World Applied Sciences Journal 9 (Special Issue of Applied Math): 32-38, 2010
ISSN 1818-4952

The Solution of Flierl-Petviashivili Equation and its Variants Using Dtm-Padé Technique

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Abstract: A numerical method for solving the Flierl–Petviashivili (FP) equation and its variants is proposed. The proposed scheme is based on differential transform method (DTM) and Padé approximants. The DTM-Padé technique introduces an alternative framework designed to overcome the difficulty of the singular point at $x = 0$. The numerical results demonstrates the validity and applicability of the method and a comparison is made with existing results.

Key words: Differential transform method · Flierl-Petviashivili equation · Padé approximants

INTRODUCTION

Consider the standard Emden–Fowler equation of the form

$$y''(x)+\frac{2}{x}y'(x)+af(x)g(y)=0,\quad y(0)=y_0,\quad y'(0)=0,\quad (1.1)$$

Where $f(x)$ and $g(y)$ are some given functions of x and y, respectively. For $f(x) = 1$ and $g(y) = y^n$, Eq. (1.1) is the standard Lane-Emden equation that was used to model the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules [1-5] and subject to the classical laws of thermodynamics. For other special forms of $g(y)$, the well-known Lane–Emden equation was used to model several phenomena in mathematical physics and astrophysics such as the theory of stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gas spheres and theory of thermionic currents. The Lane-Emden equation has recently been solved by means of Adomian's decomposition method which provides a convergent series solution [6], the quasilinearization method of Bellman and Kalaba [7], a piecewise linearization technique [8] based on the piecewise linearization of the Lane-Emden equation and the analytical solution of the resulting piecewise constant-coefficients odes, the homotopy analysis method [9], a variational approach which uses a semi-inverse method to obtain a variational principle [10] and may employ the Ritz technique to obtain approximate analytical solutions [11-13] and series solution method [14]. The series solution method considered in Ref. [14] is also compared with the homotopy perturbation method [15-17].

The Flierl-Petviashivili equation can be obtained from (1.1) by setting $f(x) = 1, \alpha =-$ and $g(y) = y + y^2$. However, in this paper, we will consider the following two variants of Flierl-Petviashivili equation [18]:

$$y''(x)+\frac{1}{x}y'(x)-y^n(x)-y^{n+1}(x)=0,\ y(0)=\alpha,\ y'(0)=0,\ n\ge 1,\quad (1.2)$$

and

$$y''(x)+\frac{1}{x}y'(x)-y(x)-y^2(x)=0,\ y(0)=\alpha,\ y'(0)=0,\quad (1.3)$$

Where the boundary condition is $y(\infty) = 0$ for variants of FP equations (1.2) and (1.3). Eqs. (1.2) and (1.3) was thoroughly investigated by Wazwaz [18] using the Adomian decomposition method and Padé approximants.

The singularity behavior at $x = 0$ is a difficult element in this type of equations. The motivation for presenting this work comes actually from the aim of introducing a reliable framework that combines the powerful Padé approximants [19] and differential transform method (DTM) established in [20] and used thoroughly in [18] and the references therein.

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The differential transform method was first applied in the engineering domain by [20]. The DTM provides an efficient explicit and numerical solution with high accuracy, minimal calculations and avoidance of physically unrealistic assumptions. However, DTM has some drawbacks. By using DTM, we get a series solution, in practice a trutncated series solution. This series solution converges in a limited interval and outside it, high errors are occurred. To overcome this drawback and improve the accurancy in larger interval, we apply the Padé approximants to the obtained series to handle the boundary conitions at infinity. The DTM method together with Padé approximants (DTM-Padé technique) extends the domain of solution and give better accuracy and better convergence than using DTM alone. This link is used before with ordinary differential equations (see [18]) and partial differential equations (see [21]).

This paper is organized as follows: In Section 2, we describe the differential transform method and give a brief discussion of Padé approximants. In Sections 3 and 4, the method is implemented to boundary value problems (1.3) and (1.4), respectivelly and conclusion remarks are presented in Section 5.

Differential Transform Method: The differential transform method is a semi-numerical-analytic-technique that formalizes Taylor series in a totally different manner. With this technique, the given differential equation and related boundary conditions are transformed into a recurrence equation that finally leads to the solution of a system of algebraic equations as coefficients of a power series solution. This method is useful to obtain exact and approximate solutions of linear and nonlinear differential equations. No need to linearization or discretization, large computational work and round-off errors are avoided. It has been used to solve effectively, easily and accurately a large class of linear and nonlinear problems with approximations. The method is well addressed in [22-31]. The basic definitions of differential transformation are introduced as follows:

Definition 2.1: If $f(t)$ is analytic in the time domain T, then

$$\varphi(t_i,k) = \left.\frac{\partial^k f(t)}{\partial t^k}\right|_{t=t_i}, \forall t \in T, \qquad (2.1)$$

Where k belongs to the set of non-negative integer, denoted as the K-domain. Therefore, Eq. (2.1) can be rewritten as

$$F(k) = \frac{1}{k!}\varphi(t_i,k) = \frac{1}{k!}\left[\frac{\partial^k f(t)}{\partial t^k}\right]_{t=t_i}, \forall k \in K, \qquad (2.2)$$

Where $F(k)$ is called the spectrum of $f(t)$ at $t = t_i$ in the K-domain.

If $f(t)$ can be represented by Taylor's series, then it can be represented as

$$f(t) = \sum_{k=0}^{\infty}(t-t_i)^k F(k) \equiv D^{-1}F(k). \qquad (2.3)$$

This equation is called the inverse of $f(t)$, with the symbol D denoting differential transform process. The particular case of Eq. (2.3) when $t_i = 0$ is referred to as the Maclaurin series of $f(t)$ and is expressed as

$$f(t) = \sum_{k=0}^{\infty} t^k F(k) \equiv D^{-1}F(k).$$

Using differential transform, a differential equation in the domain of interest can be transformed to an algebraic equation in the K-domain and $f(t)$ can be obtained by finite-term Taylor series expansion plus a remainder, as

$$f(t) = \sum_{k=0}^{N}(t-t_i)^k F(k) + R_{m+1}(t).$$

The fundamental mathematical operations performed by differential transform method are listed in Table 1.

In addition to the above operations, the following theorem that can be deduced from Eqs.(2.2) and (2.3) is given below:

Table 1: The fundamental operations of differential transform method

Time function	Transformed function
$w(t) = \alpha\, u(t) \pm \beta v(t)$	$W(k) = \alpha U(k) \pm \beta V(k)$
$w(t) = d^m u(t) / dt^m$	$W(k) = \frac{(k+m)!}{k!}U(k+m)$
$w(t) = u(t)v(t)$	$W(k) = \sum_{l=0}^{k} U(l)V(k-l)$
$w(t) = t^m$	$W(k) = \delta(k-m) = \begin{cases} 1, & if\ k = m, \\ 0, & if\ k \neq m. \end{cases}$
$w(t) = \exp(t)$	$W(k) = 1/k!$
$W(t) = \sin(\omega t + \alpha)$	$W(k) = (\omega^k/k!) \sin(k\pi/2 + \alpha)$
$W(t) = \cos(\omega t + \alpha)$	$W(k) = (\omega^k/k!) \cos(k\pi/2 + \alpha)$

Theorem

If $f,(x) = g_1(x)g_2(x)...g_{m-1}(x)g_m(x)$ then

$$F(k)=\sum_{k_{n-1}=0}^{k}\sum_{k_{n-2}=0}^{k_{n-1}}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}G_1(k_1)G_2(k_2-k_1)\ldots G_{m-1}(k_{n-1}-k_{m-2})G_m(k-k_{m-1}). \tag{2.4}$$

Padé Approximants: A Padé approximant is the ratio of two polynomials constructed from the coefficients of the Taylor series expansion of a function $y(x)$. The [L/M] Padé approximants to a function $y(x)$ are given in Ref. [19] as follows:

$$\left[\frac{L}{M}\right]=\frac{P_L(x)}{Q_M(x)}, \tag{2.5}$$

Where $P_L(x)$ is a polynomial of degree at most L and $Q_M(x)$ is a polynomial of degree at most M. The formal power series

$$(x)=\sum_{i=1}^{\infty}a_i x^j, \tag{2.6}$$

$$y(x)-\frac{P_L(x)}{Q_M(x)}=O(x^{L+M+1}), \tag{2.7}$$

determine the coefficients of $P_L(x)$ and $Q_M(x)$ by the equation.

Since we can obviously multiply the numerator and denominator by constant and leave $[L/M]$ unchanged, we impose the normalization condition

$$Q_M(0) = 10. \tag{2.8}$$

Finally we require the that $P_L(x)$ and $Q_M(x)$ have no common factors.

If we write the coefficient of $P_L(x)$ and $Q_M(x)$ as

$$\left.\begin{aligned} P_L(x)&=p_0+p_1x+p_2x^2+\cdots+p_Lx^L,\\ Q_M(x)&=q_0+q_1x+q_2x^2+\cdots+q_Mx^M.\end{aligned}\right\} \tag{2.9}$$

Then by (2.8) and (2.9) we may multiply (3.3) by $Q_M(x)$, which linearizes the coefficient equations. We can write out (2.7) in more detail as

$$\left.\begin{aligned} &a_{L+1}+a_Lq_1+\cdots+a_{L-M+1}q_M=0,\\ &a_{L+2}+a_{L+1}q_1+\cdots+a_{L-M+2}q_M=0,\\ &\vdots\\ &a_{L+M}+a_{L+M-1}q_1+\cdots+a_Lq_M=0,\end{aligned}\right\} \tag{2.10}$$

$$\left.\begin{aligned} &a_0=p_0,\\ &a_0+a_0q_1=p_1,\\ &a_2+a_1q_1+a_0q_2=p_2,\\ &\vdots\\ &a_L+a_{L-1}q_1+\cdots+a_0q_L=p_L.\end{aligned}\right\} \tag{2.11}$$

To solve these equations, one starts with Eqs. (2.10), which is a set of linear equations for all the unknown q's. Once the q's are known, then Eq. (2.11) gives an explicit formula for the unknown p's, which complete the solution.

If Eqs. (2.10) and (2.11) are nonsingular, then we can solve them directly and obtain Eq. (2.12) (see Ref.[19]), where Eq. (2.12) holds and, if the lower index on a sum exceeds the upper, the sum is replaced by zero:

$$\left[\frac{L}{M}\right]=\frac{\det\begin{bmatrix} a_{L-M+1} & a_{L-M+2} & \cdots & a_{L+1}\\ \vdots & \vdots & \ddots & \vdots\\ a_L & a_{L+1} & \cdots & a_{L+M}\\ \sum_{j=M}^{L}a_{j-M}x^j & \sum_{j=M-1}^{L}a_{j-M+1}x^j & \cdots & \sum_{j=0}^{L}a_jx^j\end{bmatrix}}{\det\begin{bmatrix} a_{L-M+1} & a_{L-M+2} & \cdots & a_{L+1}\\ \vdots & \vdots & \ddots & \\ a_L & a_{L+1} & \cdots & a_{L+M}\\ x^M & x^{M-1} & \cdots & 1\end{bmatrix}} \tag{3.12}$$

To obtain diagonal Padé approximants of different order like, [2/2], [4/4] or [6/6] we can use Mathematica.

First Generalization of the FP Equation: In this section we will discuss the first generalized variant of the FP equation

$$y''(x)+\frac{1}{x}y'(x)-y^n(x)-y^{n+1}(x)=0, \tag{3.1}$$

With the boundary conditions

$$y(0)=\alpha, \quad y'(0)=0, \quad y(\infty)=0. \tag{3.2}$$

For $n = 1$, Eq. (3.1) reduces to the standard FP equation and one can see that the differential transform of Eq. (3.1) can be evaluated by using the above operations as follows:

$$Y(k+1)=\frac{1}{(k+1)^2}\left[\begin{array}{l}\sum_{k_n=0}^{k}\sum_{k_{n-1}=0}^{k_n}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}Y(k_1)Y(k_2-k_1)\ldots Y(k_n-k_{n-1})\delta(k-k_n-1)\\ +\sum_{k_n=0}^{k}\sum_{k_{n-1}=0}^{k_n}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}Y(k_1)Y(k_2-k_1)\ldots Y(k_n-k_{n-1})\delta(k-k_{n+1}-1)\end{array}\right] \tag{3.3}$$

From the boundary conditions given in Eq. (3.2) at $x = 0$, the boundary conditions are transformed as follows:

$$Y(0)=\alpha, \ Y(1)=0. \tag{3.4}$$

When $n = 1$, Eqs.(3.1) and (3.3) become

$$y''(x)+\frac{1}{x}y'(x)-y(x)-y^2(x)=0, \tag{3.5}$$

and

$$Y(k+1)=\frac{1}{(k+1)^2}\left(\sum_{l=0}^{k}\delta(l-1)Y(k-l)+\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}Y(k_1)Y(k_2-k_1)\delta(k-k_2-1)\right). \tag{3.6}$$

Using Eqs. (3.4) and (3.6) and by taking $N = 17$, we get the following series solution

$$\begin{aligned} y(x) = {} & \alpha+\left(\frac{\alpha}{4}+\frac{\alpha^2}{4}\right)x^2+\left(\frac{\alpha}{64}+\frac{3\alpha^2}{64}+\frac{\alpha^3}{32}\right)x^4+\left(\frac{\alpha}{2304}+\frac{\alpha^2}{256}+\frac{\alpha^3}{144}+\frac{\alpha^4}{288}\right)x^6\\ & +\left(\frac{\alpha}{147456}+\frac{29\alpha^2}{147456}+\frac{53\alpha^3}{73728}+\frac{65\alpha^4}{73728}+\frac{13\alpha^5}{36864}\right)x^8\\ & +\left(\frac{\alpha}{14745600}+\frac{11\alpha^2}{1638400}+\frac{7\alpha^3}{147456}+\frac{161\alpha^4}{1474530}+\frac{21\alpha^5}{204800}+\frac{7\alpha^6}{204800}\right)x^{10}\\ & +\left(\begin{array}{l}\frac{\alpha}{2123366400}+\frac{13\alpha^2}{78643200}+\frac{2399\alpha^3}{1061683200}+\frac{469\alpha^4}{53084160}+\frac{2611\alpha^5}{176947200}\\ +\frac{5957\alpha^6}{530841600}+\frac{851\alpha^7}{265420800}\end{array}\right)x^{12}\\ & +\left(\begin{array}{l}\frac{\alpha}{416179814400}+\frac{17\alpha^2}{5549064192}+\frac{7\alpha^3}{84934656}+\frac{2233\alpha^4}{4246732800}+\frac{497\alpha^5}{353894400}\\ +\frac{217\alpha^6}{117964800}+\frac{13\alpha^7}{110559200}+\frac{13\alpha^8}{44236800}\end{array}\right)x^{14}\\ & +\left(\begin{array}{l}\frac{\alpha}{106542032486400}+\frac{523\alpha^2}{11838003609600}+\frac{25441\alpha^3}{10654203248640}+\frac{3331\alpha^4}{135895449600}+\frac{1801\alpha^5}{18119393280}\\ +\frac{10913\alpha^6}{54358179840}+\frac{29377\alpha^7}{135895449600}+\frac{1199\alpha^8}{10066329600}+\frac{1199\alpha^9}{45298483200}\end{array}\right)x^{16}\\ & +O(x^{18}). \end{aligned} \tag{3.7}$$

The series solution (3.7) is used to obtain various Padé approximants [2/2], [4/4], [6/6] and [8/8]. Roots of the Padé approximants to the FP monopole α were obtained. The roots were obtained by using the limit of the Padé approximant $[m/m]$ as $x \to \infty$ is a^8/b^8, where a_m and b_m are the leading coefficients of the numerator and denominator, respectively. For $n = 1$, that is related to the FP equation, the table can be found in [32], hence we just summarize all results for $n \geq 1$.

Table 2: Roots of the Padé approximants monopole [32] α,n = 1

Degree	Roots Ref. [18]	DTM (N = 17)
[2/2]	-1.5	-1.5
[4/4]	-2.50746	-2.50746
[6/6]	-2.390278	-2.390278
[8/8]	-2.392214	-2.392214

Table 3: Roots of the Padé approximants monopole α,n = 2

Degree	Roots Ref. [18]	DTM (N = 17)
[2/2]	-2.0	-2.0
[4/4]	-2.0	-2.0
[6/6]	-2.0	-2.0
[8/8]	-2.0	-2.0

Table 4: Roots of the Padé approximants monopole α,n = 3

Degree	Roots Ref. [18]	DTM (N = 17)
[2/2]	0.0	0.0
[4/4]	-2.197575908	-2.217927737
[6/6]	-1.918424398	-1.918424398
[8/8]	-1.848997181	-1.848997181

For n = 2, we discard the complex roots and other real roots that do not meet physical grounds. Continuing in the same manner, we obtained Tables 2-5. Table 5 shows that the roots of the monopole α converge to -1 as n increases. The results in Tables 2-5 are in good agreement with the results obtained in [18] using Adomian decomposition method.

Table 5: Roots of the Padé approximants [8/8] monopole α for several values of n

N	[8/8] Roots Ref. [18]	DTM (N = 17)	N	[8/8] Roots Ref. [18]	DTM (N = 17)
1	-2.392213866	-2.392213866	6	-1.000861533	-1.149086031
2	-2.0	-2.0	7	-1.000708285	-1.119364959
3	-1.848997181	-1.848997181	8	-1.000601615	-1.099345401
4	-1.286025892	-1.286025892	$n \to \infty$	-1.00	-1.00
5	-1.001101141	-1.197243010			

Table 6: Roots of the Padé approximants [8/8] monopole α for several values of r

r	[8/8] Roots Ref. [18]	DTM (N = 17)	r	[8/8] Roots Ref. [18]	DTM (N = 17)	r	[8/8] Roots Ref. [18]	DTM (N = 17)
1	-2.39221386	-2.39221386	8	-13.7032879	-13.7032879	15	-8.065872112	-8.065872113
2	-4.71692095	-4.18730004	9	-13.1254951	-13.1254951	16	-6.054278105	-6.054278106
3	-8.66481910	-8.66481910	10	-13.1817438	-13.1817438	17	-4.225824659	-4.225824659
4	-16.5044406	-16.5044406	11	-3.17173988	-13.7173988	18	-3.445689828	-3.445689828
5	-47.0521025	-47.0521025	12	-14.7846216	-14.7846216	19	-3.009982510	-3.009982510
6	-20.9828920	-20.9828920	13	-16.7179134	-16.71791340	20	-2.409546162	-2.734318768
7	-15.4011414	-15.5011414	14	-20.9142364	-20.91425641	$n \to \infty$	-2.392213866	-2.392213866

Second Generalization of the FP Equation: In this section we will discuss a second generalized variant of the FP equation

$$y''(x)+\frac{r}{x}y'(x)-y(x)-y^2(x)=0. \tag{3.8}$$

The boundary conditions are

$$y(0)=\alpha, \quad y'(0)=0, \quad y(\infty)=0. \tag{3.9}$$

For r = 1, Eq. (3.8) reduces to the standard FP equation. The general series solution for Eq. (3.8) is to be constructed for all possible values of $r \geq 1$.

Taking differential transform of (3.8) and using the fundamental operations of differential transform method, we obtain the following recurence relation:

$$Y(k+1)=\frac{1}{(k+1)(k+r)}\left(\sum_{l=0}^{k}\delta(l-1)Y(k-l)+\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}Y(k_1)Y(k_2-k_1)\delta(k-k_2-1)\right) \tag{3.10}$$

From the boundary conditions given in Eq. (3.9) at x = 0, the boundary conditions are transformed as follows:

$$Y(0) = \alpha,\ Y(1) = 0. \tag{3.11}$$

Using Eqs. (3.10) and (3.11) and by taking $N = 17$, the following series solution is obtained:

$$\begin{aligned} y(x) &= \alpha + \frac{\alpha(\alpha+1)}{2(r+1)}x^2 + \frac{\alpha(\alpha+1)(2\alpha+1)}{8(r+1)(r+3)}x^4 \\ &+ \frac{\alpha(\alpha+1)\left[2(3r+5)\alpha(\alpha+1)+(r+1)\right]}{48(r+1)^2(r+3)(r+5)}x^6 \\ &+ \frac{\alpha(\alpha+1)(2\alpha+1)\left[4(3r+10)\alpha(\alpha+1)+(r+1)\right]}{384(r+1)^2(r+3)(r+5)(r+7)}x^8 + \cdots, \end{aligned} \tag{3.12}$$

Which is exactly the same as in [18] by using the Adomian decomposition method.

The series solution (3.12) is used to obtain various Pad approximants [2/2], [4/4], [6/6] and [8/8]. Roots of the Padé approximants to the FP monopole α were obtained. The roots were obtained by using the limit of the Padé approximant $[m/m]$ as $x \to \infty$ is a_8/b_8, where a_m and b_m are the leading coefficients of the numerator and denominator, respectively. The following table summarizes the results for values of $r = 1,2,...,20$.

Table 6 shows that the roots exhibit a fast decrease reaching a minimum -47.05210256 at $r = 5$, then followed by a fast increase to converge to the starting value -2.392213866 as shown above. Our results are in good agreement with the results obtained in [18] using Adomian decomposition method.

CONCLUSION

The DTM-Padé technique is an efficient method for calculating approximate solutions for FP equation and its variants. Due to the existence of singular point at $x = 0$, the difficulty in type of equation can be overcomed using this technique. The results show that this technique increases efficiently the accuracy of approximate solution and leads to convergence with a rate faster than using DTM alone. The DTM-Padé technique can be used in solving other types of ordinary differential equations with singular coefficients. The use of Mathematica facilatates the calculations of the DTM-Padé technique.

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