimators of Pr{X>Y} in the exponential case, IEEE Trans. Reliab., 26: 389-392.
3. Chaturvedi, A. and K. Surinder, 1999. Further remarks on estimating the reliability function of exponential distribution under type I and type II censorings, Brazilian Jour. Prob. Statist., 13: 29-39.
4. Chaturvedi, A. and S.K. Tomer, 2002. Classical and Bayesian reliability estimation of the negative binomial distribution, Jour. Applied Statist Sci., 11: 33-43.
5. Constantine, K., M. Karson and S.K. Tse, 1986. Estimators of P(Y<X) in the gamma case. Commun. Statist. -Simul. Comp., 15: 365-388.
6. Enis, P. and S. Geisser, 1971. Estimation of the probability that Y<X. Jour. Amer. Statist. Assoc., 66: 162-168.
7. Gradshteyn, I.S. and I.M. Ryzhik, 1980. Tables of Integrals, Series and Products, Academic Press, London.
8. Johnson, N.L., 1975. Letter to the editor. Technometrics, 17: 393.
9. Kelly, G.D., J.A. Kelly and W.R. Schucany, 1976. Efficient estimation of P(Y<X) in the exponential case. Technometrics, 18: 359-360.
10. Martz, H.F. and R.A. Waller, 1982. Bayesian Reliability And Analysis. John Wiley and Sons, New York.
11. Moore, A.H. and J.E. Bilikam, 1978. Bayesian estimation of the parameters of life distribution and reliability from type II censored samples. IEEE Trans. Reliab., 27: 64-67.
12. Mudholkar, G.S. and D.K. Srivastava, 1993. Exponentiated Weibull family for analyzing bathtub failure-real data. IEEE Transaction Reliability, 42: 299-302.
13. Pugh, E.L., 1963. The best estimate of reliability in the exponential case, Operations Research, 11: 57-61.
14. Sathe, Y.S. and S.P. Shah, 1981. on estimating P(X<Y) for the exponential distribution. Commun. Statist.-Theor. Meth., 10: 39-47.
15. Singh, U., P.K. Gupta and S.K. Upadhyay, 2002. Estimation of exponentiated Weibull shape parameters under LINEX loss function. Commun. Statist. -Simulation Comput., 31: 523-537.
16. Singh, U., P.K. Gupta and S.K. Upadhyay, 2004. Estimation of three-parameter exponentiated Weibull distribution under type-II censoring, J. Stat Plan Inference, 134: 350-372.