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Couette and Poiseuille Flows for Fourth Grade Fluids Using Optimal Homotopy Asymptotic Method

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Abstract: In this paper, the steady flow of a fourth grade fluid, between two parallel plates is considered. Depending upon the relative motion of the plates we analyze four types of flows: Couette flow, plug flow, Poiseuille and generalized Couette flow. The nonlinear differential equation describing the velocity field is solved using Optimal Homotopy Asymptotic Method (OHAM). It is observed that the Optimal Homotopy Asymptotic Method is more efficient and flexible than the Perturbation and Homotopy Analyses Method.

Key words: Optimal Homotopy Asymptotic Method · Couette flow · Plug flow · Poiseuille and generalized Couette flow

INTRODUCTION

Plane Couette flows are generated by the action of boundaries in relative motion, i.e., fully-developed flows between parallel flat plates of infinite dimensions, driven by the steady motion of one of the plates. Flows between two parallel plates, or two coaxial cylinders or a flat plate and a convex cone with its top stirring the plate are the familiar examples of Couette flows [1, 2].

In recent years, there has been a great deal of interest in considerate the behavior of non-Newtonian fluids as they are used in many engineering processes. Also, non-Newtonian fluids are intensively studied by mathematicians, essentially from the point of view of differential equations theory. On the other hand, in applied sciences such as rheology or physics of the atmosphere, the approach to fluid mechanics is in an experimental setup leading to the measurement of material coefficients. Moreover, in theoretically studying how to predict the weather, ordinary differential equations represent the main tool. Further, since the failures in the predictions are strictly related to a chaotic behavior, one may find it unessential to ask whether the fluids are really Newtonian. Fluids which do not obey the Newton's law of viscosity are called as non-Newtonian fluids. Generally non-Newtonian fluids are complex mixtures slurries, pastes, plastics, gels, polymer solutions etc [3-5].

In the class of non-Newtonian fluids as second grade or higher orders fluids have different features. Rheological properties of such fluids are specified in general by their so-called constitutive equations. The Rivlin-Ericksen model [6] and Noll model [7] are among those that have established a considerable attention. Here, we accept the importance and simple model that has been used to describe the rheological characteristics exhibited by certain fluids is the fourth-grade fluid given in [8].

The fundamental governing equations for fluid motions are the Navier-Stokes equation. This inherently non-linear partial differential equation has no general solution and only a small number of exact solutions have been found because the non-linear inertial terms do not disappear automatically. These problems become even tricky to solve if non-Newtonian fluid flows are considered, since the equations of motion become highly non-linear. To solve practical problems, different perturbation techniques have been widely used in engineering and science [9 and the references therein]. Mostly, these perturbation techniques lead us some important and attractive results. However they can not be applied to all nonlinear problems. Therefore in the past few years, some new techniques have been developed to eliminate the “small parameter” assumption, such as the artificial parameter method proposed by Liu [10], the Homotopy analysis method by Liao [11], the variational iteration and Homotopy perturbation methods introduced

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by He [12-18]. Recently, Marinca *et al*. [20-23] developed a new technique. It is known as the Optimal Homotopy Asymptotic Method (OHAM). In their several papers Marinca et al. applied this technique to study different nonlinear boundary value problems of physical and engineering interest.

In this paper, the fluid between the plates is of fourth grade fluid for plane Couette flows having been studied. Four different problems depending upon the sliding plates are considered, i.e., Couette flow, plug flow, Poiseuille flow and generalized plane Couette flow. The best of our knowledge no efforts have been made using OHAM for the title problem. The paper is organized as follows: Section 2 contains the basic governing equations. In section 3 basic idea of Optimal Homotopy Asymptotic Method (OHAM) and the solution of the four mentioned problems using (OHAM) are given. Section 4 is reserved for conclusion.

Basic Equations: The basic equations governing the motion of an incompressible fluid in the absence of body forces and thermal effects are:

$$\mathrm{div}\mathbf{v} = 0 \tag{1}$$

$$\rho\frac{D\mathbf{v}}{\mathrm{dt}} = -\nabla p + \mathrm{div}\boldsymbol{\tau}, \tag{2}$$

Where: ρ is the constant density of the fluid, v is the velocity vector, p is the pressure, τ is the stress tensor and $\frac{D}{\mathrm{Dt}}$ denotes the total derivative.

The stress tensor τ for the fourth grade fluid is given by.

$$\tau = \sum_{k=1}^{4} D_k, \tag{3}$$

Where:

$$\begin{aligned}
&D_1 = \mu B_1, \quad D_2 = \alpha_1 B_2 + \alpha_2 B_1^2,\\
&D_3 = \beta_1 B_3 + \beta_2\left(B_1B_2 + B_2B_1\right) + \beta_3\left(trB_2\right)B_1,\\
&D_4 = \gamma_1 B_4 + \gamma_2\left(B_3B_1 + B_1B_3\right) + \gamma_3\left(B_2^2\right)\\
&+\gamma_4\left(B_2B_1^2 + B_1^2B_2\right) + \gamma_5\left(\left(trB_2\right)B_2\right)\\
&+\gamma_6\left(trB_2\right)B_1^2 + \left(\gamma_7 trB_3 + \gamma_8 tr\left(B_2B_3\right)\right)B_1,
\end{aligned} \tag{4}$$

in which μ is the coefficient of viscosity, $\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7$ and γ_8 are the material constants. The Rivlin-Ericksen tensors B_n are defined by $B_0 = I$, is the identity tensor and

$$\begin{aligned}
B_n &= \frac{D\boldsymbol{B}_{n-1}}{Dt} + \boldsymbol{B}_{n-1}\left(\nabla\mathbf{v}\right)\\
&+\left(\nabla\mathbf{v}\right)^{t}\boldsymbol{B}_{n-1}, \quad n \geq 1.
\end{aligned} \tag{5}$$

Since the flow is one dimensional, the velocity field is v = (u(y),0,0). The momentum Eq. (2) in component form becomes:

x- component of momentum equation:

$$\mu\frac{d^2u}{dx^2} + 6\left(\beta_2+\beta_3\right)\left(\frac{du}{dy}\right)^2\frac{d^2u}{dy^2} = \frac{dp}{dx}, \tag{6}$$

y-component:

$$\left(2\alpha_1+\alpha_2\right)\frac{d}{dy}\left(\left(\frac{du}{dy}\right)^2\right) + 4\left(\gamma_3+\gamma_4+\gamma_5+\frac{\gamma_6}{2}\right)\frac{d}{dy}\left(\left(\frac{du}{dy}\right)^4\right) = \frac{dp}{dy} \tag{7}$$

We define the generalized pressure p^* by the following relation:

$$p^* = -p + \left(2\alpha_1+\alpha_2\right)\left(\frac{du}{dy}\right)^2 + 4\left(\gamma_3+\gamma_4+\gamma_5+\frac{\gamma_6}{2}\right)\left(\frac{du}{dy}\right)^4 \tag{8}$$

Making use of Eq. (8) in Eq. (7) gives,

$$\frac{dp^*}{dy} = 0. \tag{9}$$

Eq. (9) shows that $p^* = p^*(x)$ Consequently, Eq. (6) reduces to the second ordinary non linear differential equation.

$$\mu\frac{d^2u}{dy^2} + 6\left(\beta_2+\beta_3\right)\left(\frac{du}{dy}\right)^2\frac{d^2u}{dy^2} = \frac{dp^*}{dx}. \tag{10}$$

Basic Idea of Oham: Here, we apply OHAM to the following differential equation:

$$L(u(y)) + g(y) + N(u(y)) = 0, \mathrm{B}\left(u, \frac{du}{dy}\right) = 0 \tag{11}$$

Where L is a linear operator, u(y) is an unknown function, g(y) is a known function, N is a nonlinear operator and B is a boundary operator.

According to OHAM we construct a homotopy $\phi(\mathrm{y}, p):\mathbb{R}\times[0,1]\rightarrow\mathbb{R}$ which satisfies

$$(1-p)[L(\phi(y,p)+g(y)]=H(p)[L(\phi(y,p) +g(y)+N(\phi(y,p))],\mathrm{B}\left(\phi(y,p),\frac{\partial\phi(y,p)}{\partial y}\right)=0 \tag{12}$$

Where $y\in\mathbb{R}$ and $p\in[0,1]$ is an embedding parameter, $H(p)$is a nonzero auxiliary function for $p\neq0$, $H(0) = 0$ and $\phi(y,p)$ is an unknown function. Obviously, when $p = 0$ and $p = 1$ it holds that $\phi(y,0) = u_\circ(y)$ and $\phi(y,1) = u(y)$ respectively.

Thus, as p varies from 0 to1, the solution $\phi(y,p)$ approaches from $u_\circ(y)$ to $u(y)$, where $u_\circ(y)$ is obtained from Eq (23) for $p = 0$:

$$L(u_\circ(y))+g(y)=0, \quad B\left(u_\circ,\frac{du_\circ}{dy}=0\right) \tag{13}$$

Next, we choose auxiliary function $H(p)$ in the form

$$H(p) = pc_1 + p^2c_2 + ... \tag{14}$$

Where $c_1,c_2,...$ are constants to be determined shortly.
To get an approximate solution, we expand $\phi(y,p,c_i)$ in Taylor's series about p in the following manner,

$$\phi(y,p,c_i)=u_\circ(y)+\sum_{k=1}^{\infty}u_k(y,c_1,c_2,...,c_k)p^k \tag{15}$$

Substituting Eq. (15) into Eq. (12) and equating the coefficient of like powers of p, we obtain the following linear equations.

Zeroth order problem is given by Eq. (13) and the first and second order problems are given by Eqs. (16-17) respectively:

$$L(u_1(y))+g(y)=c_1N_\circ(u_\circ(y)), \quad \mathrm{B}\left(u_1,\frac{du_1}{dy}\right)=0 \tag{16}$$

$$L(u_2(y))-L(u_1(y))=c_2N_\circ(u_\circ(y)) +c_1\left[L(u_1(y))+N_1(u_1(y),u_1(y))\right], \quad \mathrm{B}\left(u_2,\frac{du_2}{dy}\right)=0 \tag{17}$$

The general governing equations for $u_k(y)$ are given by:

$$L(u_k(y))-L(u_{k-1}(y))=c_kN_\circ(u_\circ(y)) +\sum_{i=1}^{k-1}c_i\begin{bmatrix}L(u_{k-i}(y))\\ +N_{k-i}\begin{pmatrix}u_\circ(y),u_1(y),...,\\ u_{k-1}(y)\end{pmatrix}\end{bmatrix}, \quad k=2,3,..., \ \mathrm{B}\left(u_k,\frac{du_k}{dy}\right)=0 \tag{18}$$

Where $N_m(u_\circ(y),u_1(y),...,u_{k-1}(y))$ is the coefficient of p^m in the expansion of $N(\phi(y,p))$ about the embedding parameter p.

$$N(\phi(y,p,c_i))=N_\circ(u_\circ(y)) +\sum_{m=1}^{\infty}N_m(u_\circ,u_1,u_2,...,u_m)p^m \tag{19}$$

It has been practical that the convergence of the series (15) depends upon the auxiliary constants $c_1,c_2,...$ If it is convergent at $p = 1$, one has

$$\tilde{u}(y,c_1,c_2,...,c_m)=u_\circ(y)+\sum_{i=1}^{m}u_i(y,c_1,c_2,...,c_i) \tag{20}$$

Substituting Eq. (20) into Eq. (10) it results the following expression for residual:

$$R(y,c_1,c_2,...,c_m)=L(\tilde{u}(y,c_1,c_2,...,c_m)) +g(y)+N(\tilde{u}(y,c_1,c_2,...,c_m)) \tag{21}$$

If $R = 0$, then $\tilde{u}$ will be the exact solution. Generally it doesn't happen, especially in non-linear problems.

There are many methods like Method of Least Squares, Galerkin's Method, Ritz Method and Collocation Method to find the optimal values of c_i, $i = 1,2,3,....$ We apply the Method of Least Squares as under:

$$J(c_1,c_2,...,c_m)=\int_a^b R^2(y,c_1,c_2,...,c_m)dy \tag{22}$$

Where a and b are properly chosen numbers in the domain of the problem.

$$\frac{\partial J}{\partial c_1}=\frac{\partial J}{\partial c_2}=...=\frac{\partial J}{\partial c_m}=0 \tag{23}$$

Where a and b are properly chosen numbers to locate the desired $c_i(i = 1,2,...,m)$. With these constants known, the approximate solution (of order m) is well-determined.

Plane Couette Flow Problem: Plane Couette flow, i.e., fully-developed flow between parallel flat plates of infinite dimensions, driven by the steady motion of one of the plates. (Such a flow is called shear-driven flow.) In this flow the upper wall is moving with constant speed U (so that it remains in the same plane) while the lower one is fixed. The pressure gradient is zero everywhere and the gravity term is neglected, so Eq. (10) becomes[1-2]:

$$\mu\frac{d^2u}{dy^2}+6(\beta_2+\beta_3)\left(\frac{du}{dy}\right)^2\frac{d^2u}{dy^2}=0, \tag{24}$$

with the boundary conditions $u(y) = 0$ at $y = 0$, $u(y) = U$ at $y = 2d$.

According to OHAM, we choose the linear and non linear operators as:

$$L\left(u(y,p)\right)=\frac{\partial^2 u(y,\ p)}{\partial y^2}, \tag{25}$$

$$N\left(u(y,\ p)\right)=\beta\left(\frac{\partial u(y,\ p)}{\partial y}\right)^2\frac{\partial^2 u(y,p)}{\partial y^2} \tag{26}$$

The boundary conditions are:

$$u(0, p) = 0,\ u(2d, p) = U \tag{27}$$

By Eq. (13) $g(y) = 0$.

We obtain the following three equations from Eqs. (18) and (19), (m=2)

$$u_\circ''(y) = 0,\ u_\circ(0) = 0,\ u_\circ(2d) = U, \tag{28}$$

$$u''_1(y)-u_\circ''(y)-c_1u_\circ''(y)-c_1\beta(u_\circ'(y))^2u_\circ''(y) = 0,$$
$$u_1(0) = 0,\ u_1(2d) = 0, \tag{29}$$

$$u_2''(y) - u_1''(y) - c_1u_1''(y) - 2c_1\beta u_\circ'(y)u_\circ''(y)u_1'(y) - c_2\beta(u_\circ'(y))^2u_\circ''(y) = 0$$
$$u_2(0) = 0, \qquad u_2(2d) = 0, \tag{30}$$

Eq. (28) has the following solution:

$$u_\circ(\mathrm{y})=\frac{U}{2d}y. \tag{31}$$

If this result is substituted into Eq. (20), we get the solution of Eq. (29):

$$u_1(y) = 0, \tag{32}$$

Using the expressions from (31) and (32) in Eq. (30), we obtained the second-order solution in the form.

$$u_2(y) = 0, \tag{33}$$

Therefore, the solution of the plane Couette flow upto second order takes the form.

$$\tilde{u}(y)=\frac{U}{2d}y. \tag{34}$$

The solution obtains from OHAM is identical with the solution obtained by PM, HPM and HAM [1, 2].

Plug Flow Problem: Plug flow is fully-developed flow induced between two infinite parallel plates, placed at a distance 2d apart, where both plates move with the same speed U. Assuming that the pressure gradient and the gravity in the x-direction are zero [1-3]. The boundary conditions are:

$$u(0) = U,\ u(2d) = U. \tag{35}$$

Here we apply the OHAM to solve problem (24) with the boundary conditions (35). Proceeding as before, we obtain Zeroth, first and second-order OHAM solutions in the form:

$$u_\circ(y) = U, \tag{36}$$

$$u_1(y) = U, \tag{37}$$

$$u_2(y) = U, \tag{38}$$

Substitution of Eqs. (36)-(38) into Eq. (20) yields the second order approximate analytic solution for the plug flow problem:

$$\tilde{u}(\mathrm{y})= U. \tag{39}$$

The solutions obtained here are similar with the solutions [1-2].

Fully Developed Plane Poiseuille Flow Problem: Plane Poiseuille flow, occurs when a liquid is forced between two stationary infinite flat plates, under constant pressure gradient $\partial p/\partial x$ and zero gravity. By taking the origin of the Cartesian coordinates to be on the plane of symmetry of the flow and by assuming that the distance between the two plates is 2d, the boundary conditions are: $\tau_{xy} = \mu\ du/dy = 0$ at $y = 0$ (Stationary) and $u = 0$ at $y = d$ (Stationary Plate). Note that the condition at $y = -d$ may be used instead of any of the above conditions [1-3]. The equation of motion (10) takes the form

$$\frac{d^2u}{dy^2} + \beta\left(\frac{du}{dy}\right)^2 \frac{d^2u}{dy^2} = A \tag{40}$$

Where $\beta = \frac{6(\beta_2+\beta_3)}{\mu}$, $A = \frac{dp^*/dx}{\mu}$ with the boundary conditions:

$$u(-d) = 0, \qquad u(d) = 0. \tag{41}$$

In this case, linear and non-linear operators are respectively:

$$L(u(y,p)) = \frac{\partial^2 u(y,p)}{\partial y^2},\ g(y) = 0 \tag{42}$$

$$N(u(y,\ p)) = \beta\left(\frac{\partial u(y,p)}{\partial y}\right)^2 \frac{\partial^2 u(y,p)}{\partial y^2}, \tag{43}$$

Zeroth-order problem given by Eq. (13) ($g(y) = 0$):

$$u_\circ''(y) - A = 0, \qquad u_\circ(-d) = u_\circ(d) = 0 \tag{44}$$

It is obtained that

$$u_\circ(y) = \frac{1}{2}\ (-A\,d^2 + A\,y^2) \tag{45}$$

First-order problem given by Eq. (16):

$$A + u_1''(y) - u_\circ''(y) - c_1 u_\circ''(y) - c_1 b(u_\circ'(y))^2 u_\circ''(y) = 0,$$
$$u_1(0) = 0,\ u_1(2d) = 0, \tag{46}$$

$$u_1(-d) = u_1(d) = 0$$

The solution of Eq. (46) is given by

$$u_1(y,\ c_1) = \frac{1}{2}\ (A^3\ y^4\ \beta\ c_1 - A^3\ d^4\ \beta\ c_1). \tag{47}$$

Second-order problem given by Eq. (17) for k=2:

$$u_2''(y) - u_1''(y) - c_1\,u_1''(y) - 2c_1\beta u_\circ'(y)\,u_\circ''(y)\,u_1'(y)$$
$$-c_2\beta\left(u_\circ'(y)\right)^2 u_\circ''(y) - c_1\beta u_1''(y)\left(u_\circ'(y)\right)^2 = 0 \tag{48}$$

$$u_2(-d) = u_2(d) = 0$$

with solution:

$$u_2(y, c_1, c_2) = \frac{1}{36}\big(-3\,A^3 d^4\beta c_1 + 3\,A^3 y^4\beta c_1 \tag{49}$$
$$-3\,A^3 d^4\beta c_1^2 + 3\,A^3 y^4\beta c_1^2 - 2\,A^5\,d^6\beta^2 c_1^2$$
$$+2\,A^5 y^6\beta^2 c_1^2 - 3\,A^3 d^4\beta c_2 + 3\,A^3\,y^4\beta c_2\big)$$

For the second order approximation solution, adding Eqs. (45), (47) and (49), we obtain:

$$\tilde{u}(y,c_1,c_2) = u_0(y) + u_1(y,c_1) + u_2(y,c_1,c_2)$$

$$\tilde{u}(y,c_1,c_2) = \frac{1}{36}\ \big(18A\left(y^2 - d^2\right) + 6A^3\beta c_1\left(y^4 - d^4\right)$$
$$+12\left(y^4 - d^4\right) + 8A^2\,\beta\left(y^6 - d^6\right)\ + 3c_2\left(y^4 - d^4\right)\big) \tag{50}$$

Substituting Eq. (50) in Eq. (21), we obtain the residual as:

$$R(z,c_1,c_2) = \tilde{u}''(y) + \beta\left(\tilde{u}'(y)\right)^2 \tilde{u}''(y) = 0 \tag{51}$$

For determination of constants ($c_i, i = 1,2$), we use the Method of Least Squares as under:

$$J(c_1,c_2) = \int_a^b R^2(y,c_1,c_2)\,dz \tag{52}$$

$$\frac{\partial J(c_1,c_2)}{\partial c_1} = \frac{\partial J(c_1,c_2)}{\partial c_2} = 0 \tag{53}$$

The solution of Fully developed plane Poiseuille flow problem upto second-order by

$A = 2,\ \beta = 0.2, d = 1,$
$c_1 = -0.5371148763117504$'

$c_2 = -0.03399665595169708.$

$$\tilde{u}(y) = \frac{1}{2}(-2 + 2y^2) + \frac{1}{12}(0.859384 - 0.859384y^4)$$
$$+\frac{1}{36}(0.618031 - 1.35657y^4 + 0.738541y^6). \tag{54}$$

Table 1

y	OHAM	PM	HAM
-1	0	0	0
-0.8	-0.310608	-0.33375	-0.335713
-0.6	-0.564425	-0.59174	-0.585074
-0.4	-0.753931	-0.7809	-0.768363
-0.2	-0.871391	-0.897987	-0.883683
0	-0.911217	-0.937778	-0.923342
0.2	-0.871391	-0.897987	-0.883683
0.4	-0.753931	-0.7809	-0.768363
0.6	-0.564425	-0.59174	-0.585074
0.8	-0.310608	-0.33375	-0.335713
1	0	0	0

Table 1 shows comparison of PM, HAM and OHAM for a fix data

$$A = 2,\ \beta = 0.2,\ \mathrm{d} = 1,$$
$$c_1 = -0.5371148763117504,$$
$$c_2 = -0.03399665595169708.$$

It is obvious that the result of OHAM (second order) is better than the method in reference [1].

Generralized Plane Couette Flow Problem: Consider again fully-developed plane Poiseuille flow with the upper plate moving with constant speed, U. This flow is called plane Couette-Poiseuille flow or general Couette flow. In contrast to the previous problem, this flow is not symmetric with respect to the centerline of the channel and, therefore, having the origin of the Cartesian coordinates on the centerline is not convenient. Therefore, the origin is moved to the lower plate. For this flow the governing problem consists of Eq. (40) and boundary conditions are:

$$u(0) = 0, \qquad u(2d) = U \tag{55}$$

Preceding as before, the zeroth-order, first-order and second order problems using OHAM are:

$$u_0''(y) - A = 0,\ \ u_0(0) = 0,\ u_0(2d) = U, \tag{56}$$

with solution

$$\begin{aligned} &A + u_1''(y) - u_0''(y) - c_1 u_0''(y) \\ &-c_1\beta(u_0'(y))^2\, u_0''(y) + c_1 A = 0, \end{aligned} \tag{57}$$

$$u(0) = 0, \qquad u_1(2d) = 0,$$

$$\begin{aligned} &u_2''(y) - u_1''(y) - c_1 u_1''(y) - 2c_1\beta\, u_0'(y) u_0''(y)\, u_1'(y) \\ &- c_2\beta(u_0'(y))^2\, u_0''(y)\, u_0''(y) - c_2\beta\, u_1''(y)\,(u_0'(y))^2 = 0, \\ &u_2(0) = 0, \qquad u_2(2d) = 0, \end{aligned} \tag{58}$$

As we applied the OHAM in the previous section, similarly in this case we follow the same steps to obtain the zeroth-order, first-order and second-order solutions:

$$u_0(y) = \frac{1}{2d}\left(A\, d\, y^2 - 2A\, d^2 y + Uy\right) \tag{59}$$

$$\begin{aligned} u_1(y, c_1) = \frac{1}{24d^2}\Big(&-8A^3 d^5 y\beta c_1 + 8A^2 d^3 U y\beta c_1 - 6AdU^2 y\beta c_1 \\ &+ 12A^3 d^4 y^2 \beta c_1 - 12A^2 d^2 U\, y^2 \beta c_1 + 3A\, U^2 y^2 \beta c_1 \\ &-8A^3\, d^3\, y^3\, \beta c_1 + 4\, A^2 d\, U\, y^3 \beta c_1 + 2A^3 d^2 y^2 \beta c_1\Big) \end{aligned} \tag{60}$$

$$\begin{aligned} u_2(y,c_1,c_2) = -\left(1/288d^4\right)\Big)A(2d-y)\,y\,\beta\Big(&12d^2\big(3U^2 \\ &+4A\, dU(-d+y) + 2A^2 d^2\big(2d^2 - 2d\,y + y^2\big)\big)c_1 \\ &+\big(16A^4 d^4\big(3d^2 - 3d\,y + y^2\big)\big(d^2 - d\,y + y^2\big)\beta \\ &-16A^3\, d^3\, U(d-y)\big(5d^2 - 6\,d\,y + 3\,y^2\big)\beta + 9U^2\big(4\,d^2 + U^2\beta\big) \\ &-12\,A\,d\,U(d-y)\big(4d^2 + 3U^2\beta\big) + 12A^2 d^2\big(2\,d^2\big(2\,d^2 - 2\,d\,y + y^2\big) \\ &+U^2\big(8\,d^2 - 10\,d\,y + 5\,y^2\big)\beta\big)\big)c_1^2 \ + 12\,d^2\big(3U^2 + 4\,A\,d\,U(-d+y) \\ &+2\,A^2\, d^2\big(2\,d^2 - 2\,d\,y + y^2\big)\big)c_2\Big) \end{aligned} \tag{61}$$

Thus, the final expression for the OHAM solution up to second order is:

$$\begin{aligned} \tilde{u}(y,c_1,c_2) = &\frac{1}{2d}\left(Ad\,y^2 - 2Ad^2 y + Uy\right) + \frac{1}{24d^2}\Big(-8A^3 d^5 y\beta c_1 \\ &+8A^2 d^3 U y\beta c_1 - 6AdU^2 y\beta c_1 + 12A^3 d^4 y^2 \beta c_1 - 12A^2 d^2 U\, y^2 \beta c_1 \\ &+3AU^2 y^2 \beta c_1 - 8A^3\, d^3\, y^3\, \beta c_1 + 4\,A^2 d\,U\,y^3 \beta c_1 + 2A^3 d^2 y^2 \beta c_1\Big) \\ &-\left(1/288d^4\right)\Big)A(2d-y)\,y\,\beta\Big(12d^2\big(3U^2 + 4AdU(-d+y) \\ &+2A^2\, d^2\big(2d^2 - 2d\,y + y^2\big)\big)c_1 \\ &+\big(16A^4 d^4\big(3d^2 - 3d\,y + y^2\big)\big(d^2 - d\,y + y^2\big)\beta \\ &-16A^3\, d^3\, U(d-y)\big(5d^2 - 6\,d\,y + 3\,y^2\big)\beta + 9U^2\big(4\,d^2 + U^2\beta\big) \\ &-12\,AdU(d-y)\big(4d^2 + 3U^2\beta\big) + 12A^2 d^2\big(2\,d^2\big(2d^2 - 2\,d\,y + y^2\big) \\ &+U^2\big(8\,d^2 - 10\,d\,y + 5\,y^2\big)\beta\big)\big)c_1^2 \ + 12d^2\big(3U^2 + 4\,AdU(-d+y) \\ &+2A^2\, d^2\big(2d^2 - 2\,d\,y + y^2\big)\big)c_2\Big) \end{aligned} \tag{62}$$

Substituting Eq. (62) in Eq. (40), we obtain the residual as:

Table 2

y	OHAM	PM	HAM
0	0	0	0
0.2	-0.222169	-0.238909	-0.236261
0.4	-0.376675	-0.403768	-0.398466
0.6	-0.456707	-0.492012	-0.485486
0.8	-0.457827	-0.500634	-0.496688
1	-0.379662	-0.430278	-0.431951
1.2	-0.226245	-0.286059	-0.29166
1.4	-0.00499515	-0.0751076	-0.0767076
1.6	0.275649	0.199168	0.211525
1.8	0.60996	0.546089	0.571189
2	1	1	1

$$R(z,c_1,c_2)=\tilde{u}''(y)+\beta\left(\tilde{u}'(y)\right)^2\tilde{u}''(z)=0 \quad (63)$$

For determination of constants (c_i, i = 1,2), we apply the procedure as in previous problem. For

$$A=2,\ \beta=0.2, d=1,\ U=1,$$
$$c_1=-0.643054,\ c_2=-0.040055.$$

$$\begin{aligned}\tilde{u}(y)=&y(-0.816185+y-0.298572\,y(5.415+(-3.8+y)\,y)\\&+0.551678(-2+y)\times(0.484+(-0.48+0.266667\,y)\,y\\&+0.0392305(3.33427+(-2.57942+y)\,y)(1.78778+(-1.12058+y)\,y)))\end{aligned} \quad (64)$$

Table 2 shows comparison of PM, HAM, OHAM For a fix data. For

$$A=2,\ \beta=0.2, d=1,\ U=1,\ c_1=$$
$$-0.643054,\ c_2=-0.040055.$$

The expressions (54) and (64) are plotted in figures (1)-(2). Figure 1 and 2 show a comparison between the results obtained using OHAM and the results obtained by Siddiqui *et al.* [1] who used PM and HAM for the same problem.

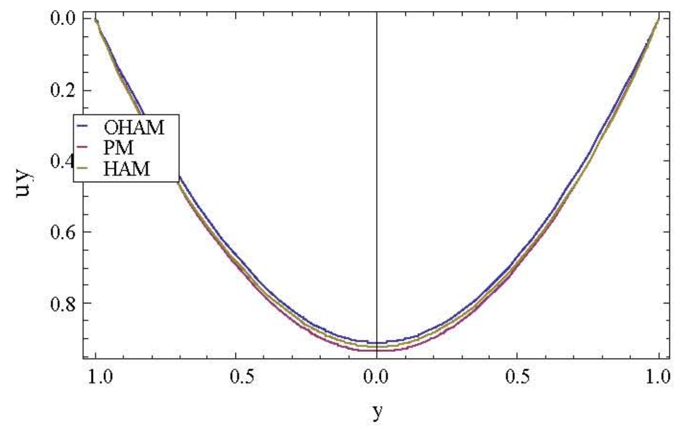

Fig. 1: Comparison between the results obtained from OHAM and Siddiqui *et al.* obtained from PM and HAM, A = 2, β = 0.2, d = 1, c_1 = -0.5371148763117504, c_2 = -0.03399665595169708.

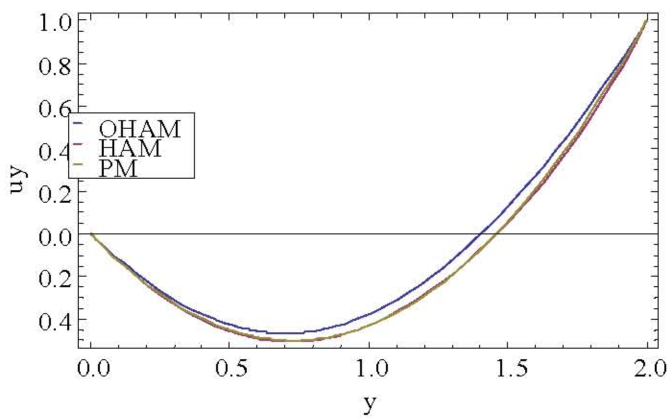

Fig. 2: Comparison between the results obtained from OHAM and Siddiqui *et al.* obtained from PM and HAM, $A=2, \beta=0.2, d=1, U=1, c_1=-0.643054, c_2=-0.040055.$

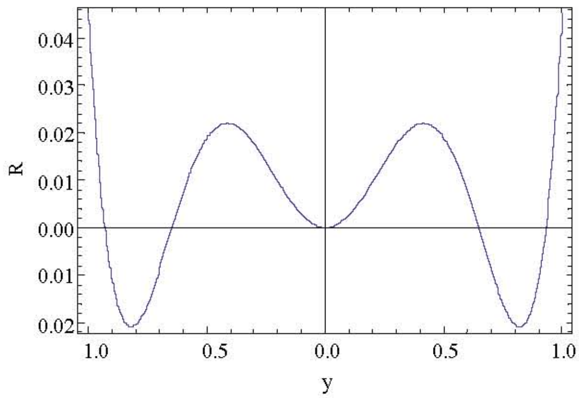

Fig. 3: Residual *R(y)* given by (51)

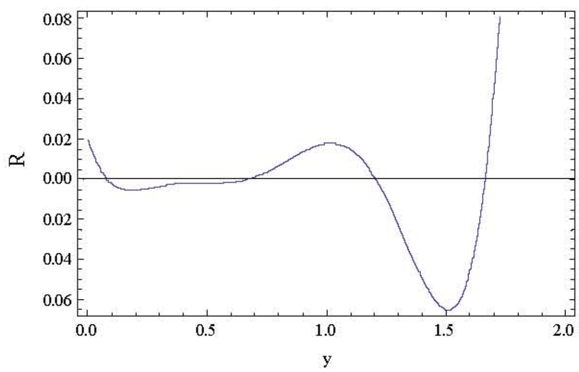

Fig. 4: Residual *R(y)* given by (61)

From Figures 3 and 4, we can observe the accuracy of the solution obtained by the title method, which in both cases are quite good. The maximum magnitude of the residual $R(y)$ is 0.045 for Poiseuille flow while 0.08 for the generalized plane Couette flow, which shows and proves the accuracy of the approximate solutions in both cases.

CONCLUSIONS

In this paper, an Optimal Homotopy Asymptotic Method is proposed for Couette flow; plug flow, Poiseuille and generalized Couette flow using fourth grade fluid, depending upon the relative motion of the plates. The results are compared with the homotopy analysis method and perturbation method and obtain satisfactory results. The result can be more improved by increasing the order.

This method provides us a convenient way to control the convergence and we can easily adjust the desired convergence regions. This technique is fast converging to the exact solution and requires less computational work. This confirms our belief that the efficiency of the OHAM gives it much wider applicability. Mathematica software is used for symbolic derivations of some of the equations.

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