World Applied Sciences Journal 9 (Special Issue of Applied Math): 44-48, 2010
ISSN 1818-4952

Analytical Solutions for Investigating Free Vibration of Cantilever Beams

[1]H.D. Kaliji, [2]A. Fereidoon, [1]M. Ghadimi and [3]M. Eftari

[1]Department of Mechanical Engineering Islamic Azad University, Semnan Branch, Semnan, Iran
[2]Department of Mechanical Engineering, Faculty of Engineering, Semnan University, Iran
[3]Department of Mechanical Engineering Islamic Azad University, South Tehran Branch, Iran

Abstract: In this paper, Energy Balance Method (EBM) and Frequency-Amplitude Formulation (FAF) proposed by He are used to obtain the behavior and frequency of the cantilever rotating beams. The application of mathematics on aerodynamic showed in the beam attached to a rigid hub which rotates along the hub axis such as helicopter blades. With simplifying some complicated parameters, vibration equation of the cantilever beam has been solved by new mathematical methods. The convenience and effectiveness of the analytical solutions have been verified by comparison of the results with Runge-Kutta 4th order method. It is predictable that by improving these methods, lots of complicated problems will solve.

Key words: Cantilever beam · Nonlinear vibration · Energy Balance Method · Frequency-Amplitude Formulation · Approximate frequency

INTRODUCTION

Recently investigating the response of the systems, especially vibration of aerodynamic systems such as helicopter blades and airplane wings is vital aspect in engineering. Because the skin panel of these structures may experience serious structural problems such as thermal buckling or panel flutters. So, studying about them should be very carefully without missing any parameter. We can assume the helicopter blades as a cantilever rotating beam. For airplane wing we have cantilever beams along longitudinal axis of the wing and both free ends beams along vertical direction of the axis.

In this paper, we solved the dynamic equation of a cantilever rotating beam. The equation derived from this problem is nonlinear. Solving equation with nonlinear terms have been one of the most time-consuming and difficult affairs among engineers. In recent years, much attention has been done to the newly approximate methods for solving nonlinear equations [1, 2]. Some of them are, He's Homotopy Perturbation Method (HPM) [3, 6], Homotopy Analysis Method (HAM) [7-9], He's Parameter-Expanding Method [10, 11], He's Variational Iteration Method (VIM) [12-14], He's Energy Balance Method (EBM) [15-17], He's Frequency-Amplitude Formulation (FAF) [18-21] and etc.

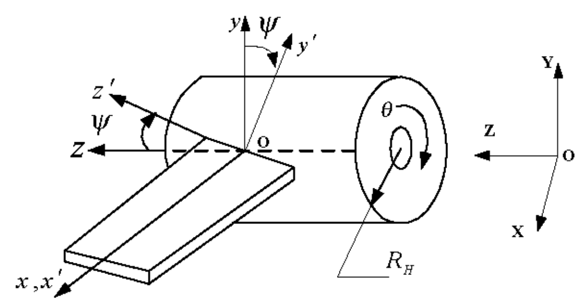

Fig. 1: Cantilever beam diagram

The cantilever beam is shown in Fig. 1 attached to the hub which is assumed to be a rigid disc with radius R, mass M and rotating at an angular velocity $\dot{\theta}$ about the Z-axis. The effect of torque T on the hub causes it to rotate only. The X, Y, Z is a system of fixed rectangular Cartesian coordinate axes with origin at the center of the hub. The x, y, z and x', y', z' are two sets of rectangular Cartesian coordinate axes rotating with the hub with common origin at the root of the beam. The setting angle Ψ is rotation of the hub about longitude axis of the beam. The beam is assumed to be initially straight along the x'-axis clamped at its base to the hub surface, having a uniform cross-sectional area A_b, flexural rigidity EI, constant length l, mass m and mass density ρ. The beam thickness is assumed to be small compared to its

Corresponding Author: H.D. Kaliji, Department of Mechanical Engineering Islamic Azad University, Semnan Branch, Semnan, Iran, Tel/Fax: +98-124-4282059, E-mail: hdadashpour@yahoo.com.

length so that the effects of shear deformation and rotary inertia can be ignored.

The mathematical model used here is a special case of equation introduced in the work of Hamdan and El-Sinawi [22]. The system Lagrangian in former work is a function of the beam transverse dimensionless deflection w, which is a continuous function of the spatial variable ξ and time t. this continuous Lagrangian is discretized using the assumed mode method as $w = w(t)\Phi(\xi)$. Where $\Phi(\xi)$ is a mode shape deflection of the beam and $w(t)$ is the corresponding time modulation modal coordinate. By substituting equation (1) in the system Lagrangian equation the coupled single mode ordinary differential equation of motion derived [23]. We performed a new format of special case equation up to third order terms by mathematical derivation as below:

$$\ddot{w} + \omega_1{}^2 w + d_1 w^2 \ddot{w} + d_1 w \dot{w}^2 + d_2 w^3 = 0, \tag{1}$$

and the initial conditions are:

$$w(0) = A, \qquad \dot{w}(0) = 0. \tag{2}$$

Where A is the amplitude of the system.

The solution and frequencies of a cantilever rotating beam obtained by applying Energy Balance Method and Frequency-Amplitude Formulation. Comparison between results achieved from former methods with numerical solutions using Runge-Kutta 4th order, showed the power and precious of these methods.

Solution Procedure: In this section, we will apply Energy Balance Method and Frequency-Amplitude Formulation to solve the nonlinear Eq. (1).

Energy Balance Method: In this method, variational formulation for Eq. (1) expressed as below:

$$J(w) = \int_0^t \left(-\frac{1}{2}\dot{w}\left(1 + d_1 w^2\right) + \frac{\omega_1{}^2}{2} w^2 + \frac{d_2}{4} w^4 \right) dt. \tag{3}$$

Hamiltonian can be written in the form:

$$H = \frac{1}{2}\dot{w}^2\left(1 + d_1 w^2\right) + \frac{\omega_1{}^2}{2} w^2 + \frac{d_2}{4} w^4 = \frac{\omega_1{}^2}{2} A^2 + \frac{d_2}{4} A^4. \tag{4}$$

Or,

$$R(t) = \frac{1}{2}\dot{w}^2\left(1 + d_1 w^2\right) + \frac{\omega_1{}^2}{2} w^2 + \frac{d_2}{4} w^4 - \frac{\omega_1{}^2}{2} A^2 - \frac{d_2}{4} A^4 = 0. \tag{5}$$

We use the following trial function to determine the angular frequency ω.

$$W(t) = A\cos(\omega t). \tag{6}$$

Substituting Eq. (6) into Eq. (5) yields:

$$R(t) = \frac{1}{2} A^2 \omega^2 \sin^2(\omega t)(1 + d_1 A^2 \cos^2(\omega t)) + \frac{\omega_1{}^2}{2} A^2 \cos^2(\omega t) + \frac{d_2}{4} A^4 \cos^4(\omega t) - \frac{\omega_1{}^2}{2} A^2 - \frac{d_2}{4} A^4 = 0. \tag{7}$$

Collocation Eq. (7) at $\omega t = \pi / 4$:

$$\omega_{EBM} = \sqrt{\frac{4\omega_1{}^2 + 3 d_2 A^2}{2(2 + d_1 A^2)}}. \tag{8}$$

Substituting Eq. (8) into Eq. (6) yields:

$$w(t) = A\cos\left(\sqrt{\frac{4\omega_1{}^2 + 3 d_2 A^2}{2(2 + d_1 A^2)}}\, t \right). \tag{9}$$

Frequency Amplitude Formulation: For solving Eq. (1) with Frequency Amplitude Formulation, we use the trial functions $w_1(t)\, A \cos t$ and $w_2(t) = A \cos \omega t$, which are the solutions of the following linear equations, respectively.

$$\ddot{w} + \tilde{\omega}_1{}^2 w = 0, \quad \tilde{\omega}_1{}^2 = 1, \tag{10}$$

$$\ddot{w} + \tilde{\omega}_2{}^2 w = 0, \qquad \tilde{\omega}_2{}^2 = \omega^2. \tag{11}$$

The residuals are:

$$R_1(t) = (-A + \omega_1{}^2 A + d_1 A^3)\cos t + (-2 d_1 A^3 + d_2 A^3)\cos^3 t, \tag{12}$$

and,

$$R_2(t) = (-A\omega^2 + \omega_1{}^2 A + d_1 A^3 \omega^2)\cos\omega t + (-2 d_1 A^3 \omega^2 + d_2 A^3)\cos^3 \omega t. \tag{13}$$

We introduce two new residual variables $\tilde{R}_1$ and $\tilde{R}_2$ define as [18, 19].

$$\tilde{R}_1 = \frac{4}{T_1}\int_0^{T_1/4} R_1(t)\cos\left(\frac{2\pi}{T_1} t\right) dt, \tag{14}$$

and,

$$\tilde{R}_2 = \frac{4}{T_2}\int_0^{T_2/4} R_2(t)\cos\left(\frac{2\pi}{T_2}t\right)dt. \tag{15}$$

We can approximately determine ω^2 in the form,

$$\omega^2 = \frac{\tilde{\omega}_1{}^2\tilde{R}_2 - \tilde{\omega}_2{}^2\tilde{R}_1}{\tilde{R}_2 - \tilde{R}_1}. \tag{16}$$

For the Eq. (1), by a simple calculation, we obtain:

$$\tilde{R}_1 = \frac{-4A + 4\omega_1{}^2 A + 3d_2A^3 - 2d_1A^3}{8}, \tag{17}$$

and,

$$\tilde{R}_2 = \frac{-4A\omega^2 + 4\omega_1 A^2 + 3d_2A^3 - 2d_1A^3\omega^2}{8}. \tag{18}$$

Applying Eq. (16), we have:

$$\omega^2 = \frac{4\omega_1{}^2 A + 3d_2A^3 - 4\omega_1 A\omega^2 - 3d_2A^3\omega^2}{-4A\omega^2 - 2d_1A^3\omega^2 + 4A + 2d_1A^3}. \tag{19}$$

Its approximate frequency write as:

$$\omega_{FAF} = \sqrt{\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}, \tag{20}$$

Where a, b and c are as follows:

$$\begin{aligned} a &= -4A - 2d_1A^3, \\ b &= 4A + 2d_1A^3 + 4\omega_1{}^2 A + 3d_2A^3, \\ c &= -4\omega_1{}^2 A - 3d_2A^3. \end{aligned} \tag{21}$$

The periodic solution as follows:

$$w\,(t) = A\cos\omega t, \tag{22}$$

Where ω is evaluated from equation (20).

According to Eq. (20), two various frequencies are obtained that one of them is the main frequency apparent through Eqs. (10, 11). The main frequency distinguishes via the coefficients of the proposed equation.

RESULTS

According to Eq. (20), two various frequencies are obtained that one of them is the main frequency apparent through Eqs. (10) and (11). The main frequency distinguishes via the coefficients of the proposed equation. We illustrate these statements for two modes.

Table 1: Values of variables in Eq. (1), for two modes

Mode	ω_1	d_1	d_2
1	1	1	1
2	0	1	1

Table 2: Comparison between present work with time marching solution

	Mode 1 for $t=5s$		Mode 2 for $t=10s$	
A	EBM or FAF	Runge-Kutta	EBM or FAF	Runge-Kutta
$\pi/18$	0.052632240	0.052829036	0.012325325	0.015003762
$\pi/9$	0.122602316	0.124664245	-0.341637273	-0.333579366
$\pi/6$	0.221170789	0.230114050	-0.232445300	-0.240549178
$\pi/3$	0.684382411	0.767884930	0.561231611	0.553621951

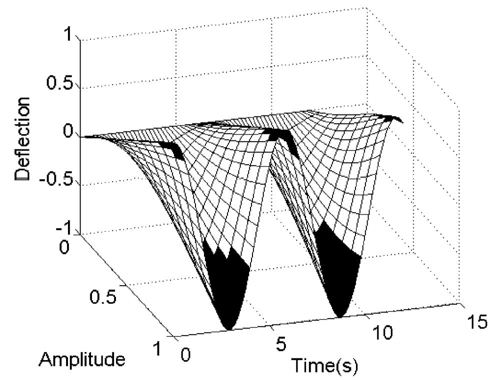

Fig. 2: Transverse vibration of the cantilever beam with EBM for first mode.

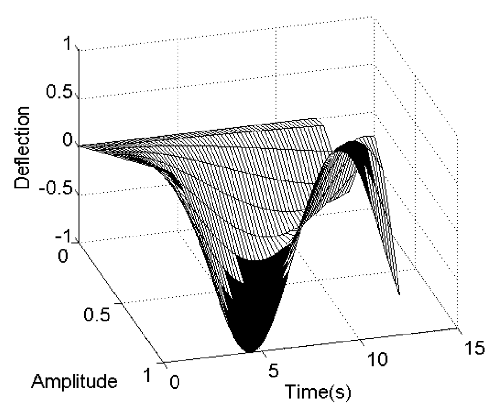

Fig. 3: Transverse vibration of the cantilever beam with FAF for second mode.

The values of parameters ω_1, d_1 and d_2 associated with each of the calculation modes are shown in Table 1. We should use alternatively negative and positive symbols in Eq. (20), for the first and second modes. To show the remarkable accuracy of the obtained result,

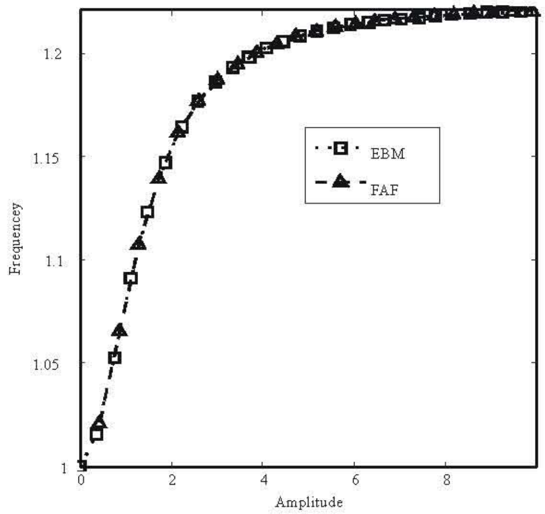

Fig. 4: Comparison between EBM and FAF for frequency versus amplitude

we compare the approximate periodic solutions with Runge-Kutta 4th order in Table 2. Because of extreme agreement between EBM and FAF we put one column for them in the table. The results for these methods are exactly the same at least until 10th order. For example for the variables of first mode at amplitude equal to $\pi/3$ and $t=5s$ the EBM result for deflection of the cantilever beam is 0.684382411994534 and for FAF result it is 0.684382411994537.

The behaviors of transverse vibration of the cantilever beam obtained by EBM for first mode and FAF for second mode are shown in Figs. 2 and 3, respectively. In Fig. 4 the frequency versus amplitude compared between the mentioned methods. It can visible that these two different approaches lead us to the similar results for the vibration of the cantilever beams system.

CONCLUSIONS

In this paper, free vibration of special case of nonlinear equation for cantilever rotating beam solved with EBM and FAF. The frequency obtained with these methods showed excellent agreement with each other. The comparison of these methods with time marching solution show agreeable results for low amplitude. It is obvious that EBM and FAF are powerful and efficient technique for finding analytical solutions thus we can use them for approximate for mechanical systems. However, further research is needed to better understanding and developing these methods on engineering problems especially mechanical affairs.

REFERENCES

1. Pirbodaghi, T., M.T. Ahmadian and M. Fesanghary, 2009. On the Homotopy Analysis Method for Non-linear Vibration of Beams. Mechanics Research Communications, 36: 143-148.
2. Hojjati, M.H. and S. Jafari, 2009. Semi-exact solution of non-uniform thickness and density rotating disks. Part II: Elastic strain hardening solution. International J. Pressure Vessels and Piping, 86: 307-318.
3. He, J.H., 1999. Homotopy Perturbation technique. Computer Methods in Applied Mechanics and Engineering, 178: 257-262.
4. He, J.H., 2005. Application of Homotopy Perturbation Method to nonlinear wave equations. Chaos Solitons & Fractals, 26: 695-700.
5. Fereidoon, A., M.R. Davoudabadi, H. Yaghoobi, D.D. Ganji, 2010. Application of Homotopy Perturbation Method and Differential Transformation Method to Damped System with Nonlinear Spring. World Appl. Sci. J., 9(6): 681-688.
6. Fereidoon, A., D.D. Ganji, H.D. Kaliji and M. Ghadimi, 2010. Analytical solution for vibration of buckled beams. International J. Research and Reviews in Appl. Sci., 4(3): 17-21.
7. Abdoul, R. Ghotbi, H. Bararnia, G. Domairry and A. Barari, 2009. Investigation of a powerful analytical method into natural convection boundary layer flow. Commun Nonlinear Sci. Numer Simulat, 14: 2222-2228.
8. Sohouli, A.R., M. Famouri, A. Kimiaeifar and G. Domairry, 2010. Application of homotopy analysis method for natural convection of Darcian fluid about a vertical full cone embedded in pours media prescribed surface heat flux. Commun Nonlinear Sci. Numer Simulat, 15: 1691-1699.
9. Fooladi, M., S.R. Abaspour, A. Kimiaeifar and M. Rahimpour, 2009. On the Analytical Solution of Kirchhoff Simplified Modelfor Beam by using of Homotopy Analysis Method. World Appl. Sci. J., 6(3): 297-302.
10. Xu, L., 2007. He's parameter-expanding methods for strongly nonlinear oscillators. Journal of Computational and Appl. Math., 207: 148-154.
11. Kimiaeifar, A., A.R. Saidi, A.R. Sohouli and D.D. Ganji, 2010. Analysis of modified Van der Pol's oscillator using He's parameter-expanding methods. Current Applied Physics, 10: 279-283.

12. Fereidoon, A., M. Ghadimi, H.D. Kaliji, M. Eftari, S. Alinia, 2010. Variational Iteration Method for Nonlinear Vibration of Systems with Linear and Nonlinear Stiffness, International J. Res. and Review in Applied Sie., 5(3): 260-263.
13. Rafei, M., D.D. Ganji, H. Daniali and H. Pashaei, 2007. The variational iteration method for nonlinear oscillators with discontinuities. J. Sound and Vibration, 305: 614-620.
14. He, J.H., 2000. Variational iteration method for autonomous ordinary differential systems. Applied Mathematics and Computation, 114: 115-123.
15. Pashaei, H., D.D. Ganji and M. Akbarzade, 2008. Applications of the energy balance method for strongly nonlinear oscillators. Progress in Electromagnetic Research M, 2: 47-56.
16. Ganji, S.S., D.D. Ganji, Z.Z. Ganji and S. Karimpour, 2008. Periodic solution for strongly nonlinear vibration system by He's energy balance method. Acta Applicandae Mathematicae, doi:10.1007/s10440-008-9283-6.
17. He, J.H., 2002. Preliminary report on the energy balance for nonlinear oscillations. Mechanics Research Communications, 29(2-3): 107-111.
18. He, J.H., 2008. Comment on He's frequency formulation for nonlinear oscillators. European J. Physics, 29: L19-L22.
19. He, J.H., 2008. An improved amplitude-frequency formulation for nonlinear oscillators, International J. Nonlinear Science and Numerical Simulation, 9(2): 211-212.
20. Hui-Li Zhang, 2009. Application of He's amplitude-frequency formulation to a nonlinear oscillator with discontinuity. Computers and Mathematics with Applications, 58: 2197-2198.
21. Xu-Chu, Cai and Wen-YingWu, 2009. He's frequency formulation for the relativistic harmonic oscillator. Computers and Mathematics with Applications, 58: 2358-2359.
22. Hamdan, M.N. and A.H. El-Sinawi, 2005. On the non-linear vibration of an inextensible rotating arm with setting angle and flexible hub. J. Sound and Vibration, 281: 375-398.
23. Jarrar, F.S.M. and M.N. Hamdan, 2007. Nonlinear vibration and buckling of a flexible rotating beam: A prescribed torque approach. Mechanism and Machine Theory, 42: 919-939.