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Sums of Powers of Integers

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Abstract: We show a procedure to obtain the recursive formula deduced by Cereceda for the sums of powers of integers.

Key words: Bernoulli and Stirling numbers · Sums of powers of integers

INTRODUCTION

Here we study the sums of powers of integers:

$$S_k(n) \equiv 1^k + 2^k + ... + n^k, \quad k \geq 0, \tag{1}$$

and we obtain the identity:

$$\sum_{m=1}^{n+1} S_{k-1}(m) = \sum_{q=2}^{n} (q-1), \binom{n+1}{q+1} S_{k-1}^{[q-]}, \quad n \geq 1, \quad k \geq 2, \tag{2}$$

such that $S_j^{[r]}$ are the Stirling numbers of the second kind [1-5].

We give a simple proof of the expression [6]:

$$n\ S_{k-1}(n) - S_k(n) = \sum_{m=1}^{n-1} S_{k-1}(m), \quad k \geq 2, \quad n \geq 1. \tag{3}$$

Finally, we exhibit that (2) and (3) imply the Cereceda's recursive formula [6]:

$$(k+1) S_k(n) = k\left(n + \frac{1}{2}\right) S_{k-1}(n) - \sum_{r=1}^{k-2} \binom{k}{r} B_{k-r}\ S_r(D), \quad n \geq 1, \quad k \geq 2, \tag{4}$$

where B_j are the Bernoulli numbers [1, 7-10].

Cereceda's Identity: We know the relation [1, 11-17]:

$$S_{k-1}(m) = \sum_{l=1}^{m} l! \binom{m+1}{l+1} S_{k-1}^{[l]}, \tag{5}$$

and the expression [1, 18, 19]:

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$$\sum_{k=j}^{N}\binom{k}{j}=\binom{N+1}{j+1}, \tag{6}$$

then:

$$\sum_{m=1}^{n-1} S_{k-1}(m) \overset{(5)}{=} \sum_{l=1}^{n-1} l!\, S_{k-1}^{[l]} \sum_{m=l}^{n-1}\binom{m+1}{l+1} \overset{(6)}{=} \sum_{l=1}^{n-1} l!\,\binom{n+1}{l+2} S_{k-1}^{[l]},$$

which is equivalent to (2), q.e.d.

On the other hand, from (1):

$$nS_{k-1}(n)-S_k(n)=n(1^{k-1}+2^{k-1}+3^{k-1}+\ldots+(n-1)^{k-1}+n^{k-1})-$$

$$-(1^k+2^k+3^k+\ldots+(n-1)^k+n^k)=(n-1)^{k-1}+(n-2)2^{k-1}+\ldots+(n-1))(n-1)^{k-1},$$

$$=1^{k-1}+(1^{k-1}+2^{k-1})+(1^{k-1}+2^{k-1}+3^{k-1})+\ldots+(1^{k-1}+2^{k-1}+3^{k-1}+\ldots+(n-1)^{k-1}),$$

thus (3) is proved.

Now we consider the following relation involving Bernoulli numbers:

$$A\equiv\sum_{r=1}^{k-2}\binom{k}{r}B_{k-r}S_r(n)=Q-\binom{k}{k-1}B_1S_{k-1}(n)-B_0S_k(n)=Q+\frac{k}{2}S_{k-1}(n)-S_k(n), \tag{7}$$

where:

$$Q\equiv\sum_{r=1}^{k}\binom{k}{r}B_{k-r}S_r(n). \tag{8}$$

We have the properties [1]:

$$\binom{k}{r}B_{k-r}=k\sum_{q=r}^{k}\frac{1}{q}S_{k-1}^{[q-1]}S_q^{(r)}, \tag{9}$$

$$\sum_{r=j}^{q}S_r^{(r)}S_r^{[j]}=\delta_{jq}, \tag{10}$$

where $S_q^{(r)}$ are the Stirling numbers of the first kind [1, 9, 20, 21], hence from (5), (8) and (9):

$$Q=k\sum_{q=1}^{k}\frac{1}{q}S_{k-1}^{[q-1]}\sum_{r=1}^{q}\sum_{r=1}^{q}\sum_{j=1}^{r}j!\binom{n+1}{j+1}S_q^{(r)}S_r^{[j]},$$

$$=k\sum_{q=1}^{k}\frac{1}{q}\,S_{k-1}^{[q-1]}+\sum_{j=1}^{q}j!\binom{n+1}{j+1}\sum_{r=j}^{q}S_q^{(r)}S_r^{[j]} \overset{(10)}{=} k\sum_{q=1}^{k}(q-1)!\binom{n+1}{q+1}S_{k-1}^{[q-1]}. \tag{11}$$

Finally, if we employ (2), (3) and (11) into (7) we obtain the Cereceda's recursive formula indicated in (4), q.e.d.

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