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Legendre Wavelets for Systems of Fredholm Integral Equations of the Second Kind

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Abstract: In this paper, systems of Fredholm integral equations of the second kind have been studied. A numerical method for solving these systems is presented. The method is based upon Legendre wavelet approximations. Some examples are presented to illustrate the ability of the method.

Key words: Systems of Fredholm integral equations of the second kind · Mother wavelet · Legendre wavelets · Operational matrix

INTRODUCTION

Orthogonal functions and polynomials have been used by many authors for solving various problems. The main idea of using orthogonal basis is that a problem reduces to solving a system of linear or nonlinear algebraic equations by truncated series of orthogonal basis functions for solution of problem and using the operational matrices. Here we use Legendre wavelets basis on interval [0, 1]. Some of its applications are nonlinear Volterra- Fredholm integral equation [2], Fredholm integral equations of the first kind [3], Abel's integral equations [4], nonlinear integral equations [5], differential equations of Lane-Emden type [6], variational problems [7] and some other problems. Systems of Fredholm integral equations of the second kind have been solved by other methods as well, Adomian decomposition method [9], Taylor-series expansion method [10], Sinc- collocation method [11], Homotopy perturbation method [12].

Legendre Wavelets and Their Properties

Wavelets and Legendre Wavelets: Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet, [1]. When the dilation parameter, α and the translation parameter, b vary continuously we have the following family of continuous wavelets as.

$$\psi_{a,b}(t)=|a|^{-\frac{1}{2}}\psi\left(\frac{t-b}{a}\right), \quad a,b\in R, \quad a\neq 0. \tag{1}$$

Legendre wavelets $\psi_{n,m}(t)=\psi(k,\hat{n},m,t)$ have four arguments $\hat{n}=2n-1,\ n=1,2,\ldots,2^{k-1}$, k is any positive integer, m is the order of Legendre polynomials and t is the normalized time.
They are defined on the interval [0, 1] as follows:

$$\psi_{nm}(t)=\psi(k,\hat{n},m,t)=\begin{cases}\sqrt{m+\frac{1}{2}}\,2^{\frac{k}{2}}P_m(2^k t-\hat{n}), & \frac{\hat{n}-1}{2^k}\le t<\frac{\hat{n}+1}{2^k},\\ 0, & otherwise\end{cases} \tag{2}$$

Where:
$m = 0,1,\ldots, M-1$ and $n = 1,2,..,2^{k-1}$. $P_m(t)$. are the famous Legendre polynomials of order m, which are orthogonal respect to the weight function $w(t)=1$, on the interval [-1.1] and satisfy the following recursive formula:

$$\begin{cases}P_0(t)=1,\\ P_1(t)=t,\\ P_{m+1}(t)=\left(\frac{2m+1}{m+1}\right)tP_m(t)-\left(\frac{m}{m+1}\right)P_{m-1}(t), & m=1,2,\ldots\end{cases} \tag{3}$$

The set of Legendre wavelets are an orthonormal set, [2-8].

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Function Approximation: A function $f(x) \in L^2([0,1])$ may be expanded as

$$f(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm} \psi_{nm}(x), \tag{4}$$

Where:
$c_{nm} = (f(x), \Psi_{nm}(x))$, stands for the inner product. We can consider truncated series in (4), as follows:

$$f(x) \cong \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm} \psi_{nm}(x) = C^T \psi(x). \tag{5}$$

Where:
C and $\Psi(x)$ are $2^{k-1}M \times 1$ matrices given by

$$\begin{aligned} C &= \left[c_{10}, c_{11}, \ldots, c_{1M-1}, c_{20}, c_{21}, \ldots, c_{2M-1}, \ldots, c_{2^{k-1}0}, \ldots, c_{2^{k-1}M-1}\right]^T \\ &= \left[c_1, c_2, \ldots, c_M, c_{M+1}, \ldots, c_{2^{k-1}M}\right]^T, \end{aligned} \tag{6}$$

and

$$\begin{aligned} \psi(x) &= \left[\psi_{10}(x), \psi_{11}(x), \ldots, \psi_{1,M-1}(x), \psi_{20}(x), \psi_{21}(x), \right. \\ &= \left. \ldots, \psi_{2,M-1}(x), \ldots, \psi_{2^{k-1}0}(x), \ldots, \psi_{2^{k-1},M-1}(x)\right]^T \\ &= \left[\psi_1(x), \psi_2(x), \ldots, \psi_M(x), \psi_{M+1}(x), \ldots, \psi_{2^{k-1}M}(x)\right]^T. \end{aligned} \tag{7}$$

Also a function $f(x,y) \in L^2([0,1]\times[0,1])$ can be approximated as

$$f(x,y) \simeq \psi^T(x) K \psi(y). \tag{8}$$

Here the elements of matrix $K = [k_{ij}]$ $2^{k-1}M \times 2^{k-1}M$ will be obtain by

$$k_{i,j} = \left(\psi_i(x), \left(f(x,y), \psi_j(y)\right)\right), \quad i,j = 1,2,\ldots,2^{k-1}M. \tag{9}$$

The Operational Matrix for Integration: The integration of the vector $\Psi(x)$, defined in (7), can be achieved as.

$$\int_0^x \psi(t)\,dt = P\psi(x). \tag{10}$$

Where:
P is the $2^{k-1}M \times 2^{k-1}M$ operational matrix for integration, [8]. This matrix was obtained as.

$$P = \frac{1}{2^k}\begin{bmatrix} L & F & F & \cdots & F \\ O & L & F & \ddots & \vdots \\ O & O & L & \ddots & F \\ \vdots & \ddots & \ddots & \ddots & F \\ O & \cdots & O & O & L \end{bmatrix}, \tag{11}$$

F,O and L are $M \times M$ matrices given by

$$F = \begin{bmatrix} 2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{12}$$

$$O = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{13}$$

$$L = \begin{bmatrix} 1 & \frac{1}{\sqrt{3}} & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3\sqrt{5}} & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{5}}{5\sqrt{3}} & 0 & \frac{\sqrt{5}}{5\sqrt{7}} & \cdots & 0 & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{7}}{7\sqrt{5}} & 0 & \frac{\sqrt{7}}{7\sqrt{9}} & \cdots & 0 & 0 \\ & & & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\frac{\sqrt{2M-3}}{(2M-3)\sqrt{2M-5}} & 0 & \frac{\sqrt{2M-3}}{(2M-3)\sqrt{2M-1}} \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{\sqrt{2M-1}}{(2M-1)\sqrt{2M-3}} & 0 \end{bmatrix}. \tag{14}$$

The integration of the product of two Legendre wavelets vector functions is derived as:

$$\int_0^1 \psi(x)\psi^T(x)\,dx = I, \tag{15}$$

Where:
I is an identity matrix.

Solution of the Systems of Fredholm Integral Equations of the Second Kind via Legandre Wavelets: Consider the following system of Fredholm integral equations of the second kind

$$u_i(x) = f_i(x) + \sum_{j=1}^{m} \int_0^1 k_{ij}(x,t)\, G_{ij}(u_1(t), u_2(t), \ldots, u_n(t))\,dt, \quad 0 \le x,t \le 1, \qquad i = 1,2,\ldots,n. \tag{16}$$

Where:
$f_i(x) \in L^2([0,1])$, $k_{ij}(x, t) \in L^2([0,1]\times[0,1])$ and G_{ij} are linear or non-linear functions of $u_1(x), u_2(x),..., u_n(x)$ for $i, j = 1,2,...,n$ and $u_i(x)$, $i = 1,2,...,n$ are the unknown functions.

We consider the following approximations for system (16) by using Legendre wavelets as:

$$u_i(x) \simeq \psi^T(x)C_i, \qquad f_i(x) \simeq \psi^T(x)F_i,$$
$$k_{ij}(x,t) \simeq \psi^T(x)K_{ij}\psi(t), \qquad G_{ij}(u_1(t), u_2(t), \dots, u_n(t)) \simeq C^{*T}_{ij}\psi(t), \quad i,j=1,2,\dots,n$$

Where:
C^*_{ij} are column vector functions of elements of the vector C_i.

By substituting these approximations into system (16), we would have:

$$\psi^T(x)C_i = \psi^T(x)F_i + \sum_{j=1}^{m}\int_0^1 \psi^T(x)K_{ij}\psi(t)\psi^T(t)C^*_{ij}\,dt$$
$$= \psi^T(x)F_i + \sum_{j=1}^{m}\psi^T(x)K_{ij}\left(\int_0^1 \psi(t)\psi^T(t)\,dt\right)C^*_{ij}, \qquad i=1,2,\dots,n.$$

Therefore, the following will be obtained:

$$\psi^T(x)C_i = \psi^T(x)\left(F_i + \sum_{j=1}^{m} K_{ij}C^*_{ij}\right), \quad i=1,2,\dots,n. \tag{17}$$

By multiplying $\Psi(x)$, in both sides of system (17), then applying $\int_0^1 (\ )dx$, we get the following linear or non-linear system.

$$C_i - \sum_{j=1}^{m} K_{ij}C^*_{ij} = F_i, \qquad i=1,2,\dots,n. \tag{18}$$

Vector functions C_i $i = 1,2...,n$ can be obtained by solving system (18).

Numerical Examples: To illustrate the method some of systems of Fredholm integral equations of the second kind have been considered. These examples are solved by $k = 1$ and $M = 6$.

Example 1: Consider the following linear system of Fredholm integral equations.

$$\begin{cases} u(x) = \frac{x}{18} + \frac{17}{36} + \int_0^1 \left(\frac{x+t}{3}\right)\left(u(t)+v(t)\right)dt, \\ v(x) = x^2 - \frac{19}{12}x + 1 + \int_0^1 xt\left(u(t)+v(t)\right)dt, \qquad 0 \le x,t \le 1. \end{cases} \tag{19}$$

With the exact solutions $u(x) = x + 1$ and $v(x) = x^2 + 1$, [9]. Put

$$u(x) \simeq \psi^T(x)C_1, \qquad v(x) \simeq \psi^T(x)C_2,$$
$$\frac{x}{18} + \frac{17}{36} \simeq \psi^T(x)F_1, \qquad x^2 - \frac{19}{12}x + 1 \simeq \psi^T(x)F_2,$$
$$\frac{x+t}{3} \simeq \psi^T(x)K_1\psi(t), \qquad xt \simeq \psi^T(x)K_2\psi(t),$$

For this system we find

$$C_1 = \left[\frac{3}{2}, \frac{\sqrt{3}}{6}, 0, 0, 0, 0\right]^T,$$
$$C_2 = \left[\frac{4}{3}, \frac{\sqrt{3}}{6}, \frac{\sqrt{5}}{30}, 0, 0, 0\right]^T.$$

The solutions would be achieved as follows

$$u(x) = \sum_{i=1}^{6} c_{1i}\psi_i(x) = x+1, \qquad v(x) = \sum_{i=1}^{6} c_{2i}\psi_i(x) = x^2+1,$$

Which are exact solutions.

Example 2: Consider the following linear system with the exact solutions $u(x) = e^x$ and $v(x) = e^{-x}$, [11].

$$\begin{cases} u(x) = 2e^x + \frac{e^{x+1}-1}{x+1} - \int_0^1 e^{x-t}u(t)\,dt - \int_0^1 e^{(x+2)t}v(t)\,dt, \\ v(x) = e^x + e^{-x} + \frac{e^{x+1}-1}{x+1} - \int_0^1 e^{xt}u(t)\,dt - \int_0^1 e^{x+t}v(t)\,dt. \end{cases} \tag{20}$$

By applying the Legendre wavelets method, we have the following approximate solutions:

$$u(x) \simeq -0.11175x^5 + 0.34468x^4 - 0.10026x^3 + 0.59824x^2 + 0.98646x + 1.0004,$$
$$v(x) \simeq 0.045106x^5 - 0.078953x^4 - 0.069991x^3 + 0.46853x^2 - 0.99646x + 0.99994.$$

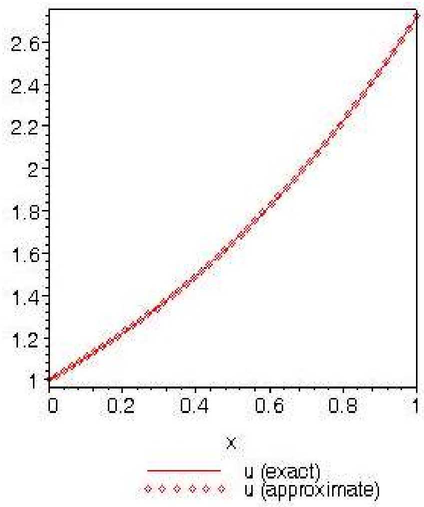

Fig. 1: Plots of exact and approximate solution for Example 2

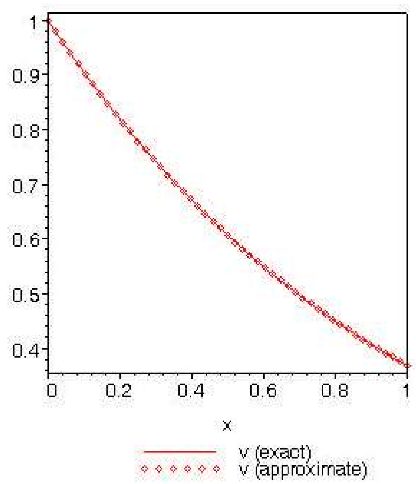

Fig. 2: Plots of exact and approximate solution for Example 2

Plots of the exact and approximate solutions are presented in Figures 1 and 2.

Example 3: consider the following non-linear system

$$\begin{cases} u(x) = x - \frac{5}{18} + \int_0^1 \frac{1}{3}\left(u(t)+v(t)\right)dt, \\ v(x) = x^2 - \frac{2}{9} + \int_0^1 \frac{1}{3}\left(u^2(t)+v(t)\right)dt, \quad 0 \le x, t \le 1. \end{cases} \tag{21}$$

with exact solution $u(x) = x$ and $v(x) = x^2$, [9].
In this example let's take

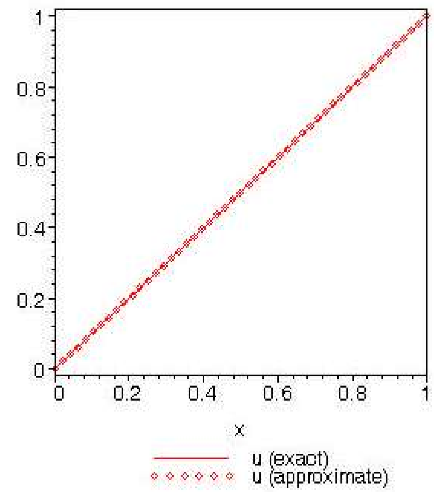

Fig. 3: Plots of exact and approximate solution for Example 4

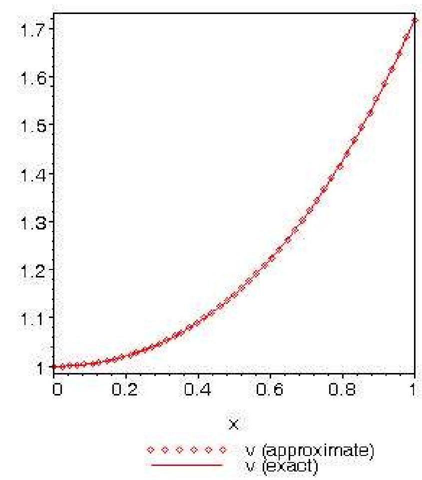

Fig. 4: Plots of exact and approximate solution for Example 4

$$u(x) \simeq \psi^T(x)C_1, \qquad v(x) \simeq \psi^T(x)C_2,$$
$$x - \frac{5}{18} \simeq \psi^T(x)F_1, \qquad x^2 - \frac{2}{9} \simeq \psi^T(x)F_2,$$
$$\frac{1}{3} \simeq \psi^T(x)K\ \psi(t), \qquad u^2(t) \simeq \psi^T(t)C_1^*,$$

Therefore by substituting into system (21), following non-linear system would be obtained:

$$\begin{cases} (I-K)C_1 - C_2 = F_1, \\ (I-K)C_2 - KC_1^* = F_2. \end{cases} \tag{22}$$

The exact solution will be obtained by solving system (22).

Example 4: In this example the following two non-linear system of Fredholm integral equations with exact solution $u(x) = x$ and $v(x) = e^{x} - x$, are studied.

$$\begin{cases} u(x) = \frac{x}{3} - \frac{1}{12} + e - \frac{1}{2}e^{2} + \int_0^1 \left((x-t)u(t)v(t) + v^{2}(t) \right) dt, \\ v(x) = e^{x} - \frac{6}{5}x - \frac{3}{2} + e + \int_0^1 \left(xt^{2}u^{2}(t) - v(t) \right) dt, \qquad 0 \le x, t \le 1. \end{cases} \tag{23}$$

By applying the Legendre wavelets method, we derive the following approximate solutions:

$$u(x) \simeq -0.11\times10^{-8} + 0.9999999968x,$$
$$v(x) \simeq 0.02771999999x^{5} - 0.00021x^{4} + 0.202048x^{3} + 0.4869719998x^{2} + 0.0018618306x + 0.999936913.$$

Plots of the exact and approximate solutions are presented in Figures 3 and 4.

CONCLUSION

In this paper Legendre wavelets method has been used to derive approximate solutions for systems of Fredholm integral equations of the second kind. It can be concluded that the method is very powerful and useful technique for finding approximate solutions of these systems. In [9] examples (1) and (3) were solved by Adomian decomposition method and approximate solutions were obtained while this method leads exact solution. In this paper, we have used the package Maple 13 to carry the computations associated with these examples.

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