World Applied Sciences Journal 9 (Special Issue of Applied Math): 01-06, 2010
ISSN 1818-4952

Exact Solutions for Some (2+1)-Dimensional Nonlinear Evolution Equations by Using Tanh-coth Method

Ahmet Bekir

Department of Mathematics, Art-Science Faculty, Dumlupinar University, Kütahya, Turkey

Abstract: In this work, we established exact solutions for some nonlinear evolution equations. The tanh-coth method was used to construct solitary and soliton solutions of (2+1)-dimensional nonlinear evolution equations. The tanh-coth method presents a wider applicability for handling nonlinear wave equations.

Key words: Tanh-coth method . (2+1)-dimensional breaking soliton (Calogero) equation . (2+1)-dimensional Burgers equations . (2+1)-dimensional Boussinesq equations

INTRODUCTION

The investigation of the travelling wave solutions for nonlinear partial differential equations plays an important role in the study of nonlinear physical phenomena. Nonlinear wave phenomena appears in various scientific and engineering fields, such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, chemical physics and geochemistry. Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in nonlinear wave equations. In recent years, new exact solutions may help to find new phenomena. A variety of powerfull methods, such as inverse scattering method [1, 2], Backlund tranformation [3], the tanh-sech method [4, 5], extended tanh method [6, 7], sine-cosine method [8, 9], homogeneous balance method [10, 11], variational iteration method [12-15] and exp-function method [16, 17] were used to develop nonlinear dispersive and dissipative problems.

The pioneer work Malfiet [18] introduced the powerful tanh method for a reliable treatment of the nonlinear wave equations. The useful tanh method is widely used by many such as in [19] and by the references therein. Later, the extended tanh method, developed by Wazwaz [20], is a direct and effective algebraic method for handling nonlinear equations. Various extensions of the method were developed as well.

Exact solutions of (2+1)-dimensional breaking soliton equation are obtained by further generalized projective Riccati equation method in [21, 22]. Geng and Cao are obtained, from which the explicit solutions, including N-soliton and algebro-geometric solutions, of the known (2+1)-dimensional breaking soliton equation and these new multi-dimensional integrable models are derived in [23].

More new double periodic and multiple soliton solutions are obtained for the generalized (2+1)-dimensional Boussinesq equation in [24]. Chen, Yan and Zhang study (2+1)-dimensional Boussinesq equation by using the new generalized transformation in homogeneous balance method [25]. Feng et al. have investigated the bifurcations and global dynamic behavior of two variants of (2+1)-dimensional Boussinesq-type equations with positive and negative exponents and obtained the sufficient conditions under which solitary, kink, breaking and periodic wave solutions appear [26]. New applications of the singular manifold method are presented to a (2+1)-dimensional Burgers equation and a higher-order Burgers equation [27].

The (2+1)-dimensional Burgers equation is chosen to illustrate the method such that many families of new rational formal Jacobi elliptic function solutions are obtained in [28]. Wang, Chen and Zhang present a new Riccati equation rational expansion method to uniformly construct a series of exact solutions for (2+1)-dimensional Burgers equation [29].

Our first interest in the present work is in implementing the tanh-coth method to stress its power in handling nonlinear equations, so that one can apply it to models of various types of nonlinearity. The next interest is in the determination of exact travelling wave solutions for (2+1)-dimensional breaking soliton (Calogero), (2+1)-dimensional Burgers and (2+1)-dimensional Boussinesq equations. Searching for exact solutions of nonlinear problems has attracted a considerable amount of research work where computer symbolic systems facilitate the computational work.

Corresponding Author: Dr. Ahmet Bekir, Department of Mathematics, Art-Science Faculty, Dumlupinar University, Kütahya, Turkey

THE TANH-COTH METHOD

Wazwaz has summarized for using tanh-coth method. A PDE

$$P(u,u_t,u_x,u_{xx},u_{tt}\ldots)=0 \tag{2.1}$$

can be converted to on ODE

$$Q(U,U',U'',U''',\ldots)=0 \tag{2.2}$$

upon using a wave variable $\xi = x+y-\beta t$, Equation (2.2) is then integrated as long as all terms contain derivatives where integration constants are considered zeros. Introducing a new independent variable

$$Y=\tan(\xi),\quad \xi = x+y-\beta t \tag{2.3}$$

leads to change of derivatives:

$$\begin{aligned}
\frac{\partial}{\partial\xi} &= (1-Y^2)\frac{\partial}{\partial Y}\\
\frac{\partial^2}{\partial\xi^2} &= (1-Y^2)\left(-2Y\frac{\partial}{\partial Y}+(1-Y^2)\frac{\partial^2}{\partial Y^2}\right)\\
\frac{\partial^3}{\partial\xi^3} &= (1-Y^2)\left(\begin{aligned}&(6Y^2-2)\frac{\partial}{\partial Y}-6Y(1-Y^2)\frac{\partial^2}{\partial Y^2}\\ &+(1-Y^2)^2\frac{\partial^3}{\partial Y^3}\end{aligned}\right)
\end{aligned} \tag{2.4}$$

The tanh-coth method [20] admits the use of the finite expansion

$$U(\xi)=S(Y)=\sum_{k=0}^{m}a_kY^k+\sum_{k=1}^{m}b_kY^{-k} \tag{2.5}$$

where m is a positive integer, for this method, that will be determined. Expansion (2.5) reduces to the standard tanh method for $b_k = 0$, $1\le k\le m$. The parameter m is usually obtained, as stated before, by balancing the linear terms of the highest order in the resulting equation with the highest order nonlinear terms. If m is not an integer, then a transformation formula should be used to overcome this difficulty. Substituting (2.5) into the ODE results is an algebraic system of equations in powers of Y that will lead to the determination of the parameters $a_k(k = 0,\ldots,m)$, $b_k(k=1,\ldots,m)$ and β.

To show the efficiency of the method described in the previous part, we present some examples.

(2+1)-DIMENSIONAL BREAKING SOLITON EQUATION

(2+1)-dimensional breaking soliton equation.

$$u_{xxxy}-2u_yu_{xx}-4u_xu_{xy}+u_{xt}=0 \tag{3.1}$$

which was first introduced by Calogero and Degasperis [30]. By using the traveling wave transformation u = u (ξ), ξ= x+y-βt, we change the Eq. (3.1) to the form

$$u''''-6u'u''-\beta u''=0 \tag{3.2}$$

obtained after integrating the ODE once and setting the constant of integration equal to zero. Balancing u'''' term with $u'u''$ in (3.2)

$$m+4=2m+3 \tag{3.3}$$

gives

$$m=1 \tag{3.4}$$

The tanh-coth method (2.5) admits the use of the finite expansion

$$u(\xi)=S(Y)=a_0+a_1Y+\frac{b_1}{Y} \tag{3.5}$$

Substituting Eq. (3.5) in Eq. (3.2), collecting all terms with the powers in Y^i (i = 0,…,6) and setting each of the obtained coefficients for Y^i to zero yields the following set of algebraic equations with respect to a_0, a_1, b_1 and β.

$$\begin{aligned}
&12a_1^2+24a_1=0\\
&40a_1-2\beta a_1-12a_1b_1-24a_1^2=0\\
&2\beta a_1+12a_1b_1+12a_1^2+16a_1=0\\
&12b_1^2+2\beta b_1+12a_1b_1+16b_1=0\\
&-12a_1b_1-24b_1^2-2\beta b_1-40b_1=0\\
&24b_1+12b_1^2=0
\end{aligned} \tag{3.6}$$

for which we get the following solution :

i) The first set:

$$a_0=\text{free parameter},\ a_1=-2,\ b_1=0,\ \beta=4 \tag{3.7}$$

ii) The second set:

$$a_0=\text{free parameter},\ a_1=0,\ b_1=-2,\ \beta=4 \tag{3.8}$$

iii) The third set:

$$a_0=\text{free parameter},\ a_1=-2,\ b_1=-2,\ \beta=16 \tag{3.9}$$

In the view of this, we can obtain the following soliton solutions

$$u_1(x,y,t) = a_0 - 2\tanh(x+y-4t) \quad (3.10)$$

$$u_2(x,y,t) = a_0 - 2\coth(x+y-4t) \quad (3.11)$$

$$u_3(x,y,t) = a_0 - 2\tanh(x+y-16t) - 2\coth(x+y-16t) \quad (3.12)$$

Comparing the above results with the relevant in [21-23] it can be seen that the some of the obtained results are new and the rest solutions are the same.

(2+1)-DIMENSIONAL BOUSSINESQ EQUATIONS

(2+1)-dimensional Boussinesq equations

$$u_{tt} - u_{xx} - u_{yy} - (u^2)_{xx} - u_{xxxx} = 0 \quad (4.1)$$

Using the wave variable ξ = x+y-βt, the system (4.1) is carried to a system of ODEs

$$(\beta^2 - 2)u'' - (u^2)'' - u'''' = 0 \quad (4.2)$$

by twice time integrating and setting the constant of integration equal to zero we find

$$(\beta^2 - 2)u - u^2 - u'' = 0 \quad (4.3)$$

Balancing u'' with u^2 in (4.3) gives

$$m+2 = 2m \quad (4.4)$$

gives

$$m = 2 \quad (4.5)$$

The tanh-coth method (2.5) admits the use of the finite expansion

$$u(\xi) = S(Y) = a_0 + a_1 Y + a_2 Y^2 + \frac{b_1}{Y} + \frac{b_2}{Y^2} \quad (4.6)$$

Substituting Eq. (4.6) in Eq. (4.3), collecting all terms with the powers in Y^i (i = 0,…,8) and setting each of the obtained coefficients for Y^i to zero yields the following set of algebraic equations with respect to a_0, a_1, a_2, b_1, b_2 and β.

$$\begin{aligned}
&-a_2^2 - 6a_2 = 0,\ 2a_2a_1 - 2a_1 = 0 \\
&\beta a_2 - 2a_0a_2 - a_1^2 + 6a_2 = 0,\ -2a_2b_1 - 2a_0a_1 + \beta a_1 = 0 \\
&\beta a_0 - 2a_2b_2 - 2a_1b_1 - 2a_2 - 2b_2 - a_0^2 - 2a_0 = 0 \\
&\beta b_1 - 2b_2a_1 - 2a_0b_1 = 0,\ 6b_2 - b_1^2 + \beta b_2 - 2a_0b_2 = 0 \\
&-2b_2b_1 - 2b_1 = 0,\ -b_2^2 - 6b_2 = 0
\end{aligned} \quad (4.7)$$

for which we get the following solution :

i) The first set:

$$a_0 = 2,\ a_1 = 0,\ a_2 = -6,\ b_1 = 0,\ b_2 = 0,\ \beta = -2 \quad (4.8)$$

ii) The second set:

$$a_0 = 2,\ a_1 = 0,\ a_2 = 0,\ b_1 = 0,\ b_2 = -6,\ \beta = -2 \quad (4.9)$$

iii) The third set:

$$a_0 = 6,\ a_1 = 0,\ a_2 = -6,\ b_1 = 0,\ b_2 = 0,\ \beta = 6 \quad (4.10)$$

iv) The fourth set:

$$a_0 = 6,\ a_1 = 0,\ a_2 = 0,\ b_1 = 0,\ b_2 = -6,\ \beta = 6 \quad (4.11)$$

v) The fifth set:

$$a_0 = -4, a_1 = 0, a_2 = -6, b_1 = 0, b_2 = -6, \beta = -6 \quad (4.12)$$

vi) The sixth set:

$$a_0 = 12, a_1 = 0, a_2 = -6, b_1 = 0, b_2 = -6, \beta = 18 \quad (4.13)$$

In the view of this, we obtain the following soliton solutions

$$u_1(x,y,t) = 2 - 6\tanh^2(x+y+2t) \quad (4.14)$$

$$u_2(x,y,t) = 2 - 6\coth^2(x+y+2t) \quad (4.15)$$

$$u_3(x,y,t) = 6 - 6\tanh^2(x+y-6t) \quad (4.16)$$

$$u_4(x,y,t) = 6 - 6\coth^2(x+y-6t) \quad (4.17)$$

$$u_5(x,y,t) = -4 - 6\left[\begin{aligned}&\tanh^2(x+y+6t)\\&+\coth^2(x+y+6t)\end{aligned}\right] \quad (4.18)$$

$$u_6(x,y,t) = 12 - 6\left[\begin{aligned}&\tanh^2(x+y-18t)\\&+\coth^2(x+y-18t)\end{aligned}\right] \quad (4.19)$$

As we know, various other types of exact solutions for the (2+1)-dimensional Boussinesq equation, such as rational solutions, polynomial solutions and the traveling wave solutions have been obtained by many authors under different approaches [24-27]. However, the soliton solutions are firstly given out to our knowledge.

(2+1)-DIMENSIONAL BURGERS EQUATIONS

(2+1)-dimensional Burgers equations

$$u_t + uu_y + avu_x + bu_{yy} + abu_{xx} = 0, u_x - v_y = 0 \qquad (5.1)$$

where a,b are real parameters.

Using the wave variable $\xi = x+y-\beta t$, the system (5.1) is carried to a system of ODEs

$$\beta u' + uu' + avu' + bu'' + abu'' = 0, u' - v' = 0 \qquad (5.2)$$

where by integrating the second equation we find

$$u = v \qquad (5.3)$$

Substituting (5.3) into the first equation of (5.2) and we obtain

$$\beta u' + (a+1)uu' + b(a+1)u'' = 0 \qquad (5.4)$$

Balancing u'' with uu' in (5.4) gives

$$m+2 = 2m+1 \qquad (5.5)$$

gives

$$m = 1 \qquad (5.6)$$

The tanh-coth method (2.5) admits the use of the finite expansion

$$u(\xi) = S(Y) = a_0 + a_1 Y + \frac{b_1}{Y} \qquad (5.7)$$

Substituting Eq. (5.7) in Eq. (5.4), collecting all terms with the powers in $Y^i (i = 0,\ldots,6)$ and setting each of the obtained coefficients for Y^i to zero yields the following set of algebraic equations with respect to a_0, a_1, b_1.

$$\begin{aligned}
&-aa_1^2 + 2ba_1 - a_1^2 + 2aba_1 = 0\\
&-a_0a_1 - \beta a_1 - aa_0a_1 = 0\\
&aa_1^2 - 2ba_1 + a_1^2 - 2aba_1 = 0\\
&a_0b_1 + \beta a_1 + a_0a_1 + \beta b_1 + a_0b_1 + aa_0a_1 = 0\\
&b_1^2 - 2bb_1 + ab_1^2 - 2abb_1 = 0\\
&-\beta b_1 - a_0b_1 - aa_0b_1 = 0\\
&2bb_1 - ab_1^2 + 2abb_1 - b_1^2 = 0
\end{aligned} \qquad (5.8)$$

for which we get the following solution :

i) The first set:

$$a_0 = -\frac{\beta}{a+1}, \; a_1 = 2b, \; b_1 = 0 \qquad (5.9)$$

ii) The second set:

$$a_0 = -\frac{\beta}{a+1}, \; a_1 = 0, \; b_1 = 2b \qquad (5.10)$$

iii) The third set:

$$a_0 = -\frac{\beta}{a+1}, \; a_1 = 2b, \; b_1 = 2b \qquad (5.11)$$

In the view of this, we can obtain the following soliton solutions

$$u_1(x,y,t) = -\frac{\beta}{a+1} + 2b\tanh(x+y-\beta t) \qquad (5.12)$$

$$u_2(x,y,t) = -\frac{\beta}{a+1} + 2b\coth(x+y-\beta t) \qquad (5.13)$$

$$u_3(x,y,t) = -\frac{\beta}{a+1} + 2b\begin{bmatrix}\tanh(x+y-\beta t)+\\ \coth(x+y-\beta t)\end{bmatrix} \qquad (5.14)$$

All the solutions reported in this paper have been verified with Maple by putting them back into the original Eqs. (3.1), (4.1) and (5.1), which can not be obtained by the methods [10, 29, 31]. To the best of our knowledge, these solutions are new and have not been reported yet. We foresee that our results can be found potentially useful for applications in mathematical physics and applied mathematics including numerical simulation.

CONCLUSION

The tanh-coth method was succesfully used to establish solitary wave solutions. The performance of the tanh-coth method is reliable and effective and gives more solutions. The applied method will be used in further works to establish more entirely new solutions for other kinds of nonlinear wave equations.

As we know, various other types of exact solutions for the nonlinear evolution equations, such as rational solutions, polynomial solutions and the traveling wave solutions have been obtained by many authors under different approaches. However, our solutions are soliton and periodic solutions. Soliton and periodic solutions have many potential applications in physics.

The availability of computer systems like Mathematica or Maple facilitates the tedious algebraic calculations. The method which we have proposed in this letter is also a standard, direct and computerizable method, which allows us to solve complicated and tedious algebraic calculation.

REFERENCES

1. Ablowitz, M.J. and H. Segur, 1981. Solitons and inverse scattering transform, SIAM, Philadelphia.
2. Vakhnenko, V.O., E.J. Parkes and A.J. Morrison, 2003. A Bäcklund transformation and the inverse scattering transform method for the generalised Vakhnenko equation, Chaos, Solitons and Fractals, 17 (4): 683-692.
3. Lu, D., B. Hong and L. Tian, 2006. Backlund Transformation and N-soliton-like Solutions to the Combined KdV-Burgers Equation with Variable Coefficients. International Journal of Nonlinear Science, 2: 3-10.
4. Malfliet, W., 1992. Solitary wave solutions of nonlinear wave equations. Am. J. Phys., 60: 650-654.
5. Wazwaz, A.M., 2004. The tanh method for travelling wave solutions of nonlinear equations. Applied Mathematics and Computation, 154 (3): 713-723.
6. Yusufoglu, E. and A. Bekir, 2007. On the Extended Tanh Method Applications of Nonlinear Equations. International Journal of Nonlinear Science, 4 (1): 10-16.
7. Fan, E., 2000. Extended tanh-function method and its applications to nonlinear equations. Phys. Lett. A, 277: 212.
8. Wazwaz, A.M., 2004. A sine-cosine method for handling nonlinear wave equations. Math. and Comput. Modelling, 40: 499-508.
9. Yusufoglu, E. and A. Bekir, 2006. Solitons and periodic solutions of coupled nonlinear evolution equations by using Sine-Cosine method. International Journal of Computer Mathematics, 83 (12): 915-924.
10. Fan, E. and H. Zhang, 1998. A note on the homogeneous balance method. Phys. Lett. A, 246: 403-406.
11. Wang, L., J. Zhou and L. Ren, 2006. The Exact Solitary Wave Solutions for a Family of BBM Equation. International Journal of Nonlinear Science, 1: 58-64.
12. Mohyud-Din, S.T., M.A. Noor and K.I. Noor, 2009. Modified Variational Iteration Method for Solving Sine-Gordon Equations. World Applied Sciences Journal, 6 (7): 999-1004.
13. Mohyud-Din, S.T., M.A. Noor and K.I. Noor, 2009. Variational Iteration Method for Solving Flierl-petviashivili Equation Using He's Polynomials and Pade´ Approximants. World Applied Sciences Journal, 6 (9): 1298-1303.
14. Noor, M.A., S.T. Mohyud-Din and M. Tahir, 2008. Modified variational iteration methods for Thomas-Fermi equation. Wd. Appl. Sci. J., 4 (4): 479-486.
15. Noor, M.A. and S.T. Mohyud-Din, 2008. Modified variational iteration method for Goursat and Laplace Problems. Wd. Appl. Sci. J., 4 (4): 487-498.
16. He, J.H., 2006. Exp -function method for nonlinear wave equations, Chaos, Solitons and Fractals, 30: 700-708.
17. He, J.H. and M.A. Abdou, 2007. New periodic solutions for nonlinear evolution equations using Exp -function method, Chaos, Solitons and Fractals, 34 (5): 1421-1429.
18. Malfliet, W. and W. Hereman, 1996. The tanh method. I: Exact solutions of nonlinear evolution and wave equations. Phys. Scr., 54: 563-568.
19. Wazwaz, A.M., 2005. The tanh method: solitons and periodic solutions for the Dodd-Bullough-Mikhailov and the Tzitzeica-Dodd-Bullough equations, Chaos, Solitons and Fractals, 25 (1): 55-63.
20. Wazwaz, A.M., 2007. The extended tanh method for new soliton solutions for many forms of the fifth-order KdV equations. Applied Mathematics and Computation, 184: 1002-1014.
21. Mei, J. and H. Zhang, 2004. New types of exact solutions for a breaking soliton equation. Chaos, Solitons and Fractals, 20: 771-777.
22. Xie, Z. and H. Zhang, 2005. Symbolic computation and construction of soliton-like solutions for a (2+1)-dimensional breaking soliton equation. Applied Mathematics and Computation, 162: 283-291.
23. Geng, Xi and C. Cao, 2004. Explicit solutions of the (2+1)-dimensional breaking soliton equation, Chaos, Solitons and Fractals, 22: 683-691.
24. Chen, H.-T. and H.-Q. Zhang, 2004. New double periodic and multiple soliton solutions of the generalized (2+1)-dimensional Boussinesq equation. Chaos, Solitons and Fractals, 20: 765-769.
25. Chen, Y., Z. Yan and H.-Q. Zhang, 2003. New explicit solitary wave solutions for (2+1)-dimensional Boussinesq equation and (3+1)-dimensional KP equation. Physics Letters A, 307: 107-113.
26. Feng, D., T. He and J. Lu, 2007. Bifurcations of travelling wave solutions for (2+1)-dimensional Boussinesq-type equation. Applied Mathematics and Computation, 185: 402-414.
27. Peng, Y. and E. Yomba, 2006. New applications of the singular manifold method to the (2+1)-dimensional Burgers equations. Applied Mathematics and Computation, 183: 61-67.

28. Wang, B., L.-N. Song and H.-Q. Zhang, 2007. A new extended elliptic equation rational expansion method and its application to (2+1)-dimensional Burgers equation. Chaos, Solitons and Fractals, 33: 1546-1551.
29. Wang, Qi, Y. Chen and H.-Q. Zhang, 2005. A new Riccati equation rational expansion method and its application to (2+1)-dimensional Burgers equation. Chaos, Solitons and Fractals, 25: 1019-1028.
30. Calogero, F. and A. Degasperis, 1977. Nuovo Cimento B, pp: 39-54.
31. Wazwaz, A.M., 2005. Variants of the two-dimensional Boussinesq equation with compactons, solitons and periodic solutions. Comput. Math. Appl., 49: 295-301.