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Exact Solutions for Accelerated Flows of a Rotating Second Grade Fluid in a Porous Medium

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Abstract: This paper describes the analysis for developing the exact solutions for magnetohydrodynamic rotating flows of a second grade fluid in a porous medium. A uniform magnetic field has been applied in a direction normal to the flow. The Laplace transform procedure has been adopted in the presentation of exact analytic solutions. Based on Modified Darcy's law, the expressions for dimensionless velocity have been developed for constant and variable flow cases. The obtained solutions are expressed as a sum of steady-state and transient solutions satisfying the imposed boundary and initial conditions. These solutions are presented graphically and discussed for various parameters of interest.

Key words: Exact solutions · Accelerated MHD flows · Rotating fluid · Porous medium

INTRODUCTION

The rotating flow of non-Newtonian fluids have stimulated the interest of researchers in fluid studies and is an area of research undergoing rapid growth in the modern fluid mechanics. This is due to their wide range of scientific applications in various fields. The specific applications of rotating non-Newtonian fluids are encountered in geophysics, specially in the study of wind generating ocean currents on rotating earth. Additionally, when the magnetohydrodynamics (MHD) effects are considered then the rotating flows are useful in cosmical fluid dynamics and solar physics particularly in studying the solar cycle and structure of rotating magnetic stars. In the presence of magnetic field, the fluid particles experience a force induced by the electric current which results in the modification of flow. The flow of non-Newtonian fluids passing through porous media are associated in the ground water flow, thermal oil recovery, food processing and in biophysical sciences where the human lungs for example are modeled as a porous layers. The literature survey revealed that, there exists a growing amount of literature on the rotating flows of Newtonian fluids (for example see the studies [1-13] and the references therein). However, such attempts are very few in case of non-Newtonian fluids [14-18].

To the best of authors' knowledge, so far no study has been reported to analyze the unsteady MHD flows of a rotating second grade fluid in a porous medium past an accelerated plate. Therefore, it is proposed here in the present investigation to make such an attempt. The rotating second grade fluid is considered to be electrically conducting and passing a porous medium. Both constant and variable flow cases are considered. The closed form solutions are obtained using Laplace transform technique. The graphical results are displayed to see the effect various parameters of interest. As a special case, the results for hydrodynamic fluid in a non-porous space and those for Newtonian fluid can be easily recovered by choosing certain values of the involved parameters to be zero. The present investigation may be useful in various practical and engineering applications, such as to study the movement of oil or gas and water through the reservoir of an oil or gas field, underground water in river beds, water purification process, to study the solar cycle and structure of rotating magnetic stars.

Constant Accelerated Flow

Mathematical Formulation: Let us consider an incompressible rotating second grade fluid bounded by a rigid plate at $z = 0$. The fluid is electrically conducting and fills the porous region $z > 0$. The $z-$ axis is taken normal to the plate. Initially, the fluid and plate are both at rest.

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At time $t = 0^+$, the fluid and plate both start solid body rotation with constant angular velocity Ω parallel to z – axis. Additionally, a constant acceleration is imposed on the lower plate and the fluid faraway from the plate is at rest. A uniform transverse magnetic field of strength B_0 is applied parallel to the axis of rotation. It is assumed that induced magnetic field, the external electric field and the electric field due to polarization of charges are negligible. The subsequent fluid motion is analyzed by the following differential equation [15].

$$\frac{\partial F}{\partial t}+\left(2i\Omega+\frac{\sigma B_0^2}{\rho}\right)F=\nu\frac{\partial^2 F}{\partial z^2}+\frac{\alpha_1}{\rho}\frac{\partial^3 F}{\partial z^2\partial t}-\frac{\nu\phi}{k}\left(1+\frac{\alpha_1}{\mu}\frac{\partial}{\partial t}\right)F, \tag{1}$$

in which $F = u + iv$ is the complex velocity and u and v are its real and imaginary parts, ρ designates the density of the fluid, ν the kinematic viscosity, μ the dynamic viscosity, α_1 viscoelasticity, $\phi(0 < \phi < 1)$ the porosity and $k > 0$ the permeability of the porous medium. The appropriate boundary and initial conditions are

$$F(0, t) = At, \quad F(z, t) \to 0 \quad \text{as } z \to \infty;\ t > 0, \tag{2}$$

$$F(z, 0) = 0; \quad Z > 0. \tag{3}$$

Introducing the following dimensionless variables

$$G=\frac{F}{(\nu A)^{1/3}},\quad \xi=\frac{zA^{1/3}}{\nu^{2/3}},\quad \tau=\frac{tA^{2/3}}{\nu^{1/3}}, \tag{4}$$

We get the following dimensionless problem

$$\frac{\partial^2 G}{\partial\xi^2}+\alpha\frac{\partial^3 G}{\partial\xi^2\partial\tau}-a_0\frac{\partial G}{\partial\tau}-b_0 G=0, \tag{5}$$

$$G(\xi, \tau) = \tau \quad G(\xi, \tau) \to 0 \quad \text{as } \xi \to \infty; \quad \tau > 0, \tag{6}$$

$$G(\xi, 0) = 0; \quad \xi > 0, \tag{7}$$

Where

$$\alpha=\frac{\alpha_1 A^{2/3}}{\rho\nu^{4/3}},\quad M^2=\frac{\sigma B_0^2\nu^{1/3}}{\rho A^{2/3}},\quad \frac{1}{K}=\frac{\phi\nu^{4/3}}{kA^{2/3}},$$

$$w=\frac{\Omega\nu^{1/3}}{A^{2/3}},\quad a_0=1+\frac{\alpha}{K},\quad b_0=\frac{1}{K}+2i\omega+M^2.$$

Solution of the Problem: The Laplace transform of Eqs. (5) and (6) in view of Eq. (7) yields the following equations in the transformed q -plane

$$\frac{d^2\overline{G}(\xi,q)}{d\xi^2}-\frac{(a_0 q+b_0)}{\alpha q+1}\overline{G}(\xi,q)=0, \tag{8}$$

$$\overline{G}(0,q)=\frac{1}{q^2}\quad \overline{G}(\xi,q)\to 0 \quad \text{as } \xi\to\infty. \tag{9}$$

Equation (8) subject to the boundary conditions (9) has the following solution

$$\overline{G}(\xi,q)=\frac{1}{q^2}\exp\left(-\xi\sqrt{\frac{a_0 q+b_0}{\alpha q+1}}\right);\quad \xi>0. \tag{10}$$

In order to determine the dimensional velocity $G(\xi,\tau)=L^{-1}\{\overline{G}(\xi,q)\}$, we write Eq. (10) in the following form

$$\overline{G}_c(\xi,q)=\overline{G}_1(q)\overline{G}_2(\xi,q), \tag{11}$$

Where

$$\overline{G}_1(q)=\frac{1}{q^2}, \tag{12}$$

$$\overline{G}_2(\xi,q)=\exp\left(-\frac{\xi}{\sqrt{\alpha}}\sqrt{W(q)}\right);\quad W(q)=\frac{a_0 q+b_0}{q+\beta},\quad \beta=\frac{1}{\alpha}. \tag{13}$$

Denoting $G_1(\tau)=L^{-1}\{\overline{G}_1(q)\}$, $G_2(\xi,\tau)=L^{-1}\{\overline{G}_2(\xi,q)\}$ and using convolution theorem [19], it is easy to write

$$G(\xi,\tau)=(G_1*G_2)(\tau)=\int_0^\tau G_1(\tau-s)G_2(\xi,s)\mathrm{d}s. \tag{14}$$

Laplace inversion of Eq. (12) leads to

$$G_1(\tau) = \tau. \tag{15}$$

In order to determine $G_2(\xi,\tau)=L^{-1}\{\overline{G}_2(\xi,q)\}$, we are using the inversion formula for compound functions [23]. Choosing $F(\xi,q)=\exp\left(-\frac{\xi}{\sqrt{\alpha}}\sqrt{q}\right)$, and $W(q)=\frac{a_0 q+b_0}{q+\beta}$, one obtains

$$f(\xi,\tau)=L^{-1}\{F(\xi,q)\}=\frac{\xi}{2\tau\sqrt{\alpha\pi\tau}}\exp\left(-\frac{\xi^2}{4\alpha\tau}\right) \tag{16}$$

and

$$G_2(\xi,\tau)=\int_0^\infty f(\xi,u)g(u,\tau)du=\frac{\xi}{2\sqrt{\alpha\pi}}\int_0^\infty \frac{1}{u\sqrt{u}}\exp\left(-\frac{\xi^2}{4\alpha u}\right)g(u,\tau)\mathrm{d}u, \tag{17}$$

Where

$$g(u,\tau)=L^{-1}\left[\exp(-uw(q))\right]=e^{-a_0u}L^{-1}\left[1+\left(\exp\left(\frac{ua_1}{q+\beta}\right)-1\right)\right]=e^{-a_0u}\left[\delta(\tau)+\sqrt{\frac{a_1u}{\tau}}e^{-\beta\tau}I_1\left(2\sqrt{a_1u\tau}\right)\right];\quad a_1=a_0\beta-b_0, \tag{18}$$

in which $I_1(.)$ is the modified Bessel function of the first kind of order one.
Hence Eq. (17) implies

$$G_2(\xi,\tau)=\frac{\xi\delta(\tau)}{2\sqrt{\alpha\pi}}\int_0^\infty \frac{1}{u\sqrt{u}}\exp\left(-\frac{\xi^2}{4\alpha u}-a_0u\right)\mathrm{d}u+\frac{\sqrt{a_1}\xi e^{-\beta\tau}}{2\sqrt{\alpha\pi\tau}}\int_0^\infty \frac{1}{u}\exp\left(-\frac{\xi^2}{4\alpha u}-a_0u\right)I_1\left(2\sqrt{a_1u\tau}\right)\mathrm{d}u. \tag{19}$$

Substitution of Eqs. (15) and (19) into Eq. (14) yields

$$G(\xi,\tau)=\frac{\xi\tau}{2\sqrt{\alpha\pi}}\int_0^\infty \frac{1}{u\sqrt{u}}\exp\left(-\frac{\xi^2}{4\alpha u}-a_0u\right)\mathrm{d}u+\frac{\sqrt{a_1}\xi}{2\sqrt{\alpha\pi}}\int_0^\tau\int_0^\infty \frac{1}{u\sqrt{s}}\exp\left(-\frac{\xi^2}{4\alpha u}-a_0u-\beta s\right)I_1\left(2\sqrt{a_1us}\right)(\tau-s)\mathrm{d}u\ \mathrm{d}s. \tag{20}$$

Using

$$\int_0^\infty \frac{1}{u\sqrt{u}}\exp\left(-\frac{\xi^2}{4\alpha u}-a_0u\right)\mathrm{d}u=\frac{2\sqrt{\alpha\pi}e^{-\frac{\xi\sqrt{a_0}}{\sqrt{\alpha}}}}{\xi},$$

Eq. (20) becomes

$$G(\xi,\tau)=\tau e^{-\frac{\xi\sqrt{a_0}}{\sqrt{\alpha}}}+\frac{\sqrt{a_1}\xi}{2\sqrt{\alpha\pi}}\int_0^\tau\int_0^\infty \frac{1}{u\sqrt{s}}\exp\left(-\frac{\xi^2}{4\alpha u}-a_0u-\beta s\right)I_1\left(2\sqrt{a_1us}\right)(\tau-s)\mathrm{d}u\ \mathrm{d}s. \tag{21}$$

The starting solution (21) holds for both small and large times. In order to write as a sum of the steady-state and transient solutions, we are using the relation

$$\int_0^\tau f(\xi,\tau,s)\mathrm{d}s=\int_0^\infty f(\xi,\tau,s)\mathrm{d}s-\int_\tau^\infty f(\xi,\tau,s)\mathrm{d}s, \tag{22}$$

Which gives

$$G(\xi,\tau)=\tau e^{-\frac{\xi\sqrt{a_0}}{\sqrt{\alpha}}}+\frac{\sqrt{a_1}\xi}{2\sqrt{\alpha\pi}}\int_0^\infty\int_0^\infty \frac{1}{u\sqrt{s}}\exp\left(-\frac{\xi^2}{4\alpha u}-a_0u-\beta s\right)I_1\left(2\sqrt{a_1us}\right)(\tau-s)\mathrm{d}u\ \mathrm{d}s$$
$$-\frac{\sqrt{a_1}\xi}{2\sqrt{\alpha\pi}}\int_\tau^\infty\int_0^\infty \frac{1}{u\sqrt{s}}\exp\left(-\frac{\xi^2}{4\alpha u}-a_0u-\beta s\right)I_1\left(2\sqrt{a_1us}\right)(\tau-s)\mathrm{d}u\ \mathrm{d}s. \tag{23}$$

Variably Accelerated Flow: Here we consider the flow situation in which the fluid motion is caused by the variable acceleration of the plate. The rest of the problem is same only the boundary condition $(2)_1$ is replaced by

$$F(0, t) = Bt^2 \quad t > 0. \tag{24}$$

Introducing the following dimensionless variables

$$G = \frac{F}{\nu^{2/5}B^{1/5}}, \quad \xi = \frac{zB^{1/5}}{\nu^{3/5}}, \quad \tau = \frac{tB^{2/5}}{\nu^{1/5}}, \tag{25}$$

the problem transforms into the following form

$$\frac{\partial^2 G}{\partial \xi^2} + \alpha\frac{\partial^3 G}{\partial \xi^2 \partial \tau} - b_1\frac{\partial G}{\partial \tau} - b_2 G = 0, \tag{26}$$

$$G(\xi, \tau) = \tau^2 \quad G(\xi, \tau) \to 0 \quad \text{as} \xi \to \infty; \quad \tau > 0, \tag{27}$$

$$G(\xi, 0) = 0; \quad \xi > 0,$$

Where

$$\alpha = \frac{\alpha_1 B^{2/5}}{\rho\nu^{6/5}}, \quad M^2 = \frac{\sigma B_0^2 \nu^{1/5}}{\rho B^{2/5}}, \quad \frac{1}{K} = \frac{\phi \nu^{4/3}}{kA^{2/3}}, \quad w = \frac{\Omega \nu^{1/3}}{A^{2/3}},$$

$$b_1 = 1 + \frac{\alpha}{K}, \quad b_2 = \frac{1}{K} + 2i\omega + M^2.$$

Adopting the procedure of the previous section, we get the following expression for dimensional velocity

$$\begin{aligned} G(\xi,\tau) = \tau^2 e^{-\frac{\xi\sqrt{b_1}}{\sqrt{\alpha}}} &+ \frac{\sqrt{b_3}\xi}{2\sqrt{\alpha\pi}}\int_0^\infty\int_0^\infty \frac{1}{u\sqrt{s}}\exp\left(-\frac{\xi^2}{4\alpha u} - b_1 u - \beta s\right) I_1\left(2\sqrt{b_3 us}\right)(\tau - s)^2 \,\mathrm{d}u\,\mathrm{d}s, \\ &- \frac{\sqrt{b_3}\xi}{2\sqrt{\alpha\pi}}\int_\tau^\infty\int_0^\infty \frac{1}{u\sqrt{s}}\exp\left(-\frac{\xi^2}{4\alpha u} - b_1 u - \beta s\right) I_1\left(2\sqrt{b_3 us}\right)(\tau - s)^2 \,\mathrm{d}u\,\mathrm{d}s, \end{aligned} \tag{28}$$

Where

$$\beta = \frac{1}{\alpha}, \quad b_3 = \beta b_1 - b_2.$$

The solution (29) thus consists of steady part minus transient one, which fades out with increasing time. Further, it is important to note that the solutions (23) and (29) satisfy both the initial and boundary conditions which provide a useful mathematical check. It is also easy to reduce the solutions (23) and (29) corresponding to hydrodynamic fluid in a non porous space by substituting $M = \frac{1}{K} = 0.$ The corresponding solutions for Newtonian fluid can also be recovered.

Graphical Results and Discussion: In this section we present the physical illustration of dimensionless complex velocity G and its dependence on the emerging parameters. Specifically, we consider the influence of the viscoelastic (second grade) parameter α, magnetic parameter M, permeability of the medium K and rotation parameter ω on the (a) real and (b) imaginary parts of G. For this purpose, Figs. (1-8) have been plotted. Here Figs. (1-4) and (5-8) have been sketched for the cases of constant and variable accelerations of the plate respectively.

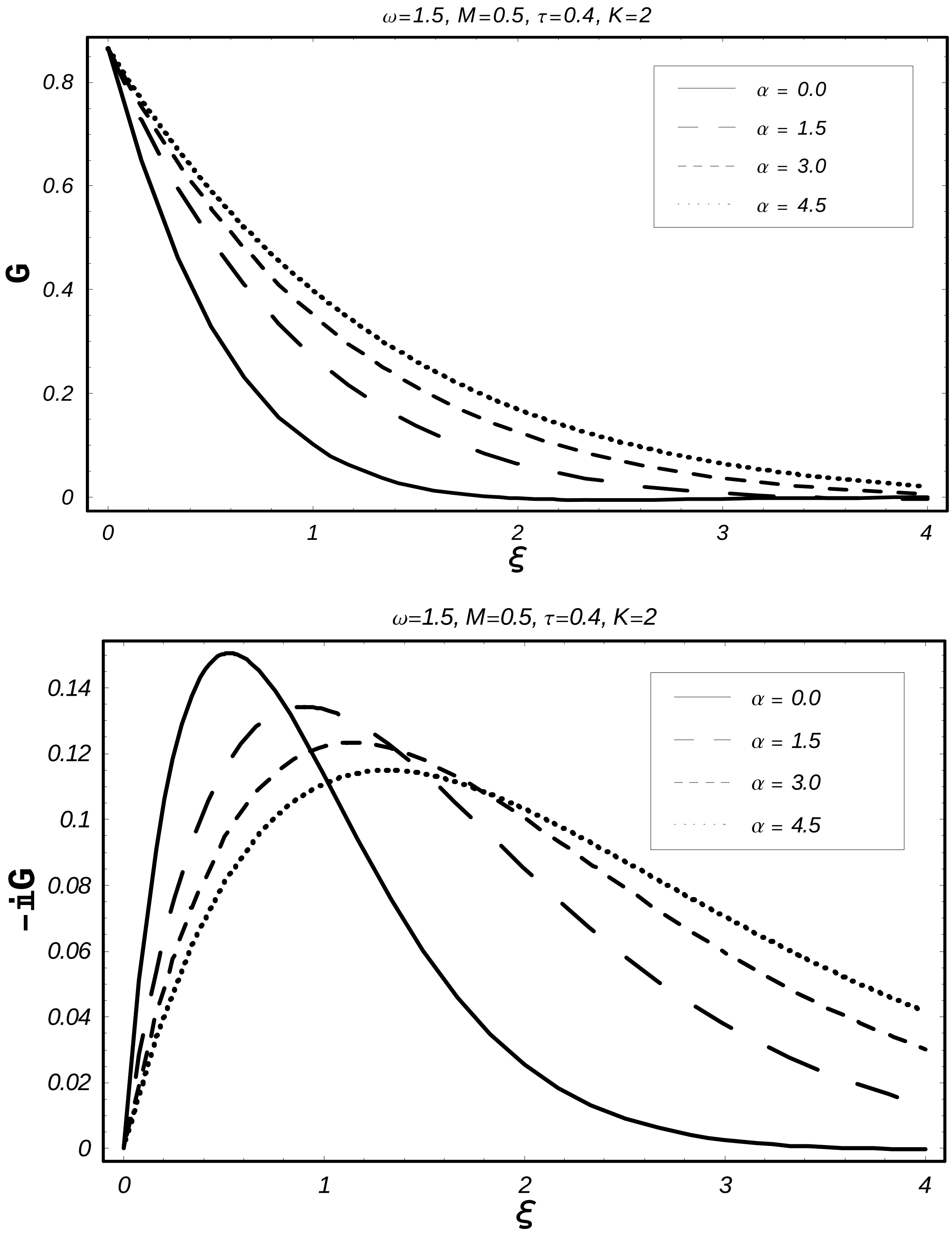


Fig.1:Velocity profiles given by Eq. (23) for different values of α.

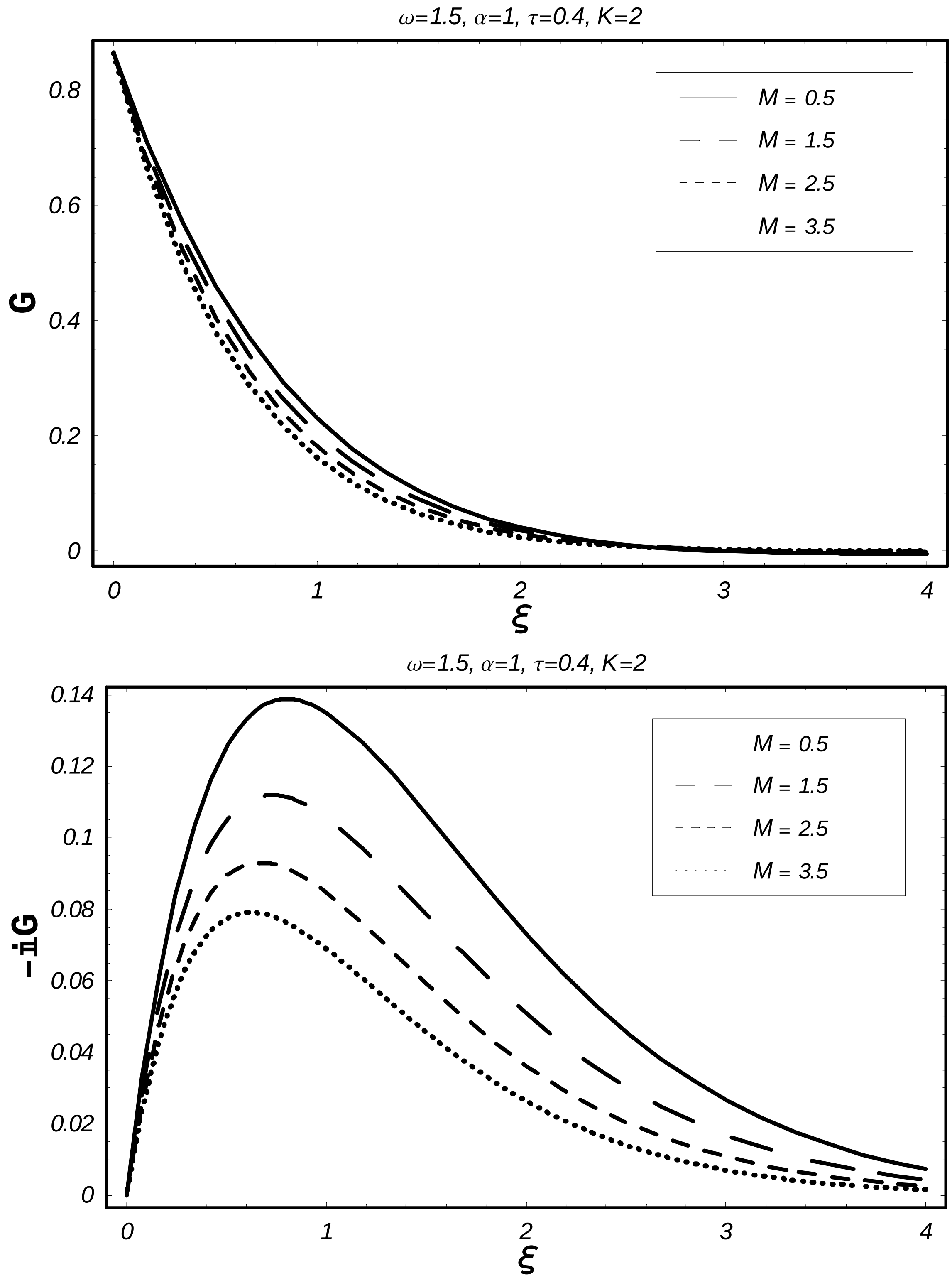
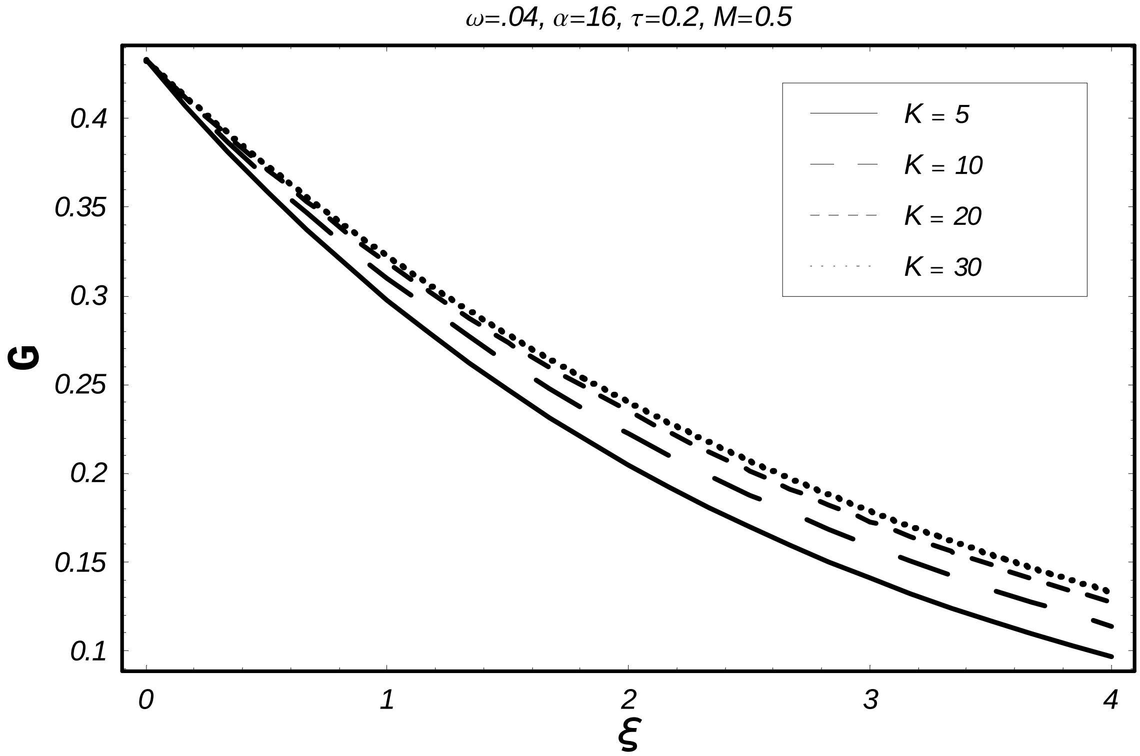


Fig. 2: Velocity profiles given by Eq. (23) for different values of *M*.



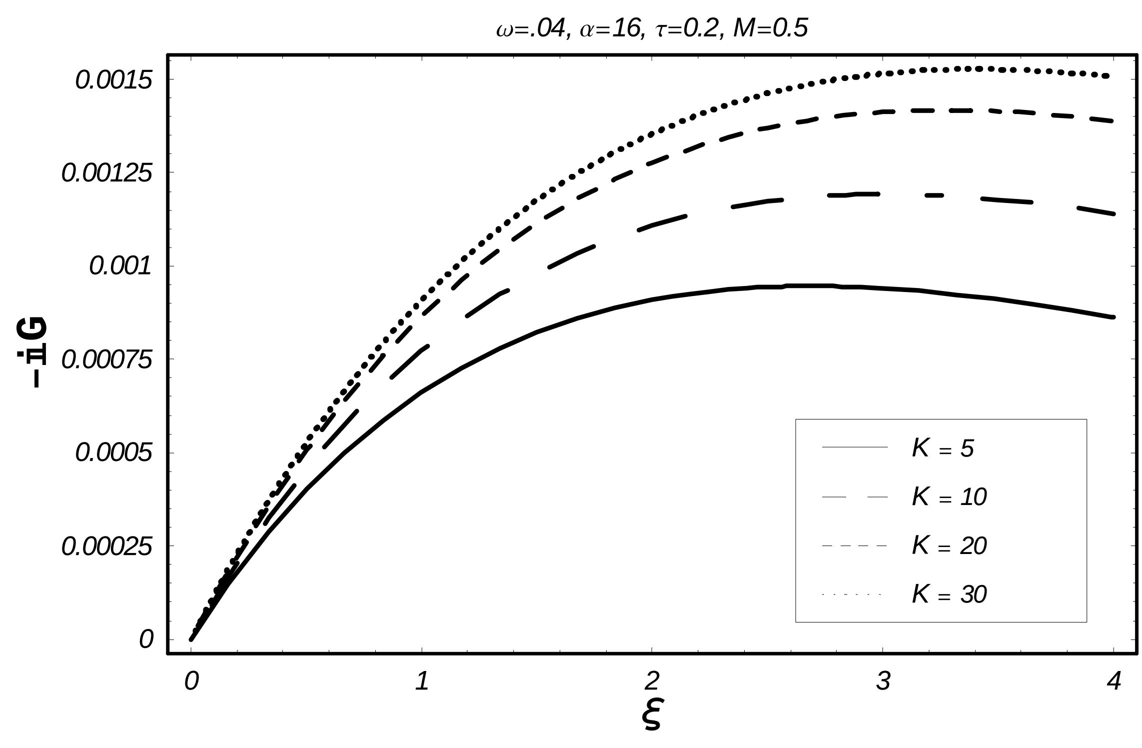


Fig. 3: Velocity profiles given by Eq. (23) for different values of K.

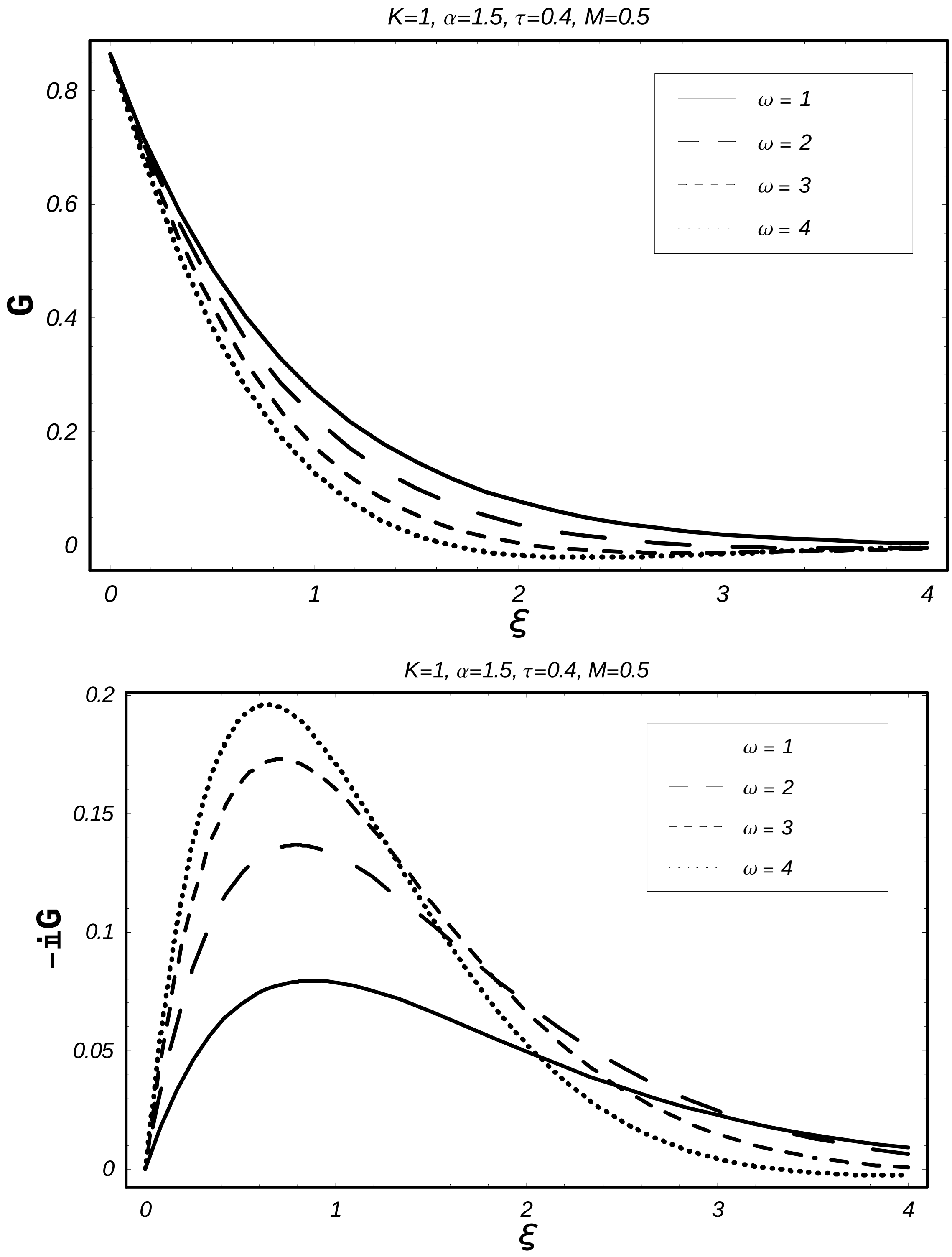


Fig. 4: Velocity profiles given by Eq. (23) for different values of ω.

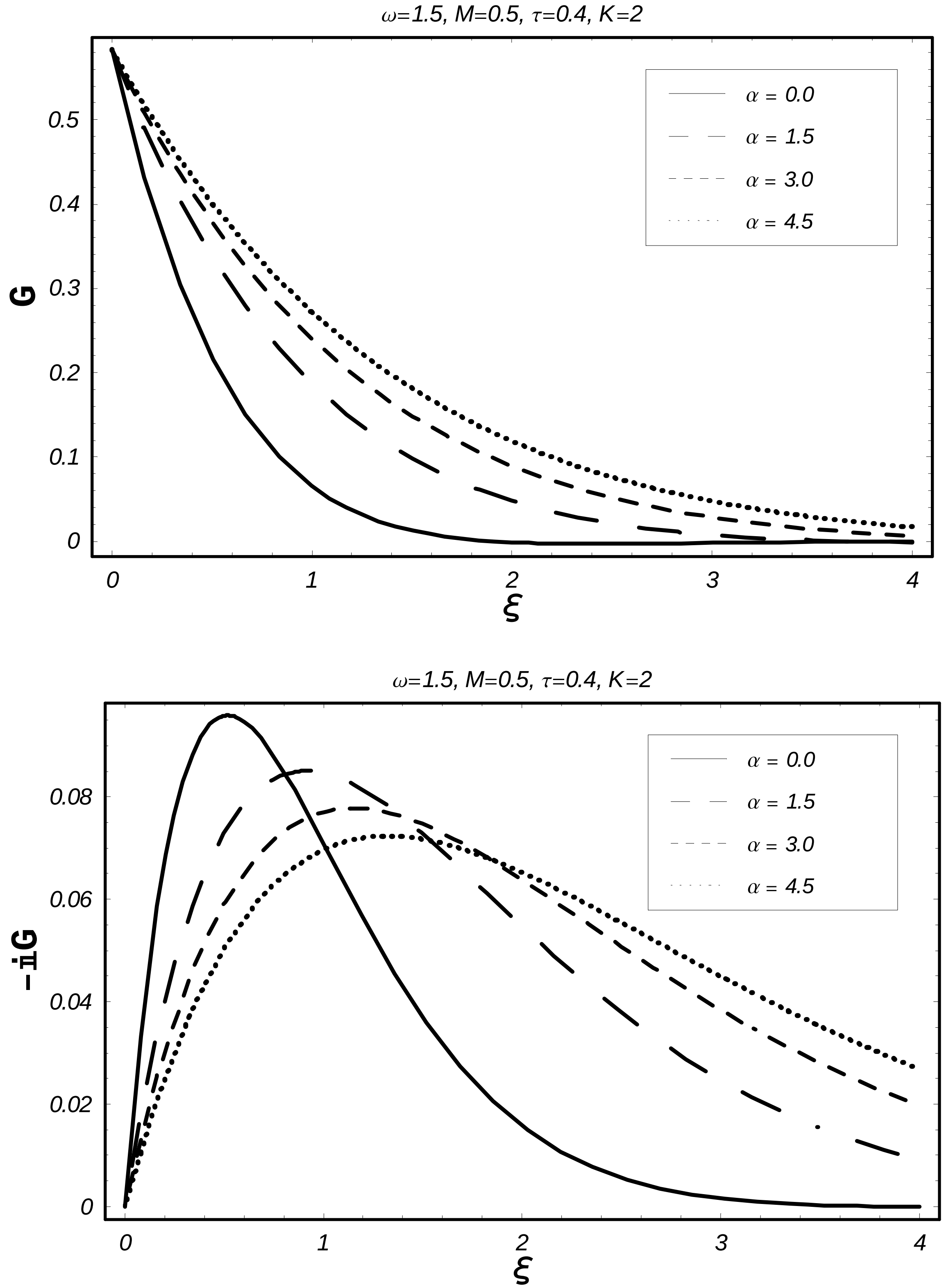


Fig. 5: Velocity profiles given by Eq. (29) for different values of α.

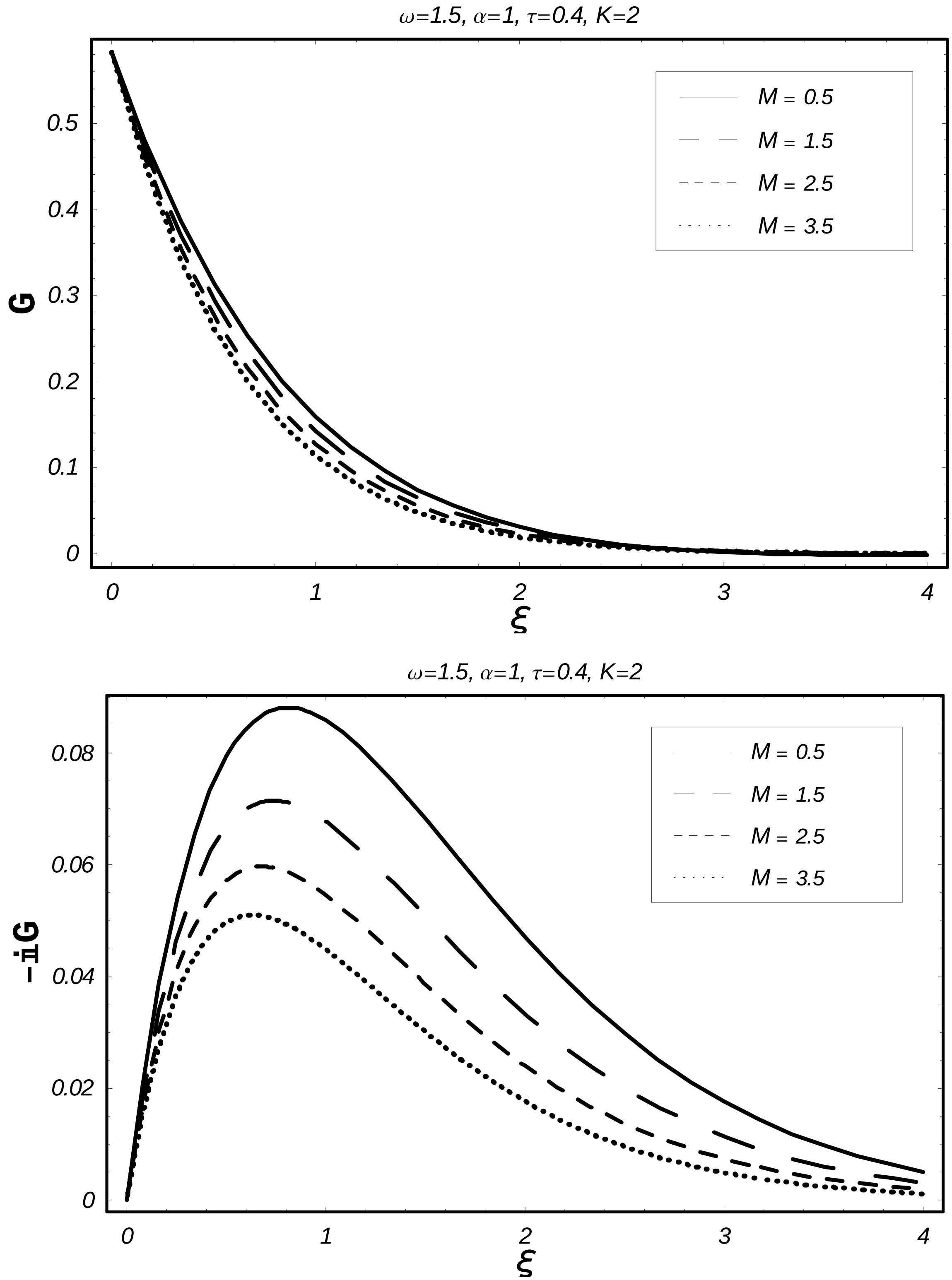


Fig. 6: Velocity profiles given by Eq. (29) for different values of *M*.

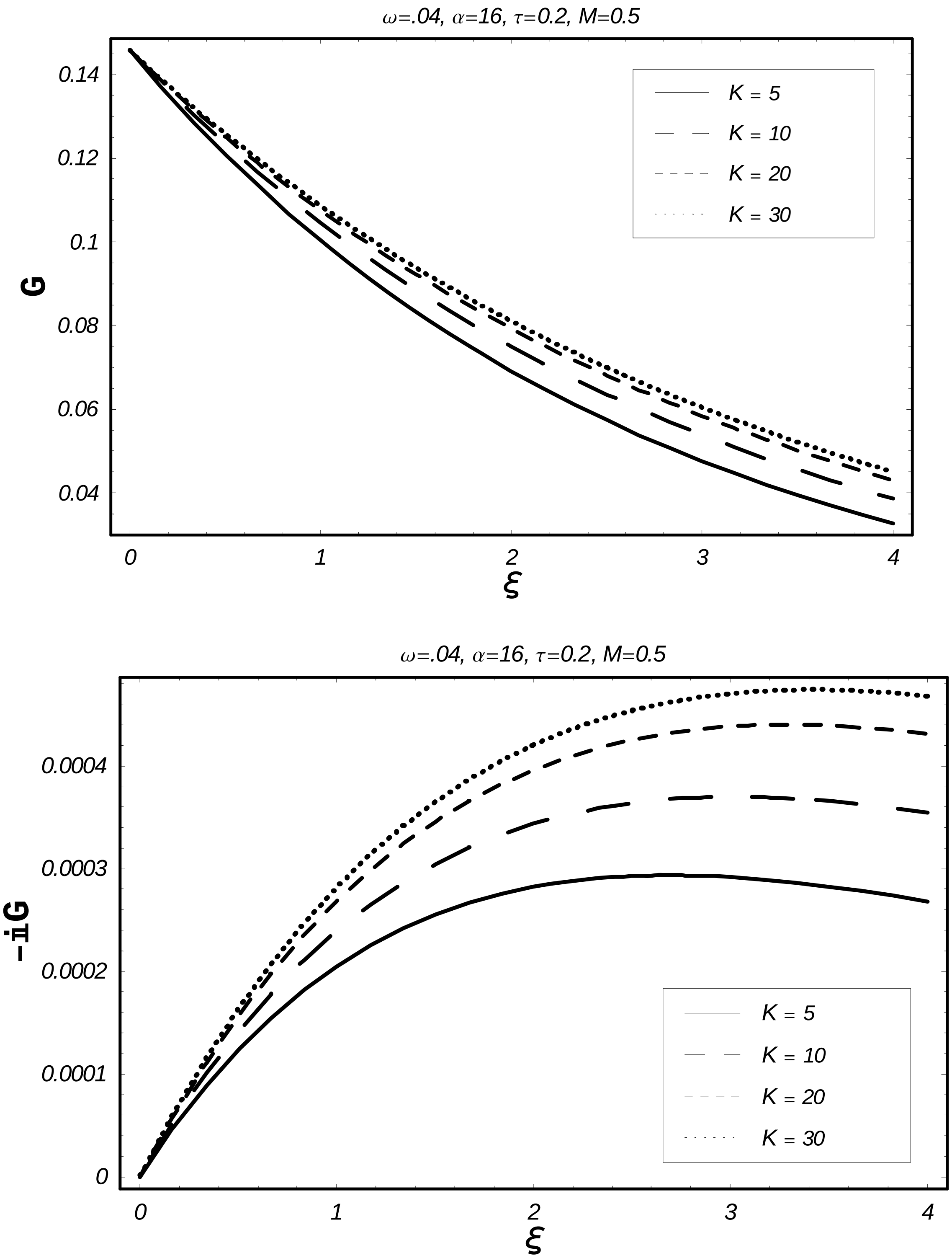


Fig. 7: Velocity profiles given by Eq. (29) for different values of K.

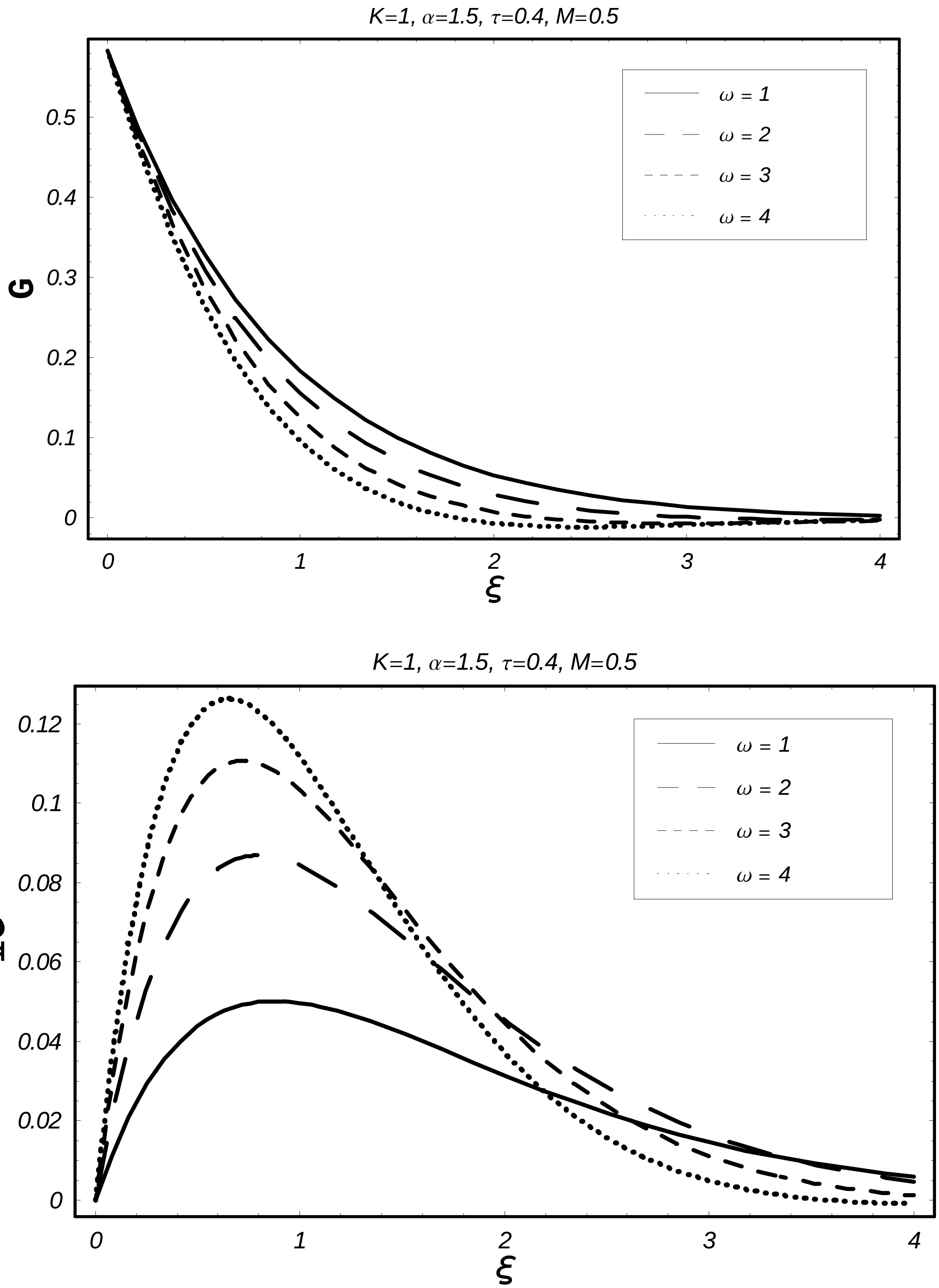


Fig. 8: Velocity profiles given by Eq. (29) for different values of ω.

Fig. (1) shows the effect of α on the real and imaginary parts of velocity. It is found that the magnitude of the real part of velocity increases while the imaginary part first increases and then decreases when α is increased. In Fig. (2), it is noted that an increase in M decreases the velocity profile monotonically for both real and imaginary parts of velocity. This is in accordance with the fact that magnetic force acts against the direction of flow and causes the velocity to slow down. The influence of K on the flow is illustrated in Fig. (3). As anticipated, an increase in permeability K of the porous medium reduces the drag force and hence causes the magnitude of velocity to increase. The variation of the velocity profiles for rotation parameter ω is shown in Fig. 4. It is obvious to see that for larger values of ω the real part of velocity is decreasing but quite opposite behavior was observed for the imaginary part of velocity. The velocity increases with increasing values of ω. As shown in Figs. (5-8) the effects of the involved parameters on the velocity profile in case of variable accelerated flow are similar to that of constant accelerated flow in a qualitative sense but quite opposite quantitatively.

Concluding Remarks: In this paper the exact analytic solutions for the magnetohydrodynamic flow of a rotating second grade in a porous medium past an accelerated plate are obtained using Laplace transform method. Both constant and variable flow cases are discussed. Graphical results are prepared to support the analytical solutions where the effects of second grade parameter α magnetic parameter M, permeability parameter K and rotation parameter ω are shown. It is found that velocity is increasing with increasing values of α whereas decreased for the larger values of rotation parameter. As expected, the effect of magnetic and permeability parameters on the velocity shows opposite behavior.

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REFERENCES

1. Soundalgekar, V.M. and I. Pop, 1970. Unsteady hydromagnetic flow in a rotating fluid, Bulletin Mathematique Roumanie, 14: 375.
2. Pop, I. and V.M. Soundalgekar, 1974. Effects of Hall current on hydromagnetic flow near a porous plate. Acta Mech., 20: 215-318.
3. Datta, N. and R.N. Jana, 1976. Hall effects on hydromagnetic Rayleigh problem in a rotating fluid. Bulletine Mathematique, 28: 63.
4. Kumar, K. and C.L. Varshney, 1984. Viscous flow through a porous medium past an oscillating plate in a rotating system, Indian J. Pure Appl. Math., 15: 1014-1019.
5. Rajeswari, V. and G. Nath, 1992. Unsteady flow over a stretching surface in a rotating fluid, Int. J. Eng. Sci., 30: 347-756.
6. Nazar, R., N. Amin and I. Pop, 2004. Unsteady boundary layer flow due to a stretching surface in a rotating fluid. Mech. Res. Comm., 31: 121-128.
7. Manna, G., S.N. Maji, M. Guria and R.N. Jana, 2007. Unsteady viscous flow past a flat plate in a rotating system, J. Phys. Sci., 11: 29-42.
8. Sajid, M., T. Javed and T. Hayat, 2008. MHD rotating flow of a viscous fluid over a shrinking surface, Nonlinear Dyn., 51: 259-265.
9. Das, S., S.L. Maji, M. Guria and R.N. Janan, 2009. Unsteady MHD Couette flow in a rotating system, Math. and Comput. Model., 50: 1211-1217.
10. Singh, A.K., N.P. Singh, U. Singh and H. Singh, 2010. Convective flow past an accelerated porous plate in a rotating system in presence of magnetic field, Int. J. Heat Mass Transfer, 52: 3390-3395.
11. Chauhan, D.S. and P. Rastogi, 2010. Radiation effects on natural convection MHD flow in a rotating vertical porous channel partially filled with porous medium, Appl. Math. Sci., 4: 643-655.
12. Sahoo, S.N., J.P. Panda and G.C. Dash, XXXX. Hydromagnetic oscillatory flow and heat transfer of a viscous liquid past a vertical porous plate in a rotating medium, Indian J. Sci. Technol., 3: 817-821.
13. Seth, G.S., M.S. Ansari and R. Nandkeolyar, XXXX. Unsteady hydromagnetic Couette flow within porous plates in a rotating system, Adv. in Appl. Math. Mech., 2: 286.
14. Asghar, S., M.M. Gulzar and T. Hayat, 2005. Rotating flow of a third grade fluid by homotopy analysis method, Appl. Math. Comput., 165: 213-221.
15. Hayat, T., C. Fetecau and M. Sajid, 2008. Analytical solution for MHD transient rotating flow of a second grade fluid in a porous space, Non-Linear Anal. Real World Appl., 9: 1619-1627.

16. Hayat, T., C. Fetecau and M. Sajid, 2008. On MHD transient flow of a Maxwell fluid in a porous medium and rotating frame, Phys. Lett. A, 372: 1639-1644.
17. Abelman, S., E. Momoniat and T. Hayat, 2009. Steady MHD flow of a third grade fluid in a rotating frame and porous space, Non-Linear Anal. Real World Appl., 10: 332-3328.
18. Tiwari, A.K and S.K. Ravi, 2009. Analytical studies on transient rotating flow of a second grade fluid in a porous medium, Adv. Theor. Appl. Mech., 2: 33-41.
19. Roberts, G.E. and H. Kaufman, 1968. Table of Laplace Transforms, W.B. Saunders Company, Philadelphia, London.