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Solution of Fractional Integro-Differential Equations via He's Variational Iteration Method

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Abstract: In this paper, we apply He's variational iteration method (VIM) to solve fractional integro differential equations which are of utmost importance in engineering and applied sciences. Numerical results reveal the complete reliability of the proposed algorithm.

Key words: Variational iteration method · Caputo fractional derivative · Fractional calculus · Integro-differential equations

INTRODUCTION

The fractional-differential equations [1-8] play a pivotal role in the modeling of number of physical phenomenon. The applications of such equations include rheology, damping laws, fluid mechanics, viscoelasticity, biology, physics, engineering and modeling of earth quakes, see [1-8] and the references therein. Several techniques [1-7] have been employed to find appropriate solutions of these equations as per their physical nature. Most of the used schemes so far encounter some inbuilt deficiencies and moreover are not compatible with the true physical nature of these problems. He [5, 8-16] developed the variational iteration method (VIM) which has been used to solve a wide range of nonlinear problems of physical nature. The basic inspiration of this paper is the implementation of He's VIM to solve fractional-integro differential equations. It has been observed that He's VIM [5, 8-31] is very efficient and is easier to apply as compare to previous schemes. Numerical results reveal the complete reliability of the proposed algorithm.

Basic Definitions: The fractional derivative of $f(x)$ in the Caputo sense is defined as

$$D_*^{\alpha} f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)dt, \tag{1}$$

For $m-1<\alpha \le m, m\in N, x>0, f \in C_{-1}^m$.

Also, we need here two of its basic properties.

If $m-1<\alpha \le m$, $m \in N$, and $f \in C_{\mu}^m, \mu \ge -1, then$

$$D_*^{\alpha} t^{\mu} = \begin{cases} 0 & (\mu \le \alpha - 1) \\ \dfrac{\Gamma(\mu+1)t^{\mu-\alpha}}{\Gamma(\mu-\alpha+1)} & (\mu > \alpha - 1) \end{cases} \tag{2}$$

Variational Iteration Method: To illustrate the basic concept of the method for the fractional integro differential equation, we consider the general fractional nonlinear integro differential equation:

$$D^{\beta} y_i(t) = f_i(t) + \int_0^t K_i(t,s) F(y(s))ds, i = 1,2,...,n \tag{3}$$

Subject to the Initial Conditions

$$y_i^k(0) = c_{ik}, k = 0,1,...,m-1, m-1<\beta \le m, m\in N \tag{4}$$

Though the system is not so complicated but it is very difficult to obtain its approximate analytical solution by traditional nonlinear analytical techniques. According to the variational iteration method, we can construct the following correction functional;

$$y_{n+1}(t) = y_n(t) + \int_0^t [\frac{dy_i(t)}{dt} - D^{\alpha}\{f_i(t) + \int_0^{\xi} K_i(t,s)F(y(s))ds\}]d\xi \tag{5}$$

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Where $y_0(t)$ is an initial approximation and $D^\alpha = d^\alpha / dt^\alpha$ is Caputo fractional derivative, defined in (7) and $\alpha + \beta = 1$.

Example 1:

$$y^{0.75}(t) = (-\frac{t^2 e^t}{5})y(t) + \frac{6t^{2.25}}{\Gamma(3.25)} + \int_0^t e^t sy(s)ds, \tag{6}$$

Where

$$y(0) = 0.$$

Eq. (6) can be written in this form

$$y^{/}(t) = -D^{0.25}[(\frac{t^2 e^t}{5})y_n - \frac{6t^{2.25}}{\Gamma(3.25)} - \int_0^t e^t sy_n(s)ds] \tag{7}$$

Using He's variational iteration method [8], the correctional functional can be constructed as follows:

$$y_{n+1}(t) = y_n(t) + \int_0^t \lambda[\frac{dy_n}{d\xi} + D^{0.25}\{(\frac{t^2 e^t}{5})y_n - \frac{6t^{2.25}}{\Gamma(3.25)} - \int_0^\xi e^\xi sy_n(s)ds\}]d\xi \tag{8}$$

and its stationary conditions can be readily obtained as

$$\frac{\partial\lambda(\xi)}{\partial\xi} = 0 \tag{9}$$

$$1 + \lambda(\xi)\big|_{\xi = t} = 0$$

So the multiplier can be identified in the following form $\lambda = 1$

As a result, we obtain the following iteration formula

$$y_{n+1}(t) = y_n(t) - \int_0^t [\frac{dy_n}{d\xi} + D^{0.25}\{(\frac{\xi^2 e^\xi}{5})y_n - \frac{6\xi^{2.25}}{\Gamma(3.25)} - \int_0^\xi e^\xi sy_n(s)ds\}]d\xi \tag{10}$$

For $n = 0$ we start with $y_0 = 0$ after substituting it in Eq. (10) we get

$$y_1(t) = \int_0^t [D^{0.25}\frac{6\xi^{2.25}}{\Gamma(3.25)}]d\xi$$

$$y_1(t)\ t^3,$$

For $n = 1$

$$y_2(t) = y_1(t) - \int_0^t [\frac{dy_1}{d\xi} + D^{0.25}\{(\frac{\xi^2 e^\xi}{5})y_1 - \frac{6\xi^{2.25}}{\Gamma(3.25)} - \int_0^\xi e^\xi sy_1(s)ds\}]d\xi$$

$$y_2(t) = t^3$$

We therefore obtain the exact solution

$$y(t) = t^3$$

Example 2:

$$y^{0.5}(t) = (\cos t - \sin t)y(t) + f(t) + \int_0^t t\sin xy(x)dx, \tag{11}$$

With Initial Condition

$$y(0) = 0.$$

and

$$f(t) = \frac{2t^{1.5}}{\Gamma(2.5)} + \frac{t^{0.5}}{\Gamma(1.5)} + t(2 - 3\cos t - t\sin t + t^2\cos t) \tag{12}$$

Eq. (11) Implies That

$$y^{/}(t) = D^{0.5}[(\cos t - \sin t)y(t) + f(t) + \int_0^t t\sin xy(x)dx], \tag{13}$$

It's Correctional Functional is

$$y_{n+1}(t) = y_n(t) + \int_0^t \lambda[y^{/}{}_n(\xi) - D^{0.5}\{(\cos\xi - \sin\xi)y_n(\xi) + f(\xi) + \int_0^\xi \xi\sin xy_n(x)dx\}]d\xi \tag{14}$$

$$\frac{\partial\lambda(\xi)}{\partial\xi} = 0$$

$$1 + \lambda(\xi)\big|_{\xi = t} = 0$$

So the multiplier can be identified as: $\lambda = -1$, put this value in Eq. (14)

$$y_{n+1}(t) = y_n(t) - \int_0^t [y^{/}{}_n(\xi) - D^{0.5}\{(\cos\xi - \sin\xi)y_n(\xi) + f(\xi) + \int_0^\xi \xi\sin xy_n(x)dx\}]d\xi \tag{15}$$

Without loss of generality we can choose its initial approximation as $y_0 = 0$,

For $n = 0$, Eq. (15) takes the form

$$y_1(t) = y_0(t) - \int_0^t [y^{/}{}_0(\xi) - D^{0.5}\{(\cos\xi - \sin\xi)y_0(\xi) + f(\xi) + \int_0^\xi \xi\sin xy_0(x)dx\}]d\xi$$

$$y_1(t) = \int_0^t [D^{0.5}f(\xi)]d\xi$$

$$y_1(t) = \int_0^t D^{0.5}\{\frac{2\xi^{1.5}}{\Gamma(2.5)} + \frac{\xi^{0.5}}{\Gamma(1.5)} + \xi(2 - 3\cos\xi - \xi\sin\xi + \xi^2\cos\xi)\}d\xi$$

In order to avoid the difficult fractional integration we can simplify the integration by taking the truncated Taylor expansions for the trigonometric terms e.g.

$$\cos t \cong 1 - \frac{t^2}{2!} + \frac{t^4}{4!} \quad \text{and} \quad \sin t \cong t - \frac{t^3}{3!} + \frac{t^5}{5!}$$

$$y_1(t) = t^2 + t,$$

For $n = 1$

$$y_2(t) = y_1(t) - \int_0^t [y'_1(\xi) - D^{0.5}\{(\cos\xi - \sin\xi)y_1(\xi) + f(\xi) + \int_0^{\xi} \xi \sin x y_1(x) dx]\}] d\xi$$

$$y_2(t) = t^2 + t, \tag{16}$$

In View of Eq. (16), the Exact Solution

$$y(t) = t^2 + t$$

Example 3:

$$D_*^{\alpha} y(t) = 1 + \int_0^t e^{-x} y^2(x) dx \tag{17}$$

$$y(0) = 0$$

We can construct a variational iteration formula for Eq. (17) in the form

$$y_{n+1}(t) = y_n(t) - \int_0^t [\frac{dy_n}{d\xi} - D^{\alpha}\{1 + \int_0^{\xi} e^{-x} y_n^2(x) dx\}] d\xi \tag{18}$$

We Can Choose Initial Approximation As $y_0 = 0$.

For $n = 0$

$$y_1(t) = y_0(t) - \int_0^t [\frac{dy_0}{d\xi} - D^{\alpha}\{1 + \int_0^{\xi} e^{-x} y_0^2(x) dx\}] d\xi$$

$$y_1(t) = \int_0^t D^{\alpha}(1) d\xi$$

$$y_1(t) = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \tag{19}$$

For $n = 1$

$$y_2(t) = y_1(t) - \int_0^t [\frac{dy_1}{d\xi} - D^{\alpha}\{1 + \int_0^{\xi} e^{-x} y_1^2(x) dx\}] d\xi$$

For simplicity we can substitute the Taylor series of

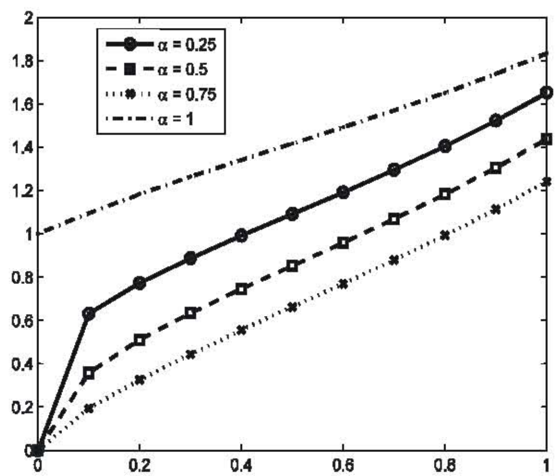
$$e^{-x} = 1 - x + \frac{x^2}{2!} + \ldots$$

$$y_2(t) = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} - \int_0^t [\frac{dy_1}{d\xi} - D^{\alpha}\{1 + \int_0^{\xi} (1 - x + \frac{x^2}{2!}) \left(\frac{x^{1-\alpha}}{\Gamma(2-\alpha)}\right)^2 dx\}] d\xi$$

$$y_2(t) = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{\Gamma(4-2\alpha)t^{4-3\alpha}}{(\Gamma(2-\alpha))^2 \Gamma(5-3\alpha)(3-2\alpha)} - \frac{\Gamma(5-2\alpha)t^{5-3\alpha}}{(\Gamma(2-\alpha))^2 \Gamma(6-3\alpha)(4-2\alpha)} + \frac{\Gamma(6-2\alpha)t^{6-3\alpha}}{(\Gamma(2-\alpha))^2 \Gamma(7-3\alpha)(5-2\alpha)} \tag{20}$$

This is good approximation of the exact solution. More iteration will give more accurate result.

Graphical Representation for different values of a is



Example 4: Consider the following system of integro differential equation

$$D_*^{\beta} y_1 = 1 + t + t^2 - y_2 - \int_0^t (y_1(x) + y_2(x)) dx, \tag{21}$$

$$D_*^{\beta} y_2 = -1 - t + y_1 - \int_0^t (y_1(x) - y_2(x)) dx,$$

Subject to the Initial Conditions

$$y_1(0) = 1, y_2(0) = -1. \tag{22}$$

Manipulating this in a way similar to that illustrated in Example 1, the correction functional can be written in the form

$$y_{1,n+1}(t) = y_{1,n}(t) + \int_0^t \lambda_1 [\frac{dy_{1,n}(\xi)}{d\xi} - D^{\alpha}(1 + t + t^2 - y_{2,n} - \int_0^{\xi} (y_{1,n}(x) + y_{2,n}(x)) dx)] d\xi$$

$$y_{2,n+1}(t) = y_{2,n}(t) + \int_0^t \lambda_2 [\frac{dy_{2,n}(\xi)}{d\xi} - D^{\alpha}(-1 - t + y_{1,n} - \int_0^{\xi} (y_{1,n}(x) - y_{2,n}(x)) dx)] d\xi \tag{23}$$

Imposing the stationary conditions on the correction functional, the Lagrange multipliers can be readily identified in the following form

$$\lambda_1 = \lambda_2 = -1.$$

As a result, we obtain the following iteration formulae

$$y_{1,n+1}(t) = y_{1,n}(t) - \int_0^t [\frac{dy_{1,n}(\xi)}{d\xi} - D^{\alpha}(1+t+t^2 - y_{2,n} - \int_0^{\xi}(y_{1,n}(x)+y_{2,n}(x))dx)]d\xi$$
$$y_{2,n+1}(t) = y_{2,n}(t) - \int_0^t [\frac{dy_{2,n}(\xi)}{d\xi} - D^{\alpha}(-1-t+y_{1,n} - \int_0^{\xi}(y_{1,n}(x)-y_{2,n}(x))dx)]d\xi \quad (24)$$

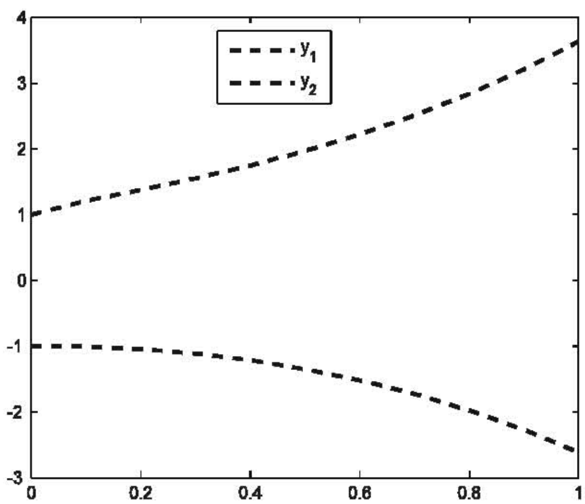
We can choose initial approximation as $y_{1,0} = 1$ and $y_{2,0} = 1-1$

For $n = 0$

$$y_{1,1}(t) = 1 + \frac{2t^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{t^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{2t^{3-\alpha}}{\Gamma(4-\alpha)} \quad (25)$$
$$y_{2,1}(t) = -1 - \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)}$$

For $n = 1$

$$y_{1,2} = 1 + \frac{2t^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{t^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{2t^{3-\alpha}}{\Gamma(4-\alpha)} + \frac{t^{3-2\alpha}}{\Gamma(4-2\alpha)} + \frac{2t^{4-2\alpha}}{\Gamma(5-2\alpha)},$$
$$y_{2,2} = -1 - \frac{3t^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{t^{2-2\alpha}}{\Gamma(3-2\alpha)} - \frac{t^{3-2\alpha}}{\Gamma(4-2\alpha)} - \frac{2t^{4-2\alpha}}{\Gamma(5-2\alpha)} - \frac{2t^{5-2\alpha}}{\Gamma(6-2\alpha)} \quad (26)$$



From above Figure, it is easy to conclude that the solution continuously depends on the time-fractional derivative.

CONCLUSION

In this paper, we have applied He's variational iteration method on some fractional integro differential equations. The obtained solutions show that He's method is very convenient and effective for fractional integro differential equation and obtained solutions has good agreement with exact ones.

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