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On the Fuzzy Subhypergroups of Some Particular Complete Hypergroups (I)

Irina Cristea

DIEA, University of Udine, Via delle Scienze 206, 33100, Udine, Italy

Abstract: Any complete hypergroup H may be represented as the union $H = \bigcup_{g \in G} A_g$, where G is a group and the subsets A_g of H satisfy certain properties. With any hypergroupoid ⟨H, o⟩ we may associate a particular fuzzy set $\tilde{\mu}$. In this paper we determine sufficient and necessary conditions for a finite complete hypergroup ⟨H, o⟩, where $G \cong (Z_p,+)$ or $G \cong (Z_{2p},+)$, p a prime number, such that the fuzzy set $\tilde{\mu}$ is a fuzzy subhypergroup of H. Moreover, as an intermediate result, we give a new decomposition of the multiplicative group Zp\{0}, with p an odd prime.

Key words: complete hypergroup; fuzzy set; fuzzy subhypergroup 2000 MSC: 20N20, 03E72

INTRODUCTION

The algebraic hyperstructures, introduced by F. Marty [19] in 1934 at the 8^{th} Congress of Scandinavian Mathematicians, are widely studied from the theoretical point of view and for their applications to pure and applied mathematics. P. Corsini and V.Leoreanu in their book [3] present some of these applications in geometry, hypergraphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, codes, median algebras, artificial intelligence, probabilities.

The study of fuzzy algebraic structures started with the concept of fuzzy subgroup of a group introduced by A. Rosenfeld [21]: he showed that many results on group theory may be extended in a natural way to fuzzy groups. Later, B. Davvaz in [12] generalized the notion of fuzzy subgroup: he defined the concept of fuzzy subhypergroup of a hypergroup [13, 14].

Other relations between hyperstructures and fuzzy sets were considered by P. Corsini; he associated a join space with a fuzzy set [4] and then a fuzzy set, denoted $\tilde{\mu}$ with a hypergroupoid [4]. These connections lead to a sequence of fuzzy sets and join spaces, which ends if two consecutive join spaces are isomorphic. This argument has been studied in depth by the author in her Ph.D thesis: in [23] she has studied, together with M. Stefanescu, the above sequence in general and in [7] for a complete hypergroup; together with P. Corsini [6] the author determined the fuzzy grade of a particular non-complete 1-hypergroup. Moreover, the author has investigated [10] the sequences of join spaces associated with a hypergroupoid using fuzzy sets endowed with two membership functions. More connections between algebraic hyperstructures and fuzzy sets can be found in [1, 15, 16, 18]. Fuzzy theory is widly applicable in information gathering, modelling, analysis, optimisation, control, decision making, etc. example [17, 20, 22].

In this paper, we deal with the fuzzy subhypergroups of some particular finite complete hypergroups. More exactly, we see when the fuzzy set $\tilde{\mu}$ is a fuzzy subhypergroup of a complete hypergroup constructed with a group G isomorphic with one of the groups $(Z_p, +)$ or $(Z_{2p}, +)$, with p an odd prime. To obtain this, we used a new decomposition of the multiplicative group Zp\{0}.

PRELIMINARIES

First we recall some basic notions which will be used through out this paper. More details can be found in the books [2, 3].

A hypergroup ⟨H, o⟩ is a non empty set H endowed with a hyperoperation $o: H \times H \to \wp^*(H)$ which satisfies the conditions:

- For any $(x,y,z) \in H^3, (x \circ y) \circ z = x \circ (y \circ z)$ (the associativity);
- For any $x \in H, x \circ H = H \circ x = H$ (the reproducibility).

With any hypergroupoid ⟨H, o⟩, P.Corsini associated in [5] a fuzzy subset $\tilde{\mu}: H \to [0,1]$ in the following way: for any $u \in H$ we consider:

$$\tilde{\mu}(u) = \frac{\sum_{(x,y) \in Q(u)} \frac{1}{|x \circ y|}}{q(u)}, \qquad (\omega)$$

Corresponding Author: Dr. Irina Cristea, DIEA, University of Udine, Via delle Scienze 206, 33100, Udine, Italy

where $Q(u) = \{(a,b) \in H^2 \mid u \in a \circ b\}, q(u) = |Q(u)|$. If $Q(u) = \emptyset$, we set $\tilde{\mu}(u) = 0$.

Then, with any hypergroupoid H endowed with a fuzzy set, we can associate a hypergroup $\langle {}^1H, o_1 \rangle$ as follows [4]:

$$\forall (x,y) \in H^2, x \circ_1 y = \{z \in H \mid \tilde{\mu}(x) \wedge \tilde{\mu}(y) \leq \tilde{\mu}(z) \leq \tilde{\mu}(x) \vee \tilde{\mu}(y)\}$$

By using this construction and those of (ω) P.Corsini defined a sequence of fuzzy sets and special commutative hypergroups, called join spaces, which has been studied in [6-9, 23].

Definition 2.1: Let $\langle H, o \rangle$ be a hypergroup and let μ be a fuzzy subset of H. Then μ is called a fuzzy subhypergroup of H if the following axioms hold:

(FSH1). For any x, y∈H, $\mu(x) \wedge \mu(y) \leq \inf\{\mu(u), u \in x \circ y\}$.

(FSH2). For all x, a∈H, there exists $y_1 \in H$ such that $x \in a \text{ o } y_1$ and $\mu(a) \wedge \mu(x) \leq \mu(y_1)$.

(FSH3). For all x, a∈H, there exists $y_2 \in H$ such that $x \in y_2 \text{ o } a$ and $\mu(a) \wedge \mu(x) \leq \mu(y_2)$.

In what follows, when we prove that a fuzzy set in a fuzzy subhypergroup, we prove the first two conditions of the definition (the third one being similar).

Proposition 2.2: (Proposition 2, [12]) Let $\langle H, o \rangle$ be a hypergroup, where, for any x, y∈H, $x \circ y = \{z \in H \mid \mu(x) \wedge \mu(y) \leq \mu(z) \leq \mu(x) \vee \mu(y)\}$ and let μ be a fuzzy subset of H. Then μ is a fuzzy subhypergroup of $\langle H, o \rangle$.

SOME PROPERTIES OF COMPLETE HYPERGROUPS

Any complete hypergroup H may be represented as the union

$$H = \bigcup_{g \in G} A_g$$

where (G, •) is a group and the subsets $A_g \subset H$, for any g∈G, satisfy the conditions:

- For any $(g, g_2) \in G^2, g_1 \neq g_2$, we have $A_{g1} \cap A_{g2} = \emptyset$.
- If $(a,b) \in A_{g1} \times A_{g2}$, then $a \text{ o } b = A_{g1g2}$.

A significant property of the complete hypergroups we use through out this paper is given by the following result.

Proposition 3.1: [5, 7] Let H be a complete hypergroup. Then, for any u∈H, we have

$$\tilde{\mu}(u) = \frac{1}{|A_{g_u}|}$$

If |H| = |G| = n, then $A_{gi} = \{a_i\}$ for all $i \in \{1,2,...,n\}, a_i \in H$ and thus, for any $(a,b) \in H^2, |a \circ b| = 1$. Therefore H is a group. Moreover, any group is a complete hypergroup.

In the following we shall consider only finite complete hypergroups which are not groups, so $m = |G| < |H| = n$.

With any complete hypergroup H of order n,

$$H = \bigcup_{g \in G} A_g$$

where the group G has the elements $G = \{g_1, g_2, ..., g_m\}$ we associate an m-tuple, $2 \leq m \leq n-1$ denoted by $[k_1, k_2, \ldots, k_m]$, where, for any $i \in \{1,2,...,m\}, k_i = |A_{g_i}|$, $k_i \geq 1$ and

$$\sum_{i=1}^{m} k_i = n.$$

FUZZY SUBHYPERGROUPS OF THE COMPLETE HYPERGROUPS

Our goal is to determine the m-tuples $[k_1, k_2, \ldots, k_m]$ associated with finite complete hypergroups H, where the group G is isomorphic with one of the groups $(Z_p, +)$ or $(Z_{2p}, +)$, of the integers modulo p or 2p, with p a prime number, such that the membership function $\tilde{\mu}$ defined by (ω) is a fuzzy subhypergroup of H.

Proposition 4.1: Let H be a complete hypergroup such that $G \cong (Z_2, +)$. Then $\tilde{\mu}$ is a fuzzy subhypergroup of $\langle h, o \rangle$ if and only if the 2-tuple associated with H is $[k_0, k_1]$, with $1 \leq k_0 \leq k_1$.

Proof: Let us consider G = {0, g}, $H = A_0 \cup A_g$, where $|A_0| = k_0$ and $|A_g| = k_1$. We know that $\tilde{\mu}$ is a fuzzy subhypergroup of H if

a) For any x, y∈H and any $u \in x \circ y, \tilde{\mu}(u) \geq \tilde{\mu}(x) \wedge \tilde{\mu}(y)$;
b) For any x, a∈H, there exists y∈H such that $x \in a \text{ o } y$ and $\tilde{\mu}(y) \geq \tilde{\mu}(a) \wedge \tilde{\mu}(x)$.

Investigating the first condition, we obtain the following situations:

- If x, y$\in A_0$, then x o y = A_0 and for any $u \in x \circ y, \tilde{\mu}(u) = \tilde{\mu}(x) = \tilde{\mu}(y)$ and then $\tilde{\mu}(u) \geq \tilde{\mu}(x) \wedge \tilde{\mu}(y)$.
- If x, y$\in A_g$, then x o y = A_{2g} = A_0 and for any $u \in x \circ y, \tilde{\mu}(u) \geq \tilde{\mu}(x) \wedge \tilde{\mu}(y)$ if and only if $\frac{1}{k_0} \geq \frac{1}{k_1}$, i.e. $k_0 \leq k_1$.
- If x$\in A_0$ and y$\in A_g$ (or conversely), then x o y = A_g and for any, u$\in$x o y we have $\tilde{\mu}(u) = \tilde{\mu}(y) \geq \tilde{\mu}(x) \wedge \tilde{\mu}(y)$.

So the condition a) is true if and only if $k_0 \leq k_1$.
Now we analyse the second condition b):

- If a$\in A_0$, there exists y = x = H, such that x $\in$ a o y and $\tilde{\mu}(y) = \tilde{\mu}(x) \geq \tilde{\mu}(a) \wedge \tilde{\mu}(x)$, for any x$\in$H.
- If a$\in A_g$, for x$\in A_0$, there exists y = a such that $x \in a \circ y = a \circ a = A_{2g} = A_0$ and $\tilde{\mu}(a) \geq \tilde{\mu}(a) \wedge \tilde{\mu}(x)$; for a$\in A_g$, there exists y$\in A_1$, such that x$\in$a o y (by reproducibility) and $\tilde{\mu}(y) \geq \tilde{\mu}(a) \wedge \tilde{\mu}(x)$ is equivalent with $\frac{1}{k_0} \geq \frac{1}{k_1}$, i.e. $k_0 \leq k_1$,

In conclusion, $\tilde{\mu}$ is a fuzzy subhypergroup of H if and only if $k_0 \leq k_1$.

In the sequel, unless otherwise stated, p always represents any odd prime number.

Theorem 4.2: Let H be a complete hypergroup such that $G \cong (Z_p,+)$. Then $\tilde{\mu}$ is a fuzzy subhypergroup of $\langle H, o\rangle$ if and only if the p-tuple associated with H is $[k_0,\underbrace{k_1,k_1,\dots,k_1}_{p-1\ \text{times}}]$, with $1 \leq k_0 \leq k_1$.

Proof: Since, $G \cong (Z_p,+)$ we can consider $H = \bigcup_{i=0}^{p-1} A_{g_i}$, $G = \{g_0, g_1, \dots, g_{p-1}\}$, with g_0 the identity of the group G. For simplicity, we denote $k_{g_i} = k_{i(\mathrm{mod}\,p)} = k_i$. By the definition of a complete hypergroup, for any x$\in$H, there exists a unique i $\in$ {0,1,…,p-1} such that $x \in A_{g_i} \overset{not}{=} A_x \overset{not}{=} A_i$.

By Gauss' theorem, the group $U(Z_p) = (Z_p^*, \cdot)$ of the units of the integers modulo p is a cyclic group and we consider $U(Z_p) = <\hat{d}>$, $d \in \{1,2,...,p-1\}, (d,p-1) = 1$; therefore, for any $\hat{i} \in U(Z_p)$, we have $i \equiv d^{t}(\mathrm{mod}\,p)$.

We suppose that $\tilde{\mu}$ is a fuzzy subhypergroup of H.

First we prove, by induction on $s \in N^*$, that $k_i \geq k_{si}$, for any i $\in$ {1,2,…, p-1}. Set x, y$\in A_i$, then, for any $u \in x \circ y = A_{2i}$ we have $\tilde{\mu}(u) \geq \tilde{\mu}(x) \wedge \tilde{\mu}(y)$, that is $k_i \geq k_{2i}$. Now we suppose $k_i \geq k_{si}$ and we show $k_i \geq k_{(s+1)i}$: for $x \in A_i, y \in A_{si}$ and for any $u \in x \circ y = A_{(s+1)i}$, the relation $\tilde{\mu}(u) \geq \tilde{\mu}(x) \wedge \tilde{\mu}(y)$ means $\frac{1}{k_{(s+1)i}} \geq \frac{1}{k_i} \wedge \frac{1}{k_{si}} = \frac{1}{k_i}$, so $k_i \geq k_{(s+1)i}$. It follows $k_i \geq k_{di}$, i.e. $k_{d^t} \geq k_{d^{t+1}}$ and then $k_d \geq k_{d^2} \geq \dots \geq k_{d^{p-1}} = k_1 \geq k_{d^p} = k_d$. We used here Fermat' theorem: if p is a prime number and a and p are coprime, then $a^p \equiv a(\mathrm{mod}\,p)$.

In conclusion, for any

$$i \in \{1,2,...,p-1\}, k_i = k_1 \tag{4}$$

Moreover, for any $g \in G \setminus \{g_0\}$, $k_g = k_{p-g} = k_1$ and then, taking x$\in A_g$ and y$\in A_{p-g}$, for any $u \in x \circ y = A_0$ the relation $\tilde{\mu}(u) \geq \tilde{\mu}(x) \wedge \tilde{\mu}(y)$ means

$$k_0 \leq k_g = k_1 \tag{4.2}$$

Resuming, the first condition in the definition of a fuzzy subhypergroup leads to the following

p-tuple associated with

$$H: [k_0,\underbrace{k_1,k_1,\dots,k_1}_{p-1\ \text{times}}], \text{ with } 1 \leq k_0 \leq k_1 \tag{4.3}$$

Now we prove that, for these complete hypergroups, the second condition in the definition of a fuzzy subhypergroup is also verified (and similarly for the third condition).

Let us consider a$\in A_0$, then x$\in$a o x = A_x and taking x = y we have $\tilde{\mu}(y) \geq \tilde{\mu}(a) \wedge \tilde{\mu}(x)$.

Let us consider now a$\notin A_0$. By the reproducibility, there exists y$\in$H such that x$\in$a o y.

- If x$\in A_0$, it is clear that y$\notin A_0$ (if we suppose y$\in A_0$, it results $x \in a \circ y = A_a \neq A_0$, false), so $\tilde{\mu}(x) = \frac{1}{k_0}$, $\tilde{\mu}(a) = \tilde{\mu}(y) = \frac{1}{k_1}$ and it is verified that $\tilde{\mu}(y) \geq \tilde{\mu}(a) \wedge \tilde{\mu}(x)$.
- If a$\notin A_0$ we have $\tilde{\mu}(x) = \tilde{\mu}(a) = \frac{1}{k_1}$ and again the relation $\tilde{\mu}(y) \geq \tilde{\mu}(a) \wedge \tilde{\mu}(x)$ is verified, since $\tilde{\mu}(y) \in \left\{\frac{1}{k_0}, \frac{1}{k_1}\right\}$ and $k_0 \leq k_1$.

So, $\tilde{\mu}$ is a fuzzy subhypergroup of H if and only if the p-tuple associated with H is like in (4.3).

Proposition 4.3: Let H be a complete hypergroup such that $G \cong (Z_4, +)$. Then $\tilde{\mu}$ is a fuzzy subhypergroup of $\langle H, o\rangle$ if and only if the 4-tuple associated with H is $[k_0, k_1, k_2, k_1]$ with $1 \le k_0 \le k_2 \le k_1$.

Proof: We consider that $\tilde{\mu}$ is a fuzzy subhypergroup of $\langle H, o\rangle$, so it verifies the relation (FSH1). For simplicity, we denote $G = \{0, 1, 2, 3\}$ and let $[k_0, k_1, k_2, k_3]$ be the 4-tuple associated with H.

We set:

$x, y \in A_1$; then, for any $u \in x \circ y = A_2$, the relation (FSH1) means $k_1 \ge k_2$;

$x, y \in A_2$; then, for any $u \in x \circ y = A_0$ the relation (FSH1) means $k_2 \ge k_0$;

$x, y \in A_3$; then, for any $u \in x \circ y = A_1$ it follows by (FSH1) that $k_3 \ge k_2$;

$x \in A_1$, $y \in A_2$; then, for any $u \in x \circ y = A_3$ by (FSH1) it results $k_1 \ge k_3$;

$x \in A_2$, $y \in A_3$; then, for any $u \in x \circ y = A_1$ by (FSH1) we obtain $k_3 \ge k_2$;

thus $k_1 = k_3 \ge k_2 \ge k_0$.

It remains to prove that, under this condition, the hypergroup H satisfies also the property (FSH2). Indeed, we have to consider the situations:

1. If $a \in A_0$, there exists $x = y \in H$ such that $x \in x \circ y$ and $\tilde{\mu}(y) = \tilde{\mu}(x) \ge \tilde{\mu}(a) \wedge \tilde{\mu}(x)$.
2. If $a \in A_2$, by the reproducibility in H, there exists $y \in H$ such that $x \in a \circ y$.
   - For $a \in A_0$, we may take $y = a \in A_0$ and then $a \circ y = A_0, x \in A_0$ and $\tilde{\mu}(a) = \tilde{\mu}(y) \ge \tilde{\mu}(a) \wedge \tilde{\mu}(x)$;
   - For $x \in A_1$ (or $x \in A_3$), we may take $y \in A_3$ (or $y \in A_1$) and then $x \in a \circ y = A_1$ (or $x \in x \circ y = A_3$) and $\tilde{\mu}(y) = \tilde{\mu}(x) \ge \tilde{\mu}(a) \wedge \tilde{\mu}(x)$;
   - For $x \in A_2$, we take $y \in A_0$ and then $x \in a \circ y = A_2$ and $\tilde{\mu}(y) \ge \tilde{\mu}(a) \wedge \tilde{\mu}(x)$ is equivalent with $k_0 \ge k_2$;
3. Similarly if $a \in A_1$ or $a \in A_3$.

Thus, for any 4-tuple, $[k_0, k_1, k_2, k_1]$ with $1 \le k_0 \le k_2 \le k_1$ associated with a complete hypergroup H, the property (FSH2) is valid.

Next we analyse the same problem for a complete hypergroup $H = \bigcup_{g \in G} A_g$, where $G \cong (Z_{2p}, +)$, p an odd prime. First we present an helpful decomposition of the group $(Z_p^*, \cdot) = U(Z_p)$ of the units modulo p, with p an odd prime; this result help us to prove the principal theorem of this section.

Lemma 4.4: [11] We consider that $ord_{(Z_p^*, \cdot)} < \overline{2} > = d < p - 1$. If $[Z_p^* : < \overline{2} >] = t$, then we have the decomposition $Z_p^* = \bigcup_{i=1}^{t} M_i$, with $M_1 = < \overline{2} >$, for any $i \ne j$, $M_i \cap M_j = \varnothing$, for any $i \in \{1,2,...,t\}$, $|M_i| = d$ and denoting $M_i = \{e_{i1}, e_{i2}, ..., e_{id}\}$, the following statements hold:

- (L_1) for any $j \in \{1, 2,\ldots,d\}$, $2e_{ij} \equiv e_{i\,j+1}(mod p)$ and $2e_{id} \equiv e_{i1}(mod p)$;
- (L_2) for any $i \in \{1, 2,\ldots,t-1\}$, there exist $j_i \ne l_i \in \{1,2,...,d\}$ such that $e_{ij_i} + e_{il_i} \in M_{i+1}$ and there exist $j_t \ne l_t \in \{1,2,...,d\}$ such that $e_{tj_t} + e_{tl_t} \in M_1$.

Remark: If p is an odd prime, we have $2Z_p^* = Z_p^*$ and then we have also a decomposition of $2Z_p^*$ as in the previous lemma.

Now we investigate the complete hypergroups $H = \bigcup_{g \in G} A_g$, when $G \cong (Z_{2p}, +)$.

For simplicity, we set

$$G = \{0,1,2,...,2p-1\} = \{0\} \cup G^e \cup G^o$$

where $G^e = \{2,4,6,...,2(p-1)\}$

(the set of the even elements) and $G^o = \{1,3,5,...,2p-1\}$ (the set of the odd elements).

Lemma 4.5: Let $\langle H, o\rangle$ be a complete hypergroup such that $G \cong (Z_{2p}, +)$. If, for any $u \in x \circ y$, $\tilde{\mu}(u) \ge \tilde{\mu}(x) \wedge \tilde{\mu}(y)$ then, for any $i \in \{1,2,...,p-1\}$, we have $k_2 = k_{2i}$.

Proof: Since p is a prime number, the group U (Z_p) is cyclic and we know that $ord_{U(Z_p)}(\overline{2}) \le p - 1$. We prove the following two cases:

I) We suppose that $ord_{U(Z_p)}(\overline{2}) = p - 1$, that is $U(Z_p) = < \overline{2} >$. Then, for any $i \in \{1, 2,\ldots,p-1\}$, there exists $t \in \{1, 2,\ldots,p-1\}$ such that $2i \equiv 2^t (mod 2p) \Leftrightarrow 2i \equiv 2^t (mod p)$.

Set $x, y \in A_{2i}$; then for any $u \in x \circ y = A_{2^{t+1}(mod p)}$ we have $\tilde{\mu}(u) \ge \tilde{\mu}(x) \wedge \tilde{\mu}(y)$ if and only if $k_{2^t} \ge k_{2^{t+1}}$. Therefore

$$k_2 \ge k_{2^2} \ge \ldots \ge k_{2^{p-1}} \ge k_{2^p(\mathrm{mod}\,p)} = k_2,$$

that is $k_2 = k_{2i}$, for any $i\in\{1, 2,\ldots,p-1\}$.

II) Now we suppose that $\mathrm{ord}_{U(Z_p)}(\overline{2}) = d < p-1$. It is clear that $G^e = 2\overline{G}$, with $\overline{G} \cong U(Z_p)$ and then there exists a partition of G^e as in Lemma 4.4:

$$G^e = \bigcup_{i=1}^{t} M_i,\ M_i = \{e_{i1}, e_{i2}, \ldots, e_{id}\},\ \text{for any } i\in\{1, 2,\ldots, t\}.$$

With these notations we have $H = \bigcup A_{e_{ij}}$, $i\in\{1, 2,\ldots, t\}$, $j\in\{1, 2,\ldots, d\}$.
We fix $\overline{i} \in \{1,2,\ldots,t\}$ and we prove that

$$k_{e_{\overline{i}1}} \ge k_{e_{\overline{i}2}} \ge \ldots \ge k_{e_{\overline{i}d}} \ge k_{e_{\overline{i}1}},$$

so, for any $j\in\{1, 2,\ldots, d\}$, $k_{e_{\overline{i}1}} \ge k_{e_{\overline{i}j}}$ (4.4)

Indeed, for any $j\in\{1,2,\ldots, d-1\}$, set $x,y\in A_{e_{\overline{i}j}}$; for $u\in x\circ y = A_{2e_{\overline{i}j}} = A_{e_{\overline{i}(j+1)}}$ (from (L1)), we have $\tilde{\mu}(u) \ge \tilde{\mu}(x)\wedge\tilde{\mu}(y)$ if and only if $k_{e_{\overline{i}j}} \ge k_{e_{\overline{i}(j+1)}}$. Now set $j = d$ and $x,y\in A_{e_{\overline{i}d}}$; again, for $u\in x\circ y = A_{2e_{\overline{i}d}} = A_{e_{\overline{i}1}}$ (from (L1)), we have $\tilde{\mu}(u) \ge \tilde{\mu}(x)\wedge\tilde{\mu}(y)$ if and only if $k_{e_{\overline{i}d}} \ge k_{e_{\overline{i}1}}$ and now, the relation (4.4) is proved.

Next, using the properties (L2), we shall prove that, for any $i\in\{1, 2,\ldots,t\}$ and any $j\in\{1, 2,\ldots,d\}$,

$$k_{e_{ij}} = k_{e_{11}} = k_2. \qquad (4.5)$$

First we prove that, for any $i \in \{1,2,\ldots,t-1\}$, $k_{e_{i1}} \ge k_{e_{(i+1)1}}$.

Since (L2), for any $i \in \{1,2,\ldots,t-1\}$, there exist $j_i \ne l_i \in \{1,2,\ldots,d\}$, such that $e_{ij_i} + e_{il_i} \in M_{i+1}$.

Set $x\in A_{e_{ij_i}}$ and $y\in A_{e_{il_i}}$; then, for $u\in x\circ y = A_{e_{ij_i}+e_{il_i}} = A_{e_{(i+1)s}}$, with $s\in\{1,2,\ldots,d\}$ (from (L1)), we have $\tilde{\mu}(u) \ge \tilde{\mu}(x)\wedge\tilde{\mu}(y)$ if and only if $k_{e_{ij_i}} \ge k_{e_{(i+1)s}}$ and since (4.4) this is equivalent with $k_{e_{i1}} \ge k_{e_{(i+1)1}}$. Now set $i = t$; since (L2), there exist $j_t \ne l_t \in \{1,2,\ldots,d\}$ such that $e_{tj_t} + e_{tl_t} \in M_1$. Set $x \in A_{e_{tj_t}}$ and $y\in A_{e_{tl_t}}$; then, for $u\in x\circ y = A_{e_{tj_t}+e_{tl_t}} = A_{e_{1s}}$, with $s\in\{1,2,\ldots,d\}$ (from (L1)), we have $\tilde{\mu}(u) \ge \tilde{\mu}(x)\wedge\tilde{\mu}(y)$ if and only if $k_{e_{tj_t}} \ge k_{e_{1s}}$ which is equivalent (from (4.4)) with $k_{e_{t1}} \ge k_{e_{11}}$.

So, we obtained that, for any $i\in\{1,2,\ldots, t\}$ and $j\in\{1,2,\ldots,d\}$, $k_{e_{ij}} = k_{e_{11}}$ We note that $M_1 = \{e_{11}, e_{12},\ldots,e_{1d}\} = \{2,4,\ldots,2^d\}$, therefore, for any $i\in\{1,2,\ldots,t\}$ and $j\in\{1,2,\ldots,d\}$, $k_{e_{ij}} = k_2$ and since $e_{ij}\in G^e$ (by (4.4) and (4.5)), we conclude that, for any $i\in\{1,2,\ldots,p-1\}$, $k_2 = k_{2i}$. Now we may give the main result of this section.

Theorem 4.6: Let $H = \bigcup_{g\in G} A_g$ be a complete hypergroup such that G is isomorphic with $(Z_{2p},+)$, p an odd prime. Then $\tilde{\mu}$ is a fuzzy subhypergroup of $\langle H, \circ\rangle$ if and only if the 2p-tuple associated with H has the form

$$[k_0, \underbrace{k_1, k_2, k_1, k_2, \ldots, k_1, k_2, k_p, k_2, k_1, k_2, \ldots, k_1, k_2, k_1}_{2p-1\ \text{times}}],$$

with

$$k_0 \le k_2 \le k_1 \le k_2 \vee k_p \qquad (4.6)$$

$$k_0 \le k_p \le k_1 \qquad (4.7)$$

Proof: Let $[k_0, k_1, k_2, \ldots, k_{2p-1}]$ be the 2p-tuple associated with the complete hypergroup H and we suppose that $\tilde{\mu}$ is a fuzzy subhypergroup of H. Then $\tilde{\mu}$ verifies the conditions (FSH1) and (FSH2) from Definition 2.5. By Lemma 4.5, for any $i\in\{1,2,\ldots,p-1\}$, $k_2 = k_{2i}$.
Set $x, y\in A_1$ and $u\in x\circ y = A_2$; using (FSH1),

we obtain $\frac{1}{k_2} \ge \frac{1}{k_1} \wedge \frac{1}{k_1}$, so $k_1 \ge k_2$ (4.8)

Set $x, y\in A_p$ and $u\in x\circ y = A_0$; since (FSH1)

We obtain $\frac{1}{k_0} \ge \frac{1}{k_p} \wedge \frac{1}{k_p}$, so $k_p \ge k_0$. (4.9)

Set $x\in A_1$, $y\in A_{2j}$, with $j\in\{1,2,\ldots,p-1\}$ and set $u\in x\circ y = A_{2j+1}$; since (FSH1) we obtain

$$\frac{1}{k_{2j+1}} \ge \frac{1}{k_1} \wedge \frac{1}{k_{2j}} = \frac{1}{k_1} \wedge \frac{1}{k_2} \text{ and thus, since } \qquad (4.8)$$

$$k_1 \ge k_{2j+1} \qquad (4.10)$$

We have proved the relation (4.7).

Set $x\in A_2$, $y\in A_{2p-2}$ and $u\in x\circ y = A_0$; then, the relation (FSH1) and Lemma 4.5 lead to

$$\frac{1}{k_0} \geq \frac{1}{k_2} \wedge \frac{1}{k_2}, \text{ so } k_2 \geq k_0 \quad (4.11)$$

Set $x, y \in A_{2j+1}$, $j < p, 2j+1 \neq p$ and set $u \in x \circ y = A_{2(2j+1)}$; again by (FSH1) we obtain $\frac{1}{k_{2(2j+1)}} \geq \frac{1}{k_{2j+1}} \wedge \frac{1}{k_{2j+1}}$ and thus, by Lemma 4.5, $k_{2j+1} \geq k_2$ (4.12)

Similarly, setting $x \in A_{2j+1}$, $y \in A_{2(p-j)}$, with $0<j<p$, we obtain by (FSH1) and (4.12) that

$$k_{2j+1} \geq k_1 \quad (4.13)$$

therefore, by (4.10) and (4.13), it results, for any $j<p$, with $2j+1 \neq p$, that

$$k_1 \geq k_{2j+1} \quad (4.14)$$

Finally, setting $x \in A_2$, $y \in A_p$ we obtain

$$k_1 \leq k_2 \vee k_p \quad (4.15)$$

By (4.11), (4.8) and (4.15), it results also the relation (4.6).

In conclusion, $\tilde{\mu}$ verifies the relation (FSH1) if and only if the 2p-tuple associated with the complete hypergroup H has the form

$$[k_0, \underbrace{k_1, k_2, k_1, k_2, \ldots, k_1, k_2, k_p, k_2, k_1, k_2, \ldots, k_1, k_2, k_1}_{2p-1 \text{ times}}]$$

under the conditions (4.6) and (4.7).

To complete the proof, we have to show that, for such complete hypergroups, also the relation (FSH2) (and similarly (FSH3)) is satisfied. We analyse the following cases :

1. If $a, x \in A_i$, with $i \in \{0,1,2,...,2p-1\}$, then $\tilde{\mu}(a) = \tilde{\mu}(x) = \frac{1}{k_i}$ and there exists $y \in A_0$ such that $x \in a \circ y = A_i$ and $\tilde{\mu}(y) = \frac{1}{k_0} \geq \frac{1}{k_i} = \tilde{\mu}(a)$
2. If $a \in A_{2i+1}$, with $2i+1 \neq p$ and $x \in A_{2j}$ (with i, $j<p$), then $\tilde{\mu}(a) = \frac{1}{k_1}$ and $\tilde{\mu}(x) = \frac{1}{k_2}$ and therefore there exists $l<p$ such that $y \in A_{2l+1}$ (so $\tilde{\mu}(y) = \frac{1}{k_1}$ or $\tilde{\mu}(y) = \frac{1}{k_p}$) and $x \in a \circ y = A_{2j}$,

$$\tilde{\mu}(y) = \frac{1}{k_1} \geq \frac{1}{k_1} \wedge \frac{1}{k_2} = \frac{1}{k_1} = \tilde{\mu}(a) \wedge \tilde{\mu}(x)$$

or

$$\tilde{\mu}(y) = \frac{1}{k_p} \geq \frac{1}{k_1} \wedge \frac{1}{k_2} = \frac{1}{k_1} = \tilde{\mu}(a) \wedge \tilde{\mu}(x)$$

equivalent with $k_p \leq k_1$, relation which is verified.

3. Similarly, if $x \in A_{2i+1}$, with $2i+1 \neq p$ and $a \in A_{2j}$ (with i, $j<p$).
4. If $a \in A_{2i}$ and $x \in A_{2j}$ (with i, $j<p$), then $\tilde{\mu}(a) = \tilde{\mu}(x) = \frac{1}{k_2}$ and there exists $l<p$ such that $y \in A_{2l}$ (so $\tilde{\mu}(y) = \frac{1}{k_2}$) and $x \in a \circ y = A_{2j}$, $\tilde{\mu}(y) = \tilde{\mu}(a) = \tilde{\mu}(x)$; thus $\tilde{\mu}(y) \geq \tilde{\mu}(a) \wedge \tilde{\mu}(x)$.
5. If $a \in A_{2i+1}$ and $x \in A_{2j+1}$ (with i, $j<p$ and $2i+1 \neq p \neq 2j+1$), then $\tilde{\mu}(a) = \tilde{\mu}(x) = \frac{1}{k_1}$ and there exists $l<p$ such that $y \in A_{2l}$ (so $\tilde{\mu}(y) = \frac{1}{k_2}$) and $x \in a \circ y = A_{2j+1}$ and thus $\tilde{\mu}(y) = \frac{1}{k_2} \geq \frac{1}{k_1} = \tilde{\mu}(a) \wedge \tilde{\mu}(x)$.
6. Finally, if $a \in A_p$, then
   - for $x \in A_{2i}$, there exists $j<p$ such that $y \in A_{2j+1}$ and $x \in a \circ y = A_{2i}$,

$$\tilde{\mu}(y) = \frac{1}{k_1} \geq \frac{1}{k_p} \wedge \frac{1}{k_2} = \tilde{\mu}(a) \wedge \tilde{\mu}(x),$$

i.e. $k_1 \leq k_2 \vee k_p$, so (4.15).

for $x \in A_{2i+1}$, there exists $j<p$ such that $y \in A_{2j}$ and $x \in a \circ y = A_{2i+1}$,

$$\tilde{\mu}(y) = \frac{1}{k_2} \geq \frac{1}{k_p} \wedge \frac{1}{k_1} = \frac{1}{k_1} = \tilde{\mu}(a) \wedge \tilde{\mu}(x), \text{ so } (4.8).$$

This ends the proof.

CONCLUSIONS AND FUTURE WORK

With any complete hypergroup H of order n, $H = \bigcup_{g \in G} A_g$, $G = \{g_1, g_2, \ldots, g_m\}$ we may associate an m-tuple, $2 \leq m \leq n-1$, denoted by $[k_1, k_2, \ldots, k_m]$. In this paper we have determined all such m-tuples such that the fuzzy set $\tilde{\mu}$ is a fuzzy subhypergroup of the hypergroup H, where the group G is isomorphic with the groups $(Z_p, +)$ or $(Z_{2p}, +)$, with p an odd prime.

It would be interesting to generalize this problem: to find the m-tuples $[k_1, k_2, \ldots, k_m]$ for other particular complete hypergroups, for example for those obtained considering the group G isomor-phic with $(Z_{pq}, +)$, with p and q odd primes, or more general, taking the group G isomorphic with $(Z_p, +)$, $n \in N$, $n \geq 3$. These generalizations will be considered in a future work [9].

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