Series Solution for Unsteady Gas Equation via MLDM-Pade´ Technique

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Abstract: In this paper, we propose a new approach to solve the unsteady gas equation. We apply the Modified Laplace Decomposition Method (MLDM) coupled with Pade´ approximation to compute a series solution of unsteady flow of gas through a porous medium. The proposed iterative scheme finds the solution without any discretization, linearization or restrictive assumptions. The nonlinear terms can be easily handled by the use of Adomian polynomials. The diagonal Pade´ approximants are used to analyze the essential behavior of y(x) and to determine the initial slope y'(0). The proposed scheme avoids the complexity provided by using perturbation and other iterative techniques.

Keywords: Modified Laplace Decomposition Method (MLDM) . unsteady flow of gas . Pade´ approximants . porous medium

INTRODUCTION

We consider the flow of gas through a semi-infinite porous medium [1-3] initially filled with gas at a uniform pressure $p_0 \geq 0$, at time t = 0, the pressure at the outflow face is suddenly reduced from p_0 to $p_1 \geq 0$ ($p_1 = 0$ is the case of diffusion into a vacuum) and is, thereafter, maintained at this lower pressure. The unsteady isothermal flow of gas is described by a nonlinear partial differential equation. The nonlinear partial differential equation that describes the unsteady flow of gas through a semi-infinite porous medium has been derived by Muskat [4] in the form

$$\nabla^2(P^2) = (2\Phi\mu/k)\frac{\partial P}{\partial t} \qquad (1)$$

where P is the pressure within porous medium, Φ the porosity, μ the viscosity, k the permeability and t the time. New variables were introduced by Kidder [2] and Davis [5] to transform the nonlinear partial differential equation (1) to the nonlinear ordinary differential equation. The nonlinear ordinary differential equation due to Kidder [2] given by (unsteady gas equation)

$$y''(x) + \frac{2x}{\sqrt{1-\alpha y}} y'(x) = 0, \qquad 0 < \alpha < 1 \qquad (2)$$

The unsteady flow of gas through a porous medium has been investigated by number of authors and several techniques including decomposition, variational

iteration using He's polynomials and Homotopy perturbation methods have been used for the analysis of this problem, [6-8]. Khuri [9] proposed a Laplace Decomposition Method (LDM) for the approximate solution of a class of nonlinear ordinary differential equations. In 2006, Agadjanov [10] developed this method for the solution of Duffing equation. The Laplace decomposition method (LDM) was proved to be compatible with the versatile nature of the physical problems and was applied to a wide class of functional equations; [11-16]. Recently a reliable modification of the Laplace decomposition algorithm has been done by Yasir [17]. The modified Laplace decomposition method is much easier compared with the Adomian decomposition method where huge complexities are involved. The fact that MLDM solves nonlinear problems without using any restricted linear highest-ordered differential operator. It can be considered as a clear advantage of this technique over the Adomian decomposition method. It is worth mentioning that the MLDM is applied without any discretization, restrictive assumption or perturbation and is free from round off errors. The objectives of this paper are three-fold: first, to introduce the new analytical method for finding the analytical solution of unsteady flow of gas through porous medium which primarily lie in its ability to avoid the unnecessary calculations of other iteration methods; second, our aim is to compare the results with solutions to the existing ones [6-8]; and third, to extend our previous approach proposed in [17] on semi-infinite domain. To make the work more concise and to get a

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better understanding of the solution behavior, the series solutions are replaced by the diagonal Pade´ approximants [19-29].

MODIFIED LAPLACE
DECOMPOSITION METHOD (MLDM)

In order to elucidate the solution procedure of the modified Laplace decomposition method, we consider the following general form of second order nonlinear ordinary differential equation with initial conditions is given by

$$f'' + b_1(x)f' + b_2(x)f = g(y) \qquad (3)$$

$$f(0) = \alpha, \ f(0) = \beta \qquad (4)$$

According to Laplace decomposition method [9, 10], we apply Laplace transform (denoted throughout this paper by L) on both sides of Eq. (3):

$$s^2 L[\ f\ ] - s^1\alpha - \beta + L[b_1(x)f'] + L[b_2(x)f] = L[g(y)] \qquad (5)$$

Using the differentiation property of Laplace transform, we have

$$L[f] = \frac{\alpha}{s} + \frac{\beta}{s^2} + \frac{1}{s^2}L[g(y)] - \frac{1}{s^2}L[b_1(x)f' + b_2(x)f] \qquad (6)$$

The Laplace decomposition method [9, 10] admits a solution in the form

$$f = \sum_{m=0}^{\infty} f_m \qquad (7)$$

The nonlinear term is decomposed as

$$g(y) = \sum_{m=0}^{\infty} A_m \qquad (8)$$

where A_m are Adomian polynomials of $g_0, g_1, g_2, g_3, ...,$ g_n and it can be calculated by the following formula

$$A_m = \frac{1}{n!}\frac{d^m}{d\lambda^m}\left[N\left(\sum_{m=0}^{\infty}\lambda^i y_i\right)\right]_{\lambda=0}, \quad m = 0,1,2,3........ \qquad (9)$$

Using Eq. (7) and Eq. (8) in Eq. (6) we get

$$L\left[\sum_{m=0}^{\infty} f_m\right] = \alpha + \frac{\beta}{s^2} + \frac{1}{s^2}L\left[\sum_{m=0}^{\infty} A_m\right]$$
$$\qquad \qquad - \frac{1}{s^2}L\left[b_1(x)\sum_{m=0}^{\infty}f_m' + b_2(x)\sum_{m=0}^{\infty}f_m\right] \qquad (10)$$

Matching both sides of Eq. (10), we have the following relation;

$$L[\ f_0\ ] = \alpha + \frac{\beta}{s^2} \qquad (11)$$

$$L[\ f_1\ ] = \frac{1}{s^2}L[A_0] - \frac{1}{s^2}L[b_1(x)f_0' + b_2(x)f_0]$$
$$L[\ f_2\ ] = \frac{1}{s^2}L[A_1] - \frac{1}{s^2}L[b_1(x)f_1' + b_2(x)f_1] \qquad (12)$$

In general the recursive relation is given by

$$L[\ f_{m+1}\ ] = \frac{1}{s^2}L[A_n] - \frac{1}{s^2}L[b_1(x)f_m' + b_2(x)f_m], \ m \geq 0$$

aking the inverse Laplace transform from both sides of Eq. (11-13), one obtains

$$f_0(x) = H(x) \qquad (14)$$

$$f_{m+1}(x) = L^{-1}\left[\begin{array}{c} \frac{1}{s^2}L[A_m] \\ -\frac{1}{s^2}L[b_1(x)f_m' + b_2(x)f_m] \end{array}\right], \quad m \geq 0 \qquad (15)$$

where H(x) represents the term arising from source term and prescribe initial condition. The modified Laplace decomposition method [17] suggests that the function H(x) defined above in (14) be decomposed into two parts, namely $H_0(x)$ and $H_1(x)$. Such that

$$H(x) = H_0(x) + H_1(x) \qquad (16)$$

he initial solution is important and the choice of Eq. (14) as the initial solution always leads to noise oscillation during the iteration procedure. Instead of the iteration procedure, Eqs. (14) and (15), we suggest the following modification

$$f_0(x) = H_0(x)$$

$$f_1(x) = H_1(x) + L^{-1}\left[\begin{array}{c} \frac{1}{s^2}L[A_0] \\ -\frac{1}{s^2}L[b_1(x)f_0' + b_2(x)f_0] \end{array}\right]$$

$$f_{m+1}(x) = L^{-1}\left[\begin{array}{c} \frac{1}{s^2}L[A_m] \\ -\frac{1}{s^2}L[b_1(x)f_m' + b_3(x)f_m] \end{array}\right], \ m \geq 1$$

The solution through the modified Laplace decomposition method highly depends upon the choice of $H_0(x)$ and $H_1(x)$.

NUMERICAL APPLICATION

In this section, we apply the Modified Laplace Decomposition Method (MLDM) for finding the analytical solution of the unsteady flow of gas through a porous medium.

$$y''(x) + \frac{2x}{\sqrt{1-\alpha y}} y'(x) = 0, \quad 0 < \alpha < 1 \quad (18)$$

With conditions

$$y(0) = 1, \quad \lim_{x \to \infty} y(x) = 0$$

where $y'(0) = B < 0$, will be examined in this work. By applying the aforesaid method subject to the initial conditions, we have

$$y(s) = \frac{1}{s} + \frac{B}{s^2} + \frac{1}{s^2} L\left[x^{-1/2} y^{3/2} \right] \quad (19)$$

The inverse of Laplace transform implies that

$$y(x) = 1 + Bx + L^{-1}\left[\frac{1}{s^2} L\left(x^{-1/2} y^{3/2} \right) \right] \quad (20)$$

Following the technique, if we assume an infinite series solution of the form (7) we obtain

$$\sum_{m=0}^{\infty} y_m(x) = 1 + Bx + L^{-1}\left[\frac{1}{s^2} L\left(x^{-1/2} \sum_{m=0}^{\infty} A_m(x) \right) \right] \quad (21)$$

Through the modified Laplace decomposition method [17] the function $H(\eta)$ can be written as

$$H(x) = 1 + Bx = H_0(x) + H_1(x) \quad (22)$$

By this consideration, we first set modified recursive relations in the form

$$y_0(x) = 1$$
$$y_1(x) = Bx + L^{-1}\left[\frac{1}{s^2} L\left(x^{-1/2} A_0 \right) \right] \quad (23)$$
$$y_{m+1}(x) = L^{-1}\left[\frac{1}{s^2} L\left(x^{-1/2} A_m(x) \right) \right], \ m \geq 1$$

In the above equation $A_m(x)$ are the Adomian polynomials [18]. For convenience, we list below the first few Adomian polynomials $A_m(x)$

$$A_0 = (1 - \alpha y_0)^{-1/2} y_0'$$

$$A_1 = (1 - \alpha y_0)^{-1/2} y_1' + \frac{\alpha}{2}(1 - \alpha y_0)^{-3/2} y_0' y_1$$

$$A_2 = (1 - \alpha y_0)^{-1/2} y_2' + \frac{\alpha}{2}(1 - \alpha y_0)^{-3/2} y_1' y_1$$
$$+ \frac{\alpha}{2}(1 - \alpha y_0)^{-3/2} y_0' y_2 + \frac{3}{8}\alpha^2(1 - \alpha y_0)^{-5/2} y_0' y_1^2,$$

$$A_4 = (1 - \alpha y_0)^{-1/2} y_3' + \frac{\alpha}{2}(1 - \alpha y_0)^{-3/2} y_2' y_1 \quad (24)$$
$$+ \frac{\alpha}{2}(1 - \alpha y_0)^{-3/2} y_1' y_2 + \frac{\alpha}{2}(1 - \alpha y_0)^{-3/2} y_0' y_3$$
$$+ \frac{3}{8}\alpha^2(1 - \alpha y_0)^{-5/2} y_1' y_1^2 + \frac{3}{4}\alpha^2(1 - \alpha y_0)^{-5/2} y_0' y_1 y_2$$
$$+ \frac{5}{16}\alpha^3(1 - \alpha y_0)^{-7/2} y_0' y_1^3$$

Using above polynomial, we calculate other components of $y(x)$

$$y_1(x) = Bx$$
$$y_2(x) = -\frac{B}{3\sqrt{1-\alpha}} x^3$$
$$y_3(x) - \frac{\alpha B^2}{12(1-\alpha)^{3/2}} x^4 + \frac{B}{10(1-\alpha)} x^5 \quad (25)$$
$$y_4(x) = -\frac{3\alpha^2 B^3}{80(1-\alpha)^{5/2}} x^5 + \frac{\alpha B^2}{15(1-\alpha)^2} x^6 + O(x^7)$$
$$y_5(x) = -\frac{\alpha^3 B^4}{48(1-\alpha)^{7/2}} x^6 + O(x^7)$$
$$\vdots$$

The series solution is given by

$$y(x) = 1 + Bx - \frac{B}{3\sqrt{1-\alpha}} x^3 - \frac{\alpha B^2}{12(1-\alpha)^{3/2}} x^4$$
$$+ \left(\frac{B}{10(1-\alpha)} - \frac{3\alpha^2 B^3}{80(1-\alpha)^{5/2}} \right) x^5 \quad (26)$$
$$+ \left(\frac{\alpha B^2}{15(1-\alpha)^2} - \frac{\alpha^3 B^4}{48(1-\alpha)^{7/2}} \right) x^6 + \cdots$$

PADÉ APPROXIMANTS

Padé approximants constitute the best approximation of a function by a rational function of a given order. Padé approximants often provide better approximation of a function than its Taylor series and they may still work in cases in which the Taylor series does not converge. For these reasons, Padé approximants are used extensively in computer calculations and it is now well known that these approximants have the advantage to manipulate polynomial approximation into the rational functions of polynomials. Through such manipulation, we can gain

Table 1: Exhibits the initial slopes B = y′(0) for various values of α

α	B[2/2] = y′(0)	B[3/3] = y′(0)
0.1	-3.556558821	-1.957208953
0.2	-2.441894334	-1.786475516
0.3	-1.928338405	-1.478270843
0.4	-1.606856838	-1.231801809
0.5	-1.373178096	-1.025529704
0.6	-1.185519607	-0.8400346085
0.7	-1.021411309	-0.6612047893
0.8	-0.8633400217	-0.4776697286
0.9	-0.6844600642	-0.2772628386

Table 2: Exhibits the values of y(x) for α = 0.5 for x = 0.1 to 1.0

x	y kidder	y[2/2]	y[3/3]
0.1	0.8816588283	0.8633060641	0.8979167028
0.2	0.7663076781	0.7301262261	0.7985228199
0.3	0.6565379995	0.6033054140	0.7041129703
0.4	0.5544024032	0.4848898717	0.6165037901
0.5	0.4613650295	0.3761603869	0.5370533796
0.6	0.3783109315	0.2777311628	0.4665625669
0.7	0.3055976546	0.1896843371	0.4062426033
0.8	0.2431325473	0.1117105165	0.3560801699
0.9	0.1904623681	0.04323673236	0.3179966614
1.0	0.1587689826	0.01646750847	0.2900255005

more information about the mathematical behavior of the solution. In addition, power series are not useful for large values of a variable, say $\eta \rightarrow \infty$, which can be attributed to the possibility of the radius of convergence not being sufficiently large to contain the boundaries of the domain. To provide an effective tool that can handle boundary value problems on an infinite or semi-infinite domain, it is therefore essential to combine the series solution, which is obtained by the iteration method or any other series solution method, with the Padé approximants.

The diagonal Pade′ approximants [19-29] can be applied to investigate the mathematical behavior of the solution y(x) to determine the initial slope y′(0).

The above tables clearly reveal that present solution method namely MLDM shows excellent agreement with the existing solutions in literature [6-8]. This analysis shows that MLDM suits for Boundary layer flow problems.

CONCLUSION

This paper presents a Modified Laplace Decomposition method, the MLDM, that can be employed to solve Unsteady gas equation. The proposed algorithm's ability to solve nonlinear problems without the use of restricted linear highest-ordered differential operator is evidence of its clear advantage over the Adomian decomposition method. It may be concluded that the MLDM is very powerful and

efficient in finding the analytical solutions for a wide class of differential equations. The method gives more realistic series solutions that converge very rapidly in physical problems. Comparison of the present solution is made with the existing solution [6-8]. An excellent agreement between the present and existing solutions is achieved.

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