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Application of Variational Homotopy Perturbation Method for Solving the Cauchy Reaction-diffusion Problem

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Abstract: In a recent paper, M.A. Noor *et al*. (Hindawi publishing corporation, Mathematical Problems in Engineering, Volume 2008, Article ID 696734, 11 pages, doi:10.1155/2008/696734) proposed the Variational Homotopy Perturbation Method (VHPM) for solving higher dimentional initial boundary value problems. In this paper, we use the proposed method to obtain the solution of Cauchy reaction-diffusion problem. Reaction-diffusion equations have special importance in engineering and sciences and constitute a good model for many systems in various fields.

Key words: Variational homotopy perturbation method . cauchy reaction-diffusion problem . time-dependent partial differential equation . lagrange multiplier

INTRODUCTION

Reaction-diffusion equations describe a wide variety of nonlinear systems in physics, chemistry, ecology, biology and engineering [1-4]. Reaction-diffusion equations are widely used as models for spatial effects in ecology. They support three important types of ecological phenomena: the existence of a minimal patch size necessary to sustain a population, the propagation of wavefronts corresponding to biological invasions and the formation of spatial patterns in the distributions of populations in homogeneous environments.

Reaction-diffusion equations can be analyzed by means of methods from the theory of partial differential equations and dynamical systems [5]. By a reaction-diffusion, we mean an equation of the following form

$$w_t = \Delta w + f(w, \nabla w; x, t) \tag{1}$$

The term Δw is diffusion term and f (w,∇w;x,t) is the reaction term. More generally the diffusion term may be of type A(w), where A is a second-order elliptic operator, which may be nonlinear and degenerate. In this paper, we consider the one-dimensional, time-dependent reaction-diffusion equation

$$w_t(x,t) = D\, w_{xx}(x,t) + r(x,t)\, w(x,t) \tag{2}$$

$$(x,t) \in \Omega \subset R^2$$

Where

$$w_t(x,t) = \frac{\partial w}{\partial t}(x,t), \quad w_{xx}(x,t) = \frac{\partial^2 w}{\partial x^2}(x,t)$$

w is the concentration, r is the Reaction parameter and D>0 is the diffusion coefficient, subject to the initial or boundary conditions

$$w(x,0) = g(x), \quad x \in R \tag{3}$$

$$w(0,t) = f_0(t), w_x(0,t) = f_1(t), \quad t \in R \tag{4}$$

The problem given by Eqs. (2) and (3) is called the characteristic Cauchy problem in the domain $\Omega = R \times R_+$, whilst the problem given by Eqs. (2) and (4) is called the non-characteristic Cauchy problem in the domain $\Omega = R_+ \times R$ [5].

In this paper, the Cauchy reaction-diffusion equation shall be solved by Variation Homotopy Perturbation Method (VHPM). The VHPM provides the solution in a rapid convergent series which may lead the solution in a closed form. It is worth mentioning that the VHPM is applied with out any discretization, restrictive assumption, or transformation and is free

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from round-off errors. Also the VHPM provides an analytical solution by using the initial conditions only and the boundary conditions can be used only to justify the obtained result. Numerical results reveal that the VHPM is easy to implement and reduces the computational work to a tangible level while still maintaining a very higher level of accuracy [6].

VARIATIONAL HOMOTOPY PERTURBATION METHOD

To convey the basic idea of the variational homotopy perturbation method, we consider the following general differential equation

$$Ly + Ny = g(x) \tag{5}$$

where L is a linear operator, N is a nonlinear operator and $g(x)$ is an inhomogeneous term. According to variational iteration method [7-11], we can construct a correct functional as follows

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\tau)\left\{L\, y_n + N\,\tilde{y}_n - g(\tau)\right\} d\tau \tag{6}$$

Where $\lambda(\tau)$ is a Lagrange multiplier [7-11] which can be identified optimally via the variational iteration method. The subscripts n denote the nth approximation, $\tilde{y}_n$ is considered as a restricted variation. That is $\delta\tilde{y}_n = 0$ and (6) is called a correct functional. Now, we apply the homotopy perturbation method

$$\sum_{i=0}^{\infty} p^i y_i = y_0 + p\int_0^x \lambda(\tau)\left\{N\left(\sum_{i=0}^{\infty} p^i\, \tilde{y}_i\right)\right\} d\tau - p\int_0^x \lambda(\tau) g(\tau)\, d\tau \tag{7}$$

which is the variational homotopy perturbation method and is formulated by the coupling of variational iteration method and Adomian's polynomials.

The embedding parameter $p \in [0,1]$ can be considered as an expanding parameter [12-18]. The homotopy perturbation method uses the homotopy parameter p as an expanding parameter [12-18] to obtain

$$f = \sum_{i=0}^{\infty} p^i y_i = y_0 + p\, y_1 + p^2\, y_2 + \cdots \tag{8}$$

If $p \to 1$, then (8) becomes the approximate solution of the form

$$y = \lim_{p\to 1} f = y_0 + y_1 + y_2 + \cdots \tag{9}$$

A comparison of like powers of p gives solutions of various orders.

VHPM FOR CAUCHY REACTION-DIFFUSION EQUATION

In order to solve Eq. (2) with initial condition (3) by means of VHPM, we choose the initial approximation

$$w_0(x,t) = g(x) \tag{10}$$

and we consider

$$L(w) = w_t(x,t) \tag{11}$$

$$N(w) = -D\, w_{xx}(x,t) - r(x,t)\, w(x,t) \tag{12}$$

where L is a linear and N is a nonlinear operator. According to the variational iteration method [7-11], we can construct a correct functional as follows

$$w_{n+1}(x,t) = w_n(x,t) + \int_0^t \lambda(\tau)\left\{w_{n_\tau}(x,\tau) - D\,\tilde{w}_{n_{xx}}(x,\tau) - r(x,\tau)\,\tilde{w}_n(x,\tau)\right\} d\tau \tag{13}$$

Where $\tilde{w}_n$ is considered as a restricted variation. Making the above functional stationary, the Lagrange multiplier can be determined as $\lambda = -1$, which yields the following iteration formula

$$w_{n+1}(x,t)=w_n(x,t)-\int_0^t \{w_{n_\tau}(x,\tau) - D\,w_{n_{xx}}(x,\tau)-r(x,\tau)w_n(x,\tau)\}\,d\tau \tag{14}$$

Applying the variational homotopy perturbation method, we have

$$\begin{aligned} w_0+p\,w_1+p^2w_2+\cdots &= w_0 \\ &+D\,p\int_0^t\left(w_{0_{xx}}+p\,w_{1_{xx}}+p^2w_{2_{xx}}+\cdots\right)d\tau \\ &+p\int_0^t r(x,\tau)\left(w_0+p\,w_1+p^2w_2+\cdots\right)d\tau \end{aligned} \tag{15}$$

Comparing the coefficient of like powers of p, we have

$$\begin{aligned} p^0 &: w_0(x,t)=g(x) \\ p^1 &: w_1(x,t)=D\int_0^t w_{0_{xx}}\,d\tau+\int_0^t r(x,\tau)w_0\,d\tau \\ p^2 &: w_2(x,t)=D\int_0^t w_{1_{xx}}\,d\tau+\int_0^t r(x,\tau)w_1\,d\tau \\ p^3 &: w_3(x,t)=D\int_0^t w_{2_{xx}}\,d\tau+\int_0^t r(x,\tau)w_2\,d\tau \\ &\vdots \end{aligned}$$

So we obtain the components which constitute $w(x,t)$, thus we will have

$$w(x,t)=w_0+w_1+w_2+\cdots$$

For later numerical computation, we let the expression

$$\varphi_n=\sum_{i=0}^{n} w_i(x,t) \tag{16}$$

to denote the n-term approximation to $w(x,t)$.

IMPLEMENTATION OF THE METHOD

In this section, we determine the reliability of the VHPM for different cases of $r(x,t)$. For the sake of comparison, we take the same examples as used in [5].

Example 1: Case r = constant. Taking $D=1$ and $r=-1$, Eq. (2) recasts as the Kolmogorov-Petrovsly-Piskunov (KPP) equation

$$w_t(x,t)=w_{xx}(x,t)-w(x,t),\quad (x,t)\in\Omega\subset R^2 \tag{17}$$

subject to the initial and boundary conditions

$$w(x,0)=e^{-x}+x=g(x),\quad x\in R \tag{18}$$

$$w(0,t)=1=f_0(t) \tag{19}$$

$$w_x(0,t)=e^{-t}-1=f_1(t),\quad t\in R \tag{20}$$

To solve equation (17) by means of VHPM, we can take an initial approximation

$$w_0(x,t)=w(x,0)=e^{-x}+x$$

Then by using the equation (15) we will have

$$\begin{aligned} w_0+p\,w_1+p^2w_2+\cdots &= e^{-x}+x \\ &+p\int_0^t\left(w_{0_{xx}}+p\,w_{1_{xx}}+p^2w_{2_{xx}}+\cdots\right)d\tau \\ &-p\int_0^t\left(w_0+p\,w_1+p^2w_2+\cdots\right)d\tau \end{aligned} \tag{21}$$

Comparing the coefficient of like powers of p, we have

$$p^0 : w_0(x,t)=e^{-x}+x$$

$$p^1 : w_1(x,t) = \int_0^t w_{0_{xx}}\, d\tau - \int_0^t w_0\, d\tau = -t\,x$$

$$p^2 : w_2(x,t) = \int_0^t w_{1_{xx}}\, d\tau - \int_0^t w_1\, d\tau = \frac{1}{2}t^2 x$$

$$p^3 : w_3(x,t) = \int_0^t w_{2_{xx}}\, d\tau - \int_0^t w_2\, d\tau = -\frac{1}{6}t^3 x$$

$$p^4 : w_4(x,t) = \int_0^t w_{3_{xx}}\, d\tau - \int_0^t w_3\, d\tau = \frac{1}{24}t^4 x$$

Table 1: The numerical results for φ_3 in comparison with the exact solution of w

tj |xi	0.1	0.2	0.3	0.4	0.5
0.1	4.0847e-007	8.1694e-007	1.2254e-006	1.6339e-006	2.0424e-006
0.2	6.4086e-006	1.2817e-005	1.9226e-005	2.5635e-005	3.2043e-005
0.3	3.1822e-005	6.3644e-005	9.5466e-005	1.2729e-004	1.5911e-004
0.4	9.8671e-005	1.9734e-004	2.9601e-004	3.9469e-004	4.9336e-004
0.5	2.3640e-004	4.7280e-004	7.0920e-004	9.4560e-004	1.2000e-003

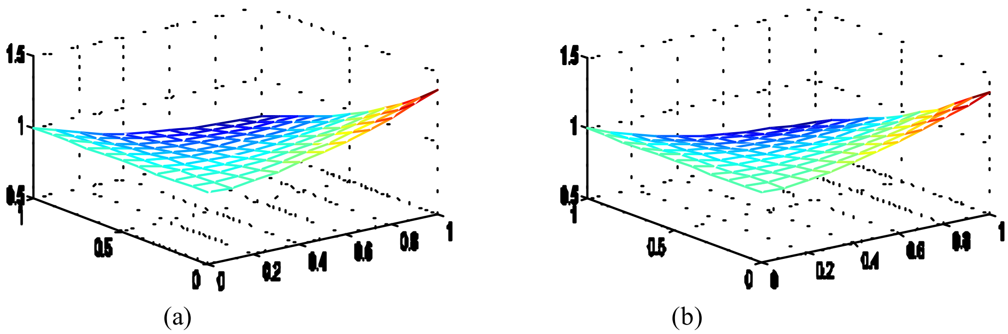


Fig. 1: Comparison between the (a) w(x,t) and (b) φ_3 for the values of t=0 (0.1) 1, x = 0 (0.1) 1 for example 1

So we obtain the components which constitute $w(x,t)$, thus we will have

$$w(x,t) = w_0(x,t) + w_1(x,t) + w_2(x,t) + \cdots$$
$$= e^{-x} + x\left(1 - t + \frac{1}{2!}t^2 - \frac{1}{3!}t^3 + \frac{1}{4!}t^4 - \cdots\right)$$

The exact value of $w(x,t)$ in a closed form is

$$w(x,t) = e^{-x} + x\,e^{-t} \tag{22}$$

as presented in [5].

In what follows, we present the absolute errors between φ_3 and the exact solution in Table 1 for the values of t=0.1 (0.1) 0.5 and x=0.1 (0.1) 0.5. Also the behavior of the solution obtained by VHPM and the exact solution is illustrated in figure 1.

Example 2: Case $r = r(t)$. Taking $D = 1$ and $r(t) = 2t$, Eq. (2) becomes

$$w_t(x,t) = w_{xx}(x,t) + 2t\,w(x,t), \quad (x,t) \in \Omega \subset R^2 \tag{23}$$

subject to the initial and boundary conditions

$$w(x,0) = e^x = g(x), \quad x \in R \tag{24}$$

$$w(0,t) = e^{t+t^2} = f_0(t) \tag{25}$$

$$w_x(0,t) = e^{t+t^2} = f_1(t), \quad t \in R \tag{26}$$

To solve equation (23) by means of VHPM, we can take an initial approximation

$$w_0(x,t) = w(x,0) = e^x$$

Then by using the equation (15) we will have

$$\begin{aligned} w_0 + p\,w_1 + p^2 w_2 + \cdots &= e^x \\ &+ p\int_0^t \left(w_{0_{xx}} + p\,w_{1_{xx}} + p^2 w_{2_{xx}} + \cdots\right) d\tau \\ &+ 2\,p\int_0^t \tau\left(w_0 + p\,w_1 + p^2 w_2 + \cdots\right) d\tau \end{aligned} \tag{27}$$

Comparing the coefficient of like powers of p, we have

$$p^0 : w_0(x,t) = e^x$$

$$p^1 : w_1(x,t) = \int_0^t w_{0_{xx}}\,d\tau + 2\int_0^t \tau\, w_0\,d\tau = e^x t(1+t)$$

$$p^2 : w_2(x,t) = \int_0^t w_{1_{xx}}\,d\tau + 2\int_0^t \tau\, w_1\,d\tau = \frac{1}{2}e^x t^2(1+t)^2$$

$$p^3 : w_3(x,t) = \int_0^t w_{2_{xx}}\,d\tau + 2\int_0^t \tau\, w_2\,d\tau = \frac{1}{6}e^x t^3(1+t)^3$$

$$p^4 : w_4(x,t) = \int_0^t w_{3_{xx}}\,d\tau + 2\int_0^t \tau\, w_3\,d\tau = \frac{1}{24}e^x t^4(1+t)^4$$

$$\vdots$$

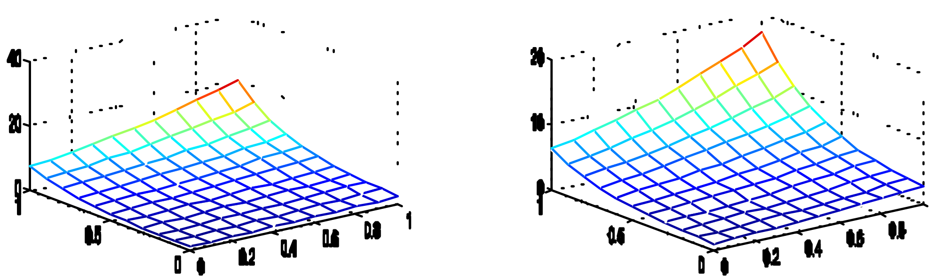
So we obtain the components which constitute $w(x,t)$, thus we will have

$$\begin{aligned} w(x,t) &= w_0(x,t) + w_1(x,t) + w_2(x,t) + \cdots \\ &= e^x\left(1 + t + \frac{3}{2!}t^2 + \frac{7}{3!}t^3 + \frac{25}{4!}t^4 + \cdots\right) \end{aligned}$$

The exact value of $w(x,t)$ in a closed form is

Table 2: The numerical results for φ_3 in comparison with the exact solution of w

tj |xi	0.1	0.2	0.3	0.4	0.5
0.1	6.8931e-006	7.6180e-006	8.4192e-006	9.3047e-006	1.0283e-005
0.2	1.6042e-004	1.7729e-004	1.9593e-004	2.1654e-004	2.3931e-004
0.3	1.2000e-003	1.3000e-003	1.4000e-003	1.6000e-003	1.7000e-003
0.4	5.1000e-003	5.6000e-003	6.2000e-003	6.9000e-003	7.6000e-003
0.5	1.7100e-002	1.8900e-002	2.0800e-002	2.3000e-002	2.5500e-002



(a) (b)

Fig. 2: Comparison between the (a) w(x,t) and (b) φ_3 for the values of t=0 (0.1) 1, x = 0 (0.1) 1 for example 2

$$w(x,t) = e^{x+t+t^2} \tag{28}$$

as presented in [5].

In what follows, we present the absolute errors between φ_3 and the exact solution in Table 2 for the values of t=0.1 (0.1) 0.5 and x=0.1 (0.1) 0.5. Also the behavior of the solution obtained by VHPM and the exact solution is illustrated in figure 2.

Example 3: Case $r = r(x)$. Taking $D = 1$ and $r(x) = -1 - 4x^2$, Eq. (2) becomes

$$w_t(x,t) = w_{xx}(x,t) - (1+4x^2)w(x,t)$$
$$(x,t) \in \Omega \subset R^2 \tag{29}$$

subject to the initial and boundary conditions

$$w(x,0) = e^{x^2} = g(x), \quad x \in R \tag{30}$$

$$w(0,t) = e^t = f_0(t) \tag{31}$$

$$w_x(0,t) = 0 = f_1(t), \quad t \in R \tag{32}$$

To solve equation (29) by means of VHPM, we can take an initial approximation

$$w_0(x,t) = w(x,0) = e^{x^2}$$

Then by using the equation (15) we will have

$$w_0 + p\,w_1 + p^2 w_2 + \cdots = e^{x^2}$$
$$+ p\int_0^t \left(w_{0_{xx}} + p\,w_{1_{xx}} + p^2 w_{2_{xx}} + \cdots\right) d\tau$$
$$- p\int_0^t (1+4x^2)\left(w_0 + p\,w_1 + p^2 w_2 + \cdots\right) d\tau \tag{33}$$

Comparing the coefficient of like powers of p, we have

$$p^0 : w_0(x,t) = e^{x^2}$$

$$p^1 : w_1(x,t) = \int_0^t w_{0_{xx}}\, d\tau - \int_0^t (1+4x^2) w_0\, d\tau = t\, e^{x^2}$$

$$p^2 : w_2(x,t) = \int_0^t w_{1_{xx}}\, d\tau - \int_0^t (1+4x^2) w_1\, d\tau = \frac{1}{2} t^2 e^{x^2}$$

$$p^3 : w_3(x,t) = \int_0^t w_{2_{xx}}\, d\tau - \int_0^t (1+4x^2) w_2\, d\tau = \frac{1}{6} t^3 e^{x^2}$$

$$p^4 : w_4(x,t) = \int_0^t w_{3_{xx}}\, d\tau - \int_0^t (1+4x^2) w_3\, d\tau = \frac{1}{24} t^4 e^{x^2}$$

$$\vdots$$

So we obtain the components which constitute $w(x,t)$, thus we will have

$$w(x,t) = w_0(x,t) + w_1(x,t) + w_2(x,t) + \cdots$$
$$= e^{x^2}\left(1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \frac{1}{4!}t^4 + \cdots\right)$$

The exact value of $w(x,t)$ in a closed form is

$$w(x,t) = e^{t+x^2} \tag{34}$$

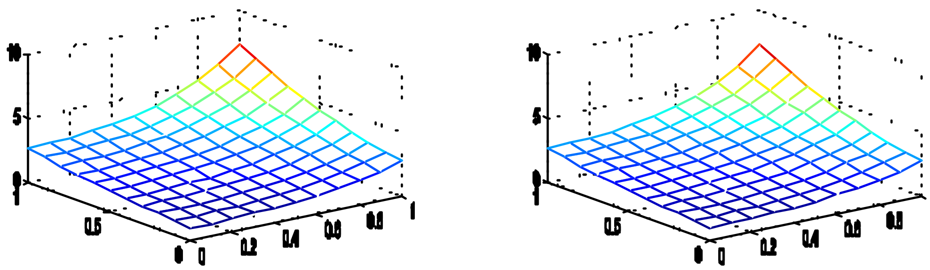
as presented in [5].

In what follows, we present the absolute errors between φ3 and the exact solution in Table 3 for the values of t=0.1 (0.1) 0.5 and x=0.1 (0.1) 0.5. Also the behavior of the solution obtained by VHPM and the exact solution is illustrated in figure 3.

Example 4: Case $r = r(x,t)$. Taking $D = 1$ and $r(x,t) = -4x^2 + 2t - 2$, Eq. (2) becomes

Table 3: The numerical results for φ_3 in comparison with the exact solution of w

tj |xi	0.1	0.2	0.3	0.4	0.5
0.1	4.2941e-006	4.4249e-006	4.6518e-006	4.9891e-006	5.4589e-006
0.2	7.0123e-005	7.2258e-005	7.5963e-005	8.1471e-005	8.9143e-005
0.3	3.6241e-004	3.7345e-004	3.9260e-004	4.2106e-004	4.6072e-004
0.4	1.2000e-003	1.2000e-003	1.3000e-003	1.4000e-003	1.5000e-003
0.5	2.9000e-003	3.0000e-003	3.2000e-003	3.4000e-003	3.7000e-003

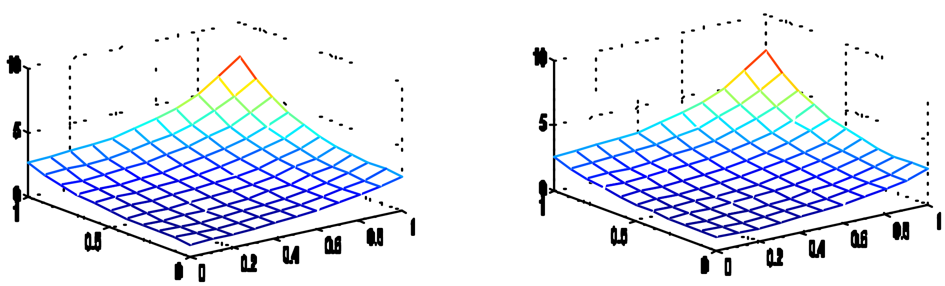


(a) (b)

Fig. 3: Comparison between the (a) w(x,t) and (b) φ_3 for the values of t=0 (0.1) 1, x=0(0.1) 1 for example 3

Table 4: The numerical results for φ_3 in comparison with the exact solution of w

tj |xi	0.1	0.2	0.3	0.4	0.5
0.1	4.2170e-010	4.3454e-010	4.5682e-010	4.8994e-010	5.3608e-010
0.2	1.0861e-007	1.1191e-007	1.1765e-007	1.2618e-007	1.3807e-007
0.3	2.8117e-006	2.8973e-006	3.0459e-006	3.2667e-006	3.5743e-006
0.4	2.8488e-005	2.9355e-005	3.0860e-005	3.3098e-005	3.6215e-005
0.5	1.7297e-004	1.7824e-004	1.8738e-004	2.0096e-004	2.1989e-004



(a) (b)

Fig. 4: Comparison between the (a) w(x,t) and (b) φ_3 for the values of t=0 (0.1) 1, x=0 (0.1) 1 for example 4

$$w_t(x,t) = w_{xx}(x,t) - \left(4x^2 - 2t + 2\right)w(x,t)$$
$$(x,t) \in \Omega \subset R^2 \tag{35}$$

subject to the initial and boundary conditions

$$w(x,0) = e^{x^2} = g(x), \quad x \in R \tag{36}$$

$$w(0,t) = e^{t^2} = f_0(t) \tag{37}$$

$$w_x(0,t) = 0 = f_1(t), \quad t \in R \tag{38}$$

To solve equation (35) by means of VHPM, we can take an initial approximation

$$w_0(x,t) = w(x,0) = e^{x^2}$$

Then by using the equation (15) we will have

$$w_0 + p\,w_1 + p^2 w_2 + \cdots = e^{x^2}$$
$$+ p\int_0^t \left(w_{0_{xx}} + p\,w_{1_{xx}} + p^2 w_{2_{xx}} + \cdots\right) d\tau$$
$$- p\int_0^t \left(4x^2 - 2\tau + 2\right)\left(w_0 + p\,w_1 + p^2 w_2 + \cdots\right) d\tau \tag{39}$$

Comparing the coefficient of like powers of p, we have

$$p^0 : w_0(x,t) = e^{x^2}$$

$$p^1 : w_1(x,t) = \int_0^t w_{0_{xx}}\, d\tau - \int_0^t \left(4x^2 - 2\tau + 2\right) w_0\, d\tau = t^2 e^{x^2}$$

$$p^2 : w_2(x,t) = \int_0^t w_{1_{xx}}\, d\tau - \int_0^t \left(4x^2 - 2\tau + 2\right) w_1\, d\tau = \frac{1}{2} t^4 e^{x^2}$$

$$p^3 : w_3(x,t) = \int_0^t w_{2_{xx}}\, d\tau - \int_0^t \left(4x^2 - 2\tau + 2\right) w_2\, d\tau = \frac{1}{6} t^6 e^{x^2}$$

$$p^4 : w_4(x,t) = \int_0^t w_{3_{xx}}\, d\tau - \int_0^t \left(4x^2 - 2\tau + 2\right) w_3\, d\tau = \frac{1}{24} t^8 e^{x^2}$$

$$\vdots$$

So we obtain the components which constitute $w(x,t)$, thus we will have

$$w(x,t) = w_0(x,t) + w_1(x,t) + w_2(x,t) + \cdots$$
$$= e^{x^2}\left(1 + t^2 + \frac{1}{2!}t^4 + \frac{1}{3!}t^6 + \frac{1}{4!}t^8 + \cdots\right)$$

The exact value of $w(x,t)$ in a closed form is

$$w(x,t) = e^{t^2 + x^2} \tag{40}$$

as presented in [5].

In what follows, we present the absolute errors between φ_3 and the exact solution in Table 4 for the values of t=0.1 (0.1) 0.5 and x=0.1 (0.1) 0.5. Also the behavior of the solution obtained by VHPM and the exact solution is illustrated in figure 4.

CONCLUSION

In this paper, Variational Homotopy Perturbation Method (VHPM) has been successfully applied to time-dependent reaction-diffusion equation. The proposed method is successfully implemented by using the initial conditions only. It is observed that the proposed scheme exploits full advantage of variational iteration method and the homotopy perturbation method. All the examples show that the results of the present method are in excellent agreement with those of exact solutions and the obtained solutions are shown graphically. Finally, we conclude that the VHPM may be considered as a nice refinement in existing numerical techniques. The computations in this paper are done by MATLAB software.

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