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Numerical Solutions of the Linear Volterra Integro-differential Equations: Homotopy Perturbation Method and Finite Difference Method

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Abstract: In the research, special type of linear volterra integro-differential equations is considered. This paper compares the Homotopy perturbation method (HPM) with finite difference method for solving these equations. HPM is an analytical procedure for finding the solutions of problems which is based on the constructing a Homotopy with an imbedding parameter p that is considered as a small parameter. The finite difference method, based upon Simpson rule and Trapezoidal rule, transforms the volterra integro-differential equation into a matrix equation. The results of applying these methods to the linear integro-differential equation show the simplicity and efficiency of these methods.

Key words: Volterra integro-differential equations . homotopy perturbation method . finite difference method

INTRODUCTION

Mathematical modeling of real-life problems usually results in functional equations, e.g. partial differential equations, integral and integro-differential equation, stochastic equations and others. Many mathematical formulations of physical phenomena contain integro-differential equations, these equations arise in fluid dynamics, biological models and chemical kinetics. Integro-differential equations are usually difficult to solve analytically so it is required to obtain an efficient approximate solution. Several numerical methods for approximating the Fredholm or volterra integro-differential equations are known. Single-term Walsh series method for volterra integro-differential equations has been proposed by Sepehrian and Razzaghi [1]. In [2], Brunner applied a collocation-type method to Volterra-Hammerstein integral equation as well as integro-differential equations. Compact finite difference method has been used for integro-differential equations by Zhao and Corless [3]. For methods using a quadrature rule, degenerate kernels, interpolation or extrapolation [4-7]. In Refs [8-10], Taylor series, Chebyshev collocation and Wavelet-Galerkin methods are used for solving such problems. In recent years, the application of homotopy perturbation method (HPM) [11-13] in nonlinear problems has been developed by scientists and engineers, because this method deforms the difficult problem under study into a simple problem which is easy to solve. Most perturbation methods assume a small parameter exists, but most nonlinear problems have no small parameter at all. Many new methods, such as the variational method [14-16], variational iterations method [17-22], various modified Lindstedt-Poincare methods [23-26] and others [27, 28] are proposed to eliminate the shortcoming arising in the small parameter assumption. A review of recently developed nonlinear analysis methods can be found in [29]. Recently, the applications of homotopy perturbation theory have appeared in the works of many scientist [30-35]; it has become a powerful mathematical tool [36, 37]. In this paper, we propose the use of HPM to solve special type of linear volterra integro-differential equations of the form:

$$\begin{cases} y'(x)+\mu(x)y(x)=f(x)+\lambda\int_a^x k(x,t)y(t)dt, a\le x\le b \\ y(a)=y_0 \end{cases} \quad (1)$$

And comparisons are made between finite difference method and homotopy perturbation method. Where the functions $f(x)$, $\mu(x)$ and the kernel $k(x, t)$ are known and $y(x)$ is the solution to be determined.

HOMOTOPY PERTURBATION METHOD

To illustrate the basic ideas of the homotopy perturbation method, we consider the following nonlinear differential equation:

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$$A(y) - f(r) = 0, \qquad r \in \Omega, (2) \tag{2}$$

with the boundary conditions

$$B\left(y, \frac{\partial y}{\partial n}\right) = 0, \qquad r \in \Omega \tag{3}$$

where A is a general differential operator, B is a boundary operator, f (r) is a known analytical function and Γ is the boundary of the domain Ω. Generally speaking, the operator A can be divided into two parts which are L and N where L is linear, but N is nonlinear. Therefore equation (2) can therefore be rewritten as follows:

$$L(y) + N(y) - f(r) = 0 \tag{4}$$

By the homotopy perturbation technique, we construct a homotopy v (r, p): $\Omega \times [0,1] \to R$ which satisfies:

$$H(v,p) = (1-p)\left[L(v) - L(y_0)\right] + p\left[A(v) - f(r)\right] = 0, \quad p \in [0,1], \quad r \in \Omega$$

where $p \in [0,1]$ is an embedding parameter and y_0 is an initial approximation of equation (2).Obviously, from these definitions we will have:

$$H(v,0) = L(v) - L(y_0) = 0$$

$$H(v,1) = A(v) - f(r) = 0$$

The changing process of p from zero to unity is just that of v (r, p) from y_0 (r) to y (r). In topology, this is called deformation and L (v) – L (y_0) and A (v)-f (r) are called homotopy. According to the HPM, we can first use the embedding parameter p as a "small parameter" and assume that the solution of (5) can be written as a power series in p:

$$v = v_0 + pv_1 + p^2 v_2 + \ldots$$

Setting p = 1, results in the approximate solution of (2):

$$y = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \ldots$$

In order to solve the equation (1) using HPM, we construct the following homotopy:

$$H(v,p) = v' - y_0' + py_0' - p\left[-\mu(x)v(x) + \lambda\int_a^x k(x,t)v(t)dx + f(x)\right] = 0$$

Substituting (6) in (7) and equating the coefficients of like powers of p, yield

$$p^0: \ v_0' - y_0' = 0 \tag{8}$$

$$p^1: \ v_1' + y_0' + \mu(x)v_0(x) - \lambda\int_a^x k(x,t)v_0(t)dt - f(x) = 0 \tag{9}$$

$$p^n: \ v_n' + \mu(x)v_{n-1}(x) - \lambda\int_a^x k(x,t)v_{n-1}(t)dt = 0, \quad n \geq 2 \tag{10}$$

Then starting with an initial approximation y_0 and solving the above equations, we can identify υ_n for n = 1, 2,… and therefore we obtain the n-th approximation of the exact solution as $y_n = v_0 + v_1 + \ldots + v_n$.
Note: In this section, we consider y (a) = y0 = β

FINITE DIFFERENCE METHOD

In this section, we consider volterra integro-differential equation in (1) and approximate to solution by numerical integration and numerical differentiation.

We will subdivide the interval of integration (a,x) into N = 2M equal subinterval of with

$$h = \frac{x_N - a}{N}, \ N \geq 1$$

where x_N is the end point we choose for x. We shall set t_0 = a and $t_j = t_0 + jh$. Since we will be using either t or x as the independent variable for the solution y. We will call $x_0 = t_0 = a$, $x = x_N = t_N$ and $x_i = a + ih = t_i$.. We will refer to the value of the functions f(x) and p (x) at x_i as $f(x_i) = f_i$ and $\mu(x_i) = \mu_i$, the value of kernel k (x, t) at (x_i, t_j) as k (x_i. t_j) = k_{ij} and the approximate value of the solution y (x) at x_i or t_i as y (t_i) = y (x_i) = y_i and y′ (x_i) = y_i'. k (x_i. t_j) Clearly vanishes for $t_j > x_i$ as the integration ends at $t_j \leq x_i$. Note that the particular value y (x_0) = y_0 according to (1).So if we use the trapezoidal rule and Simpson rule with n subinterval to approximate the integral in the volterra integro-differential equation (1), we have

$$\begin{aligned}
&\int_a^{x_1} k(x,t)y(t)dt \approx h\left[\frac{1}{2}k(x_1,t_0)y(t_0) + \frac{1}{2}k(x_1,t_1)y(t_1)\right] \ \text{(T. rule)} \\
&\int_a^{x_2} k(x,t)y(t)dt \approx \frac{h}{3}\left[k(x_2,t_0)y(t_0) + 4k(x_2,t_1)y(t_1) + k(x_2,t_2)y(t_2)\right] \ \text{(S. rule)} \\
&\int_a^{x_3} k(x,t)y(t)dt \approx h\left[\frac{1}{2}k(x_3,t_0)y(t_0) + k(x_3,t_1)y(t_1) + k(x_3,t_2)y(t_2) + \frac{1}{2}k(x_3,t_3)y(t_3)\right] \ \text{(T. rule)} \\
&\int_a^{x_4} k(x,t)y(t)dt \approx \frac{h}{3}\left[k(x_4,t_0)y(t_0) + 4k(x_4,t_1)y(t_1) + 2k(x_4,t_2)y(t_2) + 4k(x_4,t_3)y(t_3) + k(x_4,t_4)y(t_4)\right] \ \text{(S. rule)}
\end{aligned} \tag{11}$$

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$$\int_a^{x_{N-1}} k(x,t)y(t)\,dt \approx h\left[\frac{1}{2}k(x_{N-1},t_0)y(t_0)+k(x_{N-1},t_1)y(t_1)+\dots+k(x_{N-1},t_{N-2})y(t_{N-2})+\frac{1}{2}k(x_{N-1},t_{N-1})y(t_{N-1})\right] \quad \text{(T. rule)}$$

$$\int_a^{x_N} k(x,t)y(t)\,dt \approx \frac{h}{3}\left[k(x_N,t_0)y(t_0)+4k(x_N,t_1)y(t_1)+2k(x_N,t_2)y(t_2)+\dots+4k(x_N,t_{N-1})y(t_{N-1})+k(x_N,t_N)y(t_N)\right] \quad \text{(S. rule)}$$

Also the integro-differential equation (1) is approximated by (compact form)

$$\begin{aligned}
&y'_1+\mu_1 y_1=f_1+\lambda h\left[\frac{1}{2}k_{10}y_0+\frac{1}{2}k_{11}y_1\right]\\
&y'_2+\mu_2 y_2=f_2+\frac{\lambda h}{3}\left[k_{20}y_0+4k_{21}y_1+k_{22}y_2\right]\\
&y'_3+\mu_3 y_3=f_3+\lambda h\left[\frac{1}{2}k_{30}y_0+k_{31}y_1+k_{32}y_2+\frac{1}{2}k_{33}y_3\right]\\
&y'_4+\mu_4 y_4=f_4+\frac{\lambda h}{3}\left[k_{40}y_0+4k_{41}y_1+2k_{42}y_2+4k_{43}y_3+k_{44}y_4\right]\\
&\quad\vdots\\
&y'_{N-1}+\mu_{N-1}y_{N-1}=f_{N-1}+\lambda h\left[\frac{1}{2}k_{N-1,0}y_0+k_{N-1,1}y_1+\dots+k_{N-1,N-2}y_{N-2}+\frac{1}{2}k_{N-1,N-1}y_{N-1}\right]\\
&y'_N+\mu_N y_N=f_N+\frac{\lambda h}{3}\left[k_{N0}y_0+4k_{N1}y_1+2k_{N2}y_2+\dots+4k_{N,N-1}y_{N-1}+k_{NN}y_N\right]
\end{aligned} \tag{12}$$

Now we take advantage of finite differentiation to get

$$\begin{aligned}
&\frac{y_2-y_0}{2h}+\mu_1 y_1=f_1+\lambda h\left[\frac{1}{2}k_{10}y_0+\frac{1}{2}k_{11}y_1\right]\\
&\frac{y_3-y_1}{2h}+\mu_2 y_2=f_2+\frac{\lambda h}{3}\left[k_{20}y_0+4k_{21}y_1+k_{22}y_2\right]\\
&\frac{y_4-y_2}{2h}+\mu_3 y_3=f_3+\lambda h\left[\frac{1}{2}k_{30}y_0+k_{31}y_1+k_{32}y_2+\frac{1}{2}k_{33}y_3\right]\\
&\frac{y_5-y_3}{2h}+\mu_4 y_4=f_4+\frac{\lambda h}{3}\left[k_{40}y_0+4k_{41}y_1+2k_{42}y_2+4k_{43}y_3+k_{44}y_4\right]\\
&\quad\vdots\\
&\frac{y_N-y_{N-2}}{2h}+\mu_{N-1}y_{N-1}=f_{N-1}+\lambda h\left[\frac{1}{2}k_{N-1,0}y_0+k_{N-1,1}y_1+\dots+k_{N-1,N-2}y_{N-2}+\frac{1}{2}k_{N-1,N-1}y_{N-1}\right]\\
&\frac{3y_N-4y_{N-1}+y_{N-2}}{2h}+\mu_N y_N=f_N+\frac{\lambda h}{3}\left[k_{N0}y_0+4k_{N1}y_1+2k_{N2}y_2+\dots+4k_{N,N-1}y_{N-1}+k_{NN}y_N\right]
\end{aligned} \tag{13}$$

The system (13) consists of N equations and can be written in the following matrix form $KY = F$, where

$$K=\begin{pmatrix}
-\left(\lambda h^2k_{11}-2h\mu_1\right) & 1 & 0 & 0 & \dots & & & 0\\
-\left(\frac{8\lambda h^2}{3}k_{21}+1\right) & -\left(\frac{2\lambda h^2}{3}k_{22}-2h\mu_2\right) & 1 & 0 & 0 & \dots & & 0\\
-2\lambda h^2k_{31} & -\left(2\lambda h^2k_{32}+1\right) & -\left(h^2k_{33}-2h\mu_3\right) & 1 & 0 & 0 & \dots & 0\\
-\frac{8\lambda h^2}{3}k_{41} & -\frac{4\lambda h^2}{3}k_{42} & -\left(\frac{8\lambda h^2}{3}k_{43}+1\right) & -\left(\frac{2\lambda h^2}{3}k_{44}-2h\mu_2\right) & 1 & 0 & 0 \;\dots & 0\\
\vdots & & & & & & & \\
-2\lambda h^2k_{N-1,1} & \dots\; -2\lambda h^2k_{N-1,N-3} & -\left(2\lambda h^2k_{N-1,N-2}+1\right) & -\left(\lambda h^2k_{N-1,N-1}-2h\mu_{N-1}\right) & 1 & & & \\
-\frac{8\lambda h^2}{3}k_{N1} & \dots\; -\frac{8\lambda h^2}{3}k_{N,N-3} & -\left(\frac{4\lambda h^2}{3}k_{N,N-2}-1\right) & -\left(\frac{8\lambda h^2}{3}k_{N,N-1}+4\right) & -\left(\frac{2\lambda h^2}{3}k_{NN}-3-2h\mu_N\right) & & &
\end{pmatrix}$$

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ . \\ . \\ . \\ y_N \end{pmatrix}$$

and

$$F = \begin{pmatrix} 2hf_1 + \left(\lambda h^2 k_{10} + 1\right) y_0 \\ 2hf_2 + \frac{2\lambda h^2}{3} k_{20} y_0 \\ 2hf_3 + \lambda h^2 k_{30} y_0 \\ 2hf_2 + \frac{2\lambda h^2}{3} k_{40} y_0 \\ . \\ . \\ . \\ 2hf_{N-1} + \lambda h^2 k_{N-1,0} y_0 \\ 2hf_N + \frac{2\lambda h^2}{3} k_{N0} y_0 \end{pmatrix}$$

ILLUSTRATIVE EXAMPLES

Now we apply the methods presented to solve the following examples:

Example 1: Consider the Volterra integro-differential equation:

$$\begin{cases} y'(x) + y(x) = \left(x^2 + 2x + 1\right) e^{-x} + 5x^2 + 8 - \int_0^x t y(t)\, dt, 0 \le x \le 1 \\ y(0) = 10 \end{cases} \quad (14)$$

which has the exact solution y (x) = $10\text{-}xe^{-x}$. The numerical results are represented in Table 1.

To apply the homotopy perturbation method to this equation, we consider y_0 (x) = 10 as initial approximation of the exact solution and regarding (8) we start with v_0 (x) = y_0 (x). Since v_0 (a) = β and y = $v_0+v_1+v_2+\ldots$ we can set $v_n(a)$ = 0, ($n\ge1$) as initial conditions for equations (9) and (10).

Example 2: As the second example consider the Volterra integro-differential equation: [1]

$$\begin{cases} y'(x) + y(x) = 1 + 2x + \int_0^x x(1+2x) e^{t(x-t)} y(t)\, dt, \ 0 \le x \le 1 \\ y(0) = 1 \end{cases} \quad (15)$$

Table 1: Numerical results for example 1

x_i	Finite difference method with N = 14	HPM with N = 4	Exact solutions
0.0714	9.933780913	9.933495	9.933495516
0.1428	9.876458584	9.876160	9.876160300
0.2142	9.827589590	9.827049	9.827046197
0.2857	9.785803615	9.785307	9.785292202
0.3571	9.750832589	9.750163	9.750116951
0.4286	9.721466032	9.720923	9.720811832
0.5000	9.697552931	9.696975	9.696734670
0.5714	9.678042251	9.677777	9.677303930
0.6428	9.662857435	9.662846	9.661993413
0.7143	9.651100634	9.651781	9.650327386
0.7857	9.642739377	9.644241	9.641876128
0.8571	9.637018786	9.639963	9.636251847
0.9286	9.633929509	9.638737	9.633104936
1.0000	9.632846912	9.640444	9.632120559

Table 2: Numerical results for example 2

x_i	Finite difference method with N = 12	HPM with N = 2	Exact solutions
0.0833	1.00616079	1.0069712	1.006968613
0.1667	1.02810545	1.0282220	1.028167177
0.2500	1.06328807	1.0648317	1.064494459
0.3333	1.11700723	1.1187838	1.117519069
0.4167	1.18751936	1.1931987	1.189592856
0.5000	1.28254044	1.2926690	1.284025417
0.5833	1.40167958	1.4237286	1.405337908
0.6667	1.55629211	1.5955060	1.559623498
0.7500	1.74852146	1.8206228	1.755054657
0.8333	1.99578448	2.1164213	2.002596211
0.9167	2.30508217	2.5066179	2.317010501
1.0000	2.70473683	3.0235181	2.718281828

Table 3: Numerical results for Example 3

x_i	Finite difference method with N = 12	HPM with N = 10	Exact solutions
0.0833	0.9409611903	0.9232408643	0.9232408623
0.1667	0.8560967849	0.8582656556	0.8582656554
0.2500	0.8221926661	0.8032653310	0.8032653300
0.3333	0.7521848166	0.7567085596	0.7567085597
0.4167	0.7379172201	0.7172991016	0.7172991041
0.5000	0.6768034806	0.6839397209	0.6839397204
0.5833	0.6667601409	0.6557016132	0.6557016122
0.6667	0.6235899136	0.6317985685	0.6317985690
0.7500	0.6149784301	0.6115650814	0.6115650802
0.8333	0.5862745171	0.5944378021	0.5944378015
0.9167	0.5770459174	0.5799398782	0.5799398730
1.0000	0.5643959592	0.5676676511	0.5676676417

With the exact solution $y(x)=e^{x^2}$. Table 2 illustrate the numerical results.

If we want to solve this equation by mean of homotopy perturbation method, Considering $y_0(x) = 1$ and regarding (8), we start with $v_0(x) = y_0(x)$ Since $v_0(a) = \beta$ and $y = v_0+v_1+v_2+\ldots$ we can set $v_n(a) = 0$, $(n\geq 1)$ as initial conditions for equations (9) and (10).

Example 3: Consider [38]

$$\begin{cases} y'(x)+y(x)=\int_0^x e^{(t-x)}y(t)dt, & 0\leq x\leq 1 \\ y(0)=1 \end{cases} \tag{16}$$

with the exact solution $y(x) = e^{-x}\cosh x$ Results are shown in Table 3.

In order to solve this equation by mean of homotopy perturbation method, We assume $y_0(x) = 1$ and set $v_0(x) = y_0(x)$. We solve the above equations with $v_0(a) = \beta$ and $v_n(a) = 0$, $(n\geq 1)$ as initial conditions.

CONCLUSION

Integro-differential equations are usually difficult to solve analytically. In many cases, it is required to obtain the approximate solutions. In this work, we proposed the homotopy perturbation method for solving linear volterra integro-differential equations and comparisons were made with the finite difference method. Illustrative examples are included to demonstrate the validity and applicability of these techniques.

REFERENCES

1. Sepehrian, B. and M. Razzaghi, 2004. Single-term Walsh series method for the Volterra integro-differential equations. Engineering Analysis with Boundary Element, 28: 1315-1319.
2. Brunner, H., 1982. Implicitly linear collocation method for nonlinear Volterra equations. J Appl Numer Math, 9: 235-247.
3. Zhao, J. and R.M. Corless, 2006. Compact finite difference method has been used for integro-differential equations. Appl. Math. Comput., 177: 271-288.
4. CTH. Baker, 1980. Structure of recurrence relations in the study of stability in the numerical treatment of Volterra integral and integro-differential equations. J. Integ. Eqn. 2, pp: 11-29.
5. Amini, S., 1987. On the stability of Volterra integral equations with separable kernel. Appl. Anal., 24: 241-251.
6. Chang, S.H., 1982. On certain extrapolation methods for the numerical solution of integro-differential equations. J. Math. Comp., 39: 165-171.
7. Linz, P., 1969. Linear multi step methods for Volterra integro-differential equations. J. Assoc. Comput. Mach., 16: 295-301.
8. Yalcinbas, S., 2002. Taylor polynomial solutions of nonlinear Volterra-Fredholm integral equations. Appl. Math. Comput., 127: 195-206.
9. Akyaz, A. and M. Sezer, 1999. A chebyshev collocation method for the solution of linear inegro-differential equations. Int. J. Comput. Math., 72: 491-507.
10. Avudainayagam, A. and C. Vani, 2000. Wavelet-Galerkin method for integro-differential equations. Comp Elect. Eng., 32: 247-254.
11. He, J.H., 2004. Asymptotology of homotopy perturbation method. Appl. Math. Comput., 156: 591-596.
12. He, J.H., 2005. Limit cycle and bifurcation of nonlinear problems. Chaos, Solutions Fractals, 26: 827-833.
13. He, J.H., 2005. Homotopy perturbation method for bifurcation of nonlinear problems. Int. J. Nonlinear Sci. Numer. Simul., 6 (2): 207-208.
14. He, J.H., 2005. Variational principles for some nonlinear partial differential equations with variable coefficients. Chaos, Solitons Fractals, 19 (4): 847-851.
15. Liu, H.M., 2005. Variational approach to nonlinear electrochemical system. Chaos, Solitons Fractals, 23 (2): 573-576.
16. Liu, H.M., 2004. Generalized variational principles for ion acoustic plasma waves by He's semi-inverse method. Int. J. Nonlinear Sci. Numer. Simul., 5 (1): 95-96.
17. He, J.H., 1998. Approximate analytical solution for seepage flow with fractional derivatives in porous media. Comput. Methods Appl. Mech. Engrg., 167 (1-2): 57-68.
18. He, J.H., 1998. Approximate solution of nonlinear differential equations with convolution product nonlinearities. Comput. Methods Appl. Mech. Engrg., 167 (1-2): 69-73.
19. He, J.H., 1999. Variational iteration method: A kind of nonlinear analytical technique: Some examples. Int. J. Nonlinear Mech., 34 (4): 699-708.
20. He, J.H., 2000. Variational iteration method for autonomous ordinary differential systems. Appl. Math. Comput. 114 (2-3): 115-123.

21. He, J.H.,Y.Q. Wan and Q. Guo, 2004. An iteration formulation for normalized diode characteristics. Int. J. Circ. Theor. Appl., 32 (6): 629-632.
22. Saadati, R., B. Raftari, H. Abibi, S.M. Vaezpour and S. Shakeri, 2008. A Comparison Between the Variational Iteration Method and Trapezoidal Rule for Solving Linear Integro-Differential Equations. World Applied Sciences Journal, 4: 321-325.
23. He, J.H., 2001. Modified Lindstedt-Poincare methods for some strongly nonlinear oscillators. Part III: Double series expansion. Int. J. Nonlinear Sci. Numer. Simul., 2 (4): 317-320.
24. He, J.H., 2002. Modified Lindstedt-Poincare methods for some strongly nonlinear oscillators. Part I: Expansion of constant. Int. J. Nonlinear Mech., 37 (2): 309-314.
25. He, J.H., 2002. Modified Lindstedt-Poincare methods for some strongly nonlinear oscillators. Part II: A new transformation. Int. J. Nonlinear Mech., 37 (2): 315-320.
26. Liu, H.M., 2005. Approximate period of nonlinear oscillators with discontinuities by modified Lindstedt-Poincare method. Chaos, Solitons Fractals, 23 (2): 577-579.
27. El-Shahed, M., 2005. Application of He's homotopy perturbation method to Volterra's integro-differential equation. Int. J. Nonlinear Sci. Numer. Simul., 6 (2): 163-168.
28. He, J.H., 2006. Some asymptotic methods for strongly nonlinear equations. Int. J. Modern Phys. B 20: 1141-1199.
29. He, J.H., 2000. A review on some new recently developed nonlinear analytical techniques. Int. J. Nonlinear Sci. Numer. Simul., 1 (1): 51-70.
30. Hillermeier, C., 2001. Generalized homotopy approach to multiobjective optimization. Int. J. Optim. Theory Appl., 110 (3): 557-583.
31. He, J.H., 2005. Application of homotopy perturbation method to nonlinear wave equations. Chaos, Solitons Fractals, 26: 695-700.
32. He, J.H., 2001. An approximate solution technique depending upon an artificial parameter. Commun. Nonlinear Sci. Simul., 3 (2): 92-97.
33. He, J.H., 1999. Homotopy perturbation technique. Comput Methods Appl. Mech. Engng., 178 (3-4): 257-262.
34. He, J.H., 2000. A coupling method of homotopy technique and perturbation technique for nonlinear problems. Int. J. Nonlinear Mech., 35 (1): 37-43.
35. Raftari, B., 2009. Application of He's homotopy perturbation method and variational iteration method for nonlinear partial integro-differential equations. World Applied Sciences Journal 7 (4): 399-404.
36. Raftari, B., A. Ahmadi and H. Adibi, 2010. The Use of Finite Difference Method, Homotopy Perturbation Method and Variational Iteration Method for a Special Type of Linear Fredholm Integro-differential Equations. Australian Journal of Basic and Applied Sciences, 4(6): 1221-1239.
37. He, J.H., 2004. Comparison of homotopy perturbation method and homotopy analysis method. Appl. Math. Comput., 156: 527-539.
38. Jingtang, M. and H. Brunner, 2006. A posteriori error methods for non-standard Volterra integro-differential equations estimates of discontinuous Galerkin. IMA Journal of Numerical Analysis, 26: 78-95.