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A Partial Backlogging Mathematical Model under Variable Inflation and Demand with Considering Deterioration Cost

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Abstract: The practical experiences reveal that the inflation is non-deterministic and variable. Therefore, a mathematical model for the optimal production for an inventory control system of deteriorating items is formulated under time-varying and stochastic inflation environment. The demand rate is a function of inflation and the time horizon is finite. In the real situation, some but not all customers will wait for backlogged items during a shortage period, such as for fashionable commodities or high-tech products with the short product life cycle. The longer the waiting time is, the smaller the backlogging rate would be. According to such phenomenon, taking the backlogging rate into account is necessary. Thus, the model incorporates partial backlogging. The total cost consists of the deterioration cost, production cost, inventory holding cost, backordering cost, lost sale cost and ordering cost is formulated as an optimal control problem. The numerical example has been provided for evaluation and validation of the theoretical results and some special cases of the model are discussed.

Key words: Inventory . inflation-dependent demand . stochastic . shortages . deterioration

INTRODUCTION

The problem of inventory systems under inflationary conditions has received attention in recent years. Due to high inflation and consequent sharp decline in the purchasing power of money, especially in the developing countries, the financial situation has been changed and so it is impossible to ignore the effect of inflation and time value of money any further. Before the 1990s, the earlier efforts have been considered simple situations. Buzacott [1] made an Economic Order Quantity (EOQ) model with inflation subject to different types of pricing policies. Misra [2] developed a discounted-cost model and included internal (company) and external (general economy) inflation rates for various costs associated with an inventory system.

Inventoried goods can be broadly classified into four meta-categories:

Deterioration: Deterioration is defined as any process that prevents an item from being used for its intended original use, such as: (1) spoilage, as in perishable foodstuffs, fruits and vegetables; (ii) Physical depletion, as in pilferage or evaporation of volatile liquids such as gasoline and alcohol; (iii) decay, as in radioactive substances, degradation, as in electronic component, or loss of potency as in photographic films and pharmaceutical drugs.

Obsolescence: It refers to items that lose their value through time due to changes in fashion, rapid changes of technology or the introduction of a new product by a competitor.

Amelioration: It refers to items whose value or utility or quantity increase with time. It is a practical experience in wine manufacturing industry that utility or value of some kind of wine increases by age. Other examples can be high breed fishes in breeding yard (fish culture facility) or fast growing animals like the broiler, pig, etc. in farming yard. Another example is Persian carpet that its value increases with time.

No deterioration/obsolescence/amelioration: If the rate of obsolescence, deterioration or amelioration is not sufficiently low, its impact on modeling of such an inventory system cannot be ignored.

There are a few papers for obsolescing and ameliorating items. Moon *et al*. [3] considered the ameliorating/deteriorating items on an inventory model with a time-varying demand pattern. Against obsolescing and ameliorating items, the many researches has founded in the deteriorating inventory area in recent years. Products such as vegetables, fish, medicine, blood, gasoline and radioactive chemicals have a finite shelf life and start to deteriorate once they are produced. The deteriorating inventory models

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under inflationary conditions are studied extensively. In a few of these works, deterioration rate is not constant. For instance, Chen [4] proposed an inflationary model with time-proportional demand and Weibull distribution for deteriorating items using dynamic programming. Balkhi [5][a] presented a production lot size inventory model that the production, demand and deterioration rates are known, continuous and differentiable functions of time. Shortages are allowed, but only a fraction of the stock out is backordered and the rest is lost. Lo *et al*. [6] developed an integrated production-inventory model with assumptions of varying rate of deterioration, partial backordering, inflation, imperfect production processes and multiple deliveries. Most of the inventory systems for deteriorating items are considered a constant deterioration rate which will state in continuance.

The inflationary inventory systems with time-varying demand rate are considered in some researches. Datta and Pal [7] investigated a finite-time-horizon inventory model with linear time-dependent demand rate when shortages are allowed. Yang *et al*. [8] extended the inventory lot-size models to allow for inflation and fluctuating demand, which is more general than constant, increasing, decreasing and log-concave demand patterns. Other works are performed by Chen [4] and Balkhi [9][b].

Several authors have considered finite replenishment rate for inflationary inventory systems. Wee and Law [10] derived a deteriorating inventory model under inflationary conditions for determining economic production lot size when the demand rate is a linear decreasing function of the selling price. Sarker and Pan [11] surveyed the effects of inflation and the time value of money on order quantity with finite replenishment rate. Balkhi [9] proposed two flexible production lot size inventory models for deteriorating items in which the production rate at any instant depends on the demand and the on-hand inventory level at that instant. Another research is performed by Lo *et al*. [6].

Some researchers are prepared the stock-dependent demand rate models. Vrat and Padmanabhan [12] determined optimal ordering quantity for stock-dependent consumption rate items and showed that as the inflation rate increases, ordering quantity and the total system cost increase. Hou and Lin [13] developed an inventory model under inflation and time discounting for deteriorating items with stock-dependent selling rate. The selling rate is assumed to be a function of the current inventory level and the rate of deterioration is assumed to be constant. Liao and Chen [14] surveyed a retailer's inventory control system for the optimal delay in payment time for initial stock-dependent consumption rate when a wholesaler permits delay in payment. The effect of inflation rate, deterioration rate, initial stock-dependent consumption rate and a wholesaler's permissible delay in payment is discussed. A deterministic Economic Order Quantity (EOQ) inventory model taking into account inflation and time value of money developed for deteriorating items with price- and stock-dependent selling rates by Hou and Lin [15]. An efficient solution procedure is presented to determine the optimal number of replenishment, the cycle time and selling price. Hou [16] prepared an inventory model for deteriorating items with the stock-dependent consumption rate. Maiti *et al*. [17] proposed an inventory model with stock-dependent demand rate and two storage facilities under inflation and time value of money where the planning horizon is stochastic in nature and follows the exponential distribution with a known mean. Yang *et al*. [18] proposed an inventory model under inflation for deteriorating items with stock-dependent consumption rate and partial backlogging shortages.

Other efforts on inventory systems under inflationary conditions are performed under the assumption of the permissible delay in payments. Chang [19] proposed an EOQ model for deteriorating items under inflation when the supplier offers a permissible delay to the purchaser if the order quantity is greater than or equal to a predetermined quantity. Shah [20] derived an inventory model by assuming the constant rate of deterioration of units in an inventory, time value of money under the conditions of permissible delay in payments. Other models are prepared by Liao and Chen [14].

In the other efforts on inflationary inventory models, two warehouses problem are proposed. Yang [21] discussed the two-warehouse inventory problem for deteriorating items with a constant demand rate and shortages. Yang [22] extended the models introduced in Yang [21] to incorporate partial backlogging and then compare the two two-warehouse inventory models based on the minimum cost approach.

The above mentioned papers have considered a constant and well-known inflation rate over the time horizon. Horowitz [23] discussed a simple EOQ model with a Normal distribution for the inflation rate and the firm's cost of capital. He showed the importance of taking into account the inflation rate and time discounting, especially when the former is relatively high or when there is considerable uncertainty as to either the inflation rate or the marginal cost of capital. Mirzazadeh and Sarfaraz [24] presented a multiple-items inventory system with a budget constraint and the uniform distribution function for the external inflation rate. Mirzazadeh [25] compared the average annual cost and the discounted cost methods in the inventory system's modeling with considering stochastic inflation. The results show that there is a negligible difference between two procedures for wide range values of the parameters. Also, Mirzazadeh [26], in another work, proposed an inventory model under time-varying inflationary conditions.

In the above mentioned research, one of these assumptions has been considered for the demand rate:

- Constant and well known
- Time-varying
- Stock dependent
- Price-dependent.

Furthermore, in some practical situations, the demand rate is dependent to the changes in the inventory system costs. Therefore, in this paper, demand is a function of the inflation rate.

In most of the previous models, the inflation rate has been considered a constant value. In a few models, stochastic inflationary conditions are considered with known pdfs over the time horizon. In the real world, especially for long-term investment and forecasting, the fluctuations in the inflation rate cannot be disregarded. Accordingly, in this model the inflation rate is stochastic meanwhile the pdfs of inflation rates may be variable over the time horizon.

In many real situations, during a shortage period, the longer the waiting time is, the smaller the backlogging rate would be. For instance, for fashionable commodities and high-tech products with the short product life cycle, the willingness for a customer to wait for backlogging is diminishing with the length of the waiting time. Therefore, the partial backlogging has been considered. Additionally, the replenishment rate is finite and deteriorating items are surveyed with considering deterioration cost.

The rest of the paper is organized as follows. Section 2 includes the assumptions and notations. In Section 3, the model formulation is derived. Section 4 explains the solution procedure. Section 5 provides the numerical example to clarify how the proposed model is applied. Special cases are discussed in Section 6. The final section is devoted to the concluding remarks.

ASSUMPTIONS AND NOTATIONS

For the developed model, following assumptions and notations are used.

Assumptions: The following assumptions have been considered in this inventory system:

1. Shortages are allowed. Unsatisfied demand is backlogged and the fraction of shortages backordered is a differentiable and decreasing function of time t, denoted by $d(t)$, where t is the waiting time up to the next replenishment, $0= d(t)=1$ with $d(0)=1$ and $d(8)=0$. Note that if $d(t)=1$ (or 0) for all t, then shortages are completely backlogged (or lost).
2. The demand rate is assumed here to vary with the inflation.
3. The inventory system costs will be increased over the time horizon with the stochastic inflation rate. The *pdf* of the inflation rate may changes over the time horizon. For example, at time zero the inflation rate may be stochastic with the Uniform distribution function and during time horizon may change to the Normal distribution.
4. Deterioration of units occurs only when the item is effectively in stock and there is no repair or replenishment of the deteriorated items during the inventory cycle.
5. Replenishment rate is finite. The production rate is higher than the sum of consumption and deterioration rates.
6. Lead time is negligible and the initial inventory level is zero.

Notations: Once change in the *pdf* of the inflation rate over a time horizon has been considered in this model. But, the model can be extended to more than one change in the *pdf* similar to the explained method. The time horizon is divided into two different inflationary periods. The following notations are used in the model:

i_m The inflation rate in the first inflationary period (for m=1) and the inflation rate in the second inflationary period (for m=2)

$f(i_m)$ The *pdf* of i_m for m=1, 2

r The discount rate

R_m The discount rate net of inflation: $R_m = r-i_m$, m = 1,2

$R(i_m)$ The demand rate in the first inflationary period (for m=1) and the demand rate in the second inflationary period (for m=2) that is a linear function of the inflation rate

$$R(i_m) = a+bi_m,\ a\rangle 0,\ b\langle 0 \text{ for } m = 1,2 \qquad (1)$$

$d(t)$ The backlogging rate which is a decreasing function of the waiting time t, we here assume that $d(t)=e^{-at}$ where $a=0$ and t is the waiting time.

P The constant annual production rate

t The constant deterioration rate per unit time, where $(0\leq\tau\langle 1)$

c_1 The ordering cost per order at time zero

c_2 Per unit cost of the item at time zero

c_3 The inventory holding cost per unit per unit time at time zero

c_4 The backlogging cost per unit per unit time, if the shortage is backlogged

c_5 The unit opportunity cost due to lost sale, if the shortage is lost.

c_6 The deterioration cost per unit of the deteriorated item

H The fixed time horizon

Additional notations will be introduced later.

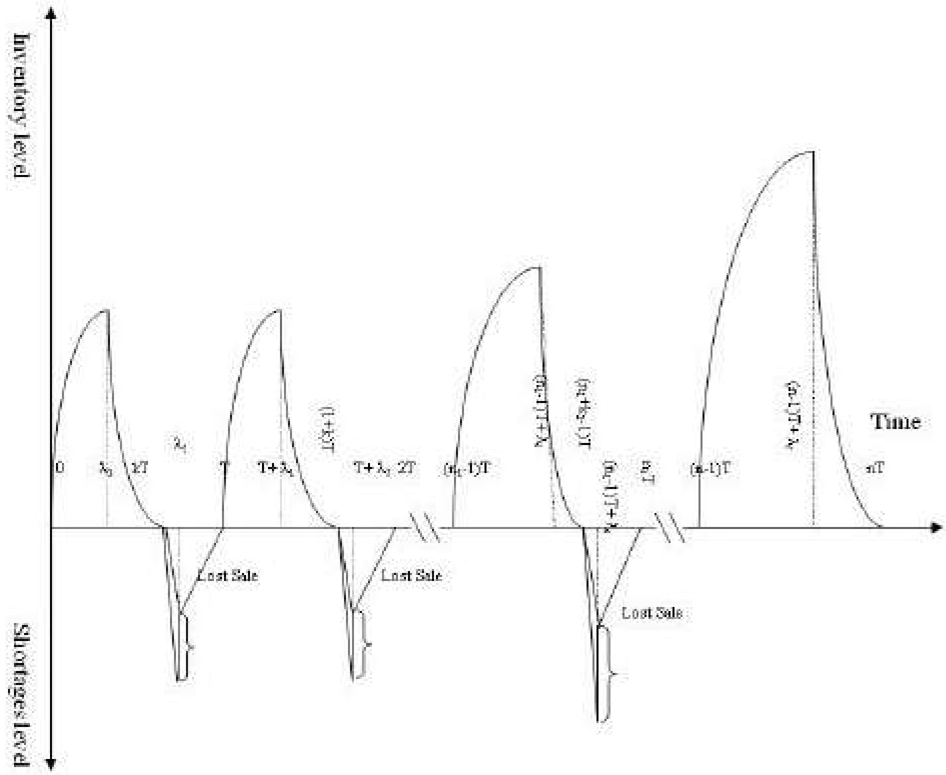

Fig. 1: Graphical representation of the inventory system

THE MODEL FORMULATION

A realization of the inventory level in the system is given in Fig. 1. In the development of the model, we assume that there are n cycles during the real time horizon H each of length T so that $T=H/n$. Initial and final inventory levels are both zero. Each inventory cycle except the last cycle can be divided into four parts. The production starts at time zero and the inventory level is increasing. This fact continues till the production stops at time λ_1. Then the level of inventory is decreasing by consumption and deterioration rates. At the moment of kT, the inventory level leads to zero and shortages occur. During the time interval $[kT,\lambda_2]$, we do not have any deterioration and therefore, shortages level linearly increases by the demand rate. At time λ_2 the production starts again and shortages level linearly decreases until the moment of T. In this moment the second cycle starts and this behavior continue till the end of the first inflationary period.

The *pdf* of inflation rate changes during the time horizon and the second inflationary period starts at time n_1T. Similar to the first inflationary period each inventory cycle can be divided into four parts. In the last cycle shortages are not allowed and the inventory cycle can be divided into two parts. The production stops at time $(n-1)T+\lambda_5$ and then the inventory level decreases until the end of the time horizon.

Let $I_i(t_i)$ denote the inventory level at any time t_i in the ith part of cycle $(i=1,2,3,4)$. The differential equations describing the inventory level at any time in the cycle are given as

$$\frac{dI_1(t_1)}{dt_1}+\tau I_1(t_1)=P-R(i_1), \qquad 0\le t_1\le\lambda_1 \qquad (2)$$

$$\frac{dI_2(t_2)}{dt_2}+\tau I_2(t_2)=-R(i_1), \qquad 0\le t_2\le kT-\lambda_1 \qquad (3)$$

$$\frac{dI_3(t_3)}{dt_3}=-\delta(\lambda_2-kT-t_3)R(i_1), \qquad 0\le t_3\le\lambda_2-kT \qquad (4)$$

$$\frac{dI_4(t_4)}{dt_4}=P-R(i_1), \qquad 0\le t_4\le T-\lambda_2 \qquad (5)$$

The inventory cycles of the second inflationary period except the last cycle can be described similar to the first inflationary period by substitution $R(i_1)$, λ_1, λ_2 and k to $R(i_2)$, λ_3, λ_4 and k_1. In the last cycle shortages are not allowed and the inventory level is governed by the following differential equations ($I_i(t_i)$ denote the inventory level at any time t_i in the *(i-4)th* part of the last cycle that $i=5,6$)

$$\frac{dI_5(t_5)}{dt_5}+\tau I_5(t_5)=P-R(i_2), \qquad 0\le t_5\le \lambda_5 \tag{6}$$

$$\frac{dI_6(t_6)}{dt_6}+\tau I_6(t_6)=-R(i_2), \qquad 0\le t_6\le T-\lambda_5 \tag{7}$$

The solutions of the above differential equations after applying the following boundary conditions: $I_1(0)=0$, $I_2(kT-\lambda_1)=0$, $I_3(0)=0$, $I_4(T-\lambda_2)=0$, $I_5(0)=0$ and $I_6(T-\lambda_5)=0$, are

$$I_1(t_1)=\frac{P-R(i_1)}{\tau}(1-e^{-\tau t_1}), \qquad 0\le t_1\le \lambda_1 \tag{8}$$

$$I_2(t_2)=\frac{-R(i_1)}{\tau}(1-e^{\tau(kT-\lambda_1-t_2)}), \qquad 0\le t_2\le kT-\lambda_1 \tag{9}$$

$$I_3(t_3)=\frac{R(i_1)e^{-(\lambda_2-kT)\alpha}}{\alpha}\left(1-e^{\alpha t_3}\right), \qquad 0\le t_3\le \lambda_2-kT \tag{10}$$

$$I_4(t_4)=(P-R(i_1))(t_4-T+\lambda_2), \qquad 0\le t_4\le T-\lambda_2 \tag{11}$$

$$I_5(t_5)=\frac{P-R(i_2)}{\tau}(1-e^{-\tau t_5}), \qquad 0\le t_5\le \lambda_5 \tag{12}$$

$$I_6(t_6)=\frac{-R(i_2)}{\tau}(1-e^{\tau(T-\lambda_5-t_6)}), \qquad 0\le t_6\le T-\lambda_5 \tag{13}$$

Using the above equations, the values of λ_1, λ_2, λ_3, λ_4 and λ_5 can be calculate with respect to *k*, k_1 and *T*. Solving $I_1(\lambda_1)=I_2(0)$ for λ_1 we have

$$\lambda_1=\frac{1}{\tau}Ln\frac{P-R(i_1)(1-e^{\tau kT})}{P} \tag{14}$$

Similarly, λ_3 is

$$\lambda_3=\frac{1}{\tau}Ln\frac{P-R(i_2)(1-e^{\tau k_1 T})}{P} \tag{15}$$

λ_2 can be calculated by solving $I_3(\lambda_2-kT)=I_4(0)$

$$\lambda_2=\frac{[P-R(i_1)(1-k)]T}{P} \tag{16}$$

Similarly λ_4 is

$$\lambda_4=\frac{[P-R(i_2)(1-k_1)]T}{P} \tag{17}$$

Finally, solving $I_5(\lambda_5)=I_6(0)$ for λ_5 we have

$$\lambda_5=\frac{1}{\tau}Ln\frac{P-R(i_2)(1-e^{\tau T})}{P} \tag{18}$$

In the special case, if the second inflationary period occurs during the last cycle, equation (18) should be revised by substitution of $R(i_2)$ with $R(i_1)$ and we have

$$\lambda_6=\frac{1}{\tau}Ln\frac{P-R(i_1)(1-e^{\tau T})}{P} \tag{19}$$

Let *ECP* as the expected present value (*EPV*) of purchasing costs, *ECH* as the *EPV* of carrying costs, *ECS* as the *EPV* of shortages costs (backordering and lost sale), *ECD* as the *EPV* of deterioration costs and *ECR* as the *EPV* of replenishment costs, respectively. The total expected present value of costs over the time horizon (*ETVC*) is

$$\begin{aligned}ETVC(n,k,k_1)&=ECR+ECP+ECH+\\ &\quad ECB+ECL+ECD\end{aligned} \tag{20}$$

The detailed analysis is given as follows.

The expected present value of ordering cost (ECR): Assume CR_1 and CR_2 as the ordering cost in the first and second inflationary period respectively. Therefore

$$CR_1=c_1\left[1+\sum_{j=0}^{y-1}e^{-R_1(jT+\lambda_2)}\right] \tag{21}$$

where, y = min {n_1, n-1}. By replacing equation (16) in equation (21) and taking the expected value we have

$$ECR_1=c_1E\left\{1+e^{-\frac{T\left[b(1-k)i_1^2-((a-rb)(k-1)+p)i_1+r(a(k-1)+p)\right]}{p}}\left[\frac{1-e^{-Ty(r-i_1)}}{(1-e^{-(r-i_1)T})}\right]\right\} \tag{22}$$

The *EPV* of the ordering cost in the second inflationary period follows from similar machinations

$$ECR_2=c_1E\left\{e^{-\frac{T\left[\left[b(1-k_1)i_2^2-((a-rb)(k_1-1)+p)i_2+r(a(k_1-1)+p)\right]\right]}{p}}\left[\frac{e^{-yT(r-i_2)}-e^{-(n-1)(r-i_2)T}}{1-e^{-(r-i_2)T}}\right]\right\} \tag{23}$$

Therefore, the total *EPV* of the ordering cost is

$$ECR = ECR_1 + ECR_2 \tag{24}$$

The expected present value of purchasing cost (ECP): Let ECP_1 and ECP_2 as the *EPV* of the purchase cost in the first and the second inflationary periods, respectively. Also, the *EPV* of the purchase cost in the last cycle is shown with ECP_{31} if the inflationary changes occur during the last cycle and ECP_{32} if the inflationary changes occur before the last cycle. The first purchase is ordered at time zero and equals to: $c_2P\lambda_1$. Then, the next purchase will occur at time λ_2 and therefore, the first cycle purchase cost is

$$c_2P\left[\lambda_1 + (T-\lambda_2)e^{-\lambda_2 R_1}\right] \tag{25}$$

The purchase cost for j-th cycle, (j=2, 3, …, n_1) is similar to the above equation with considering the discount factor, therefore, the *EPV* of the purchase cost in the first inflationary period calculation is

$$ECP_1 = c_2P\sum_{j=1}^{y} E\left[\frac{Ln\left[\frac{P-R(i_1)(1-e^{\tau kT})}{P}\right]}{\tau}e^{-(j-1)R_1T} + \left[T - \frac{\big(P-R(i_1)(1-k)\big)T}{P}\right]e^{-\left[\frac{(P-R(i_1)(1-k))T}{P}+(j-1)T\right]R_1}\right] \tag{26}$$

For the second period, the *EPV* can be calculated with similar machinations

$$ECP_2 = c_2P\sum_{j=y+1}^{n-1} E\left[\frac{Ln\left[\frac{P-R(i_2)(1-e^{\tau k_1T})}{P}\right]}{\tau}e^{-(j-1)R_2T} + \left[T - \frac{\big(P-R(i_2)(1-k_1)\big)T}{P}\right]e^{-\left[\frac{(P-R(i_2)(1-k_1))T}{P}+(j-1)T\right]R_2}\right] \tag{27}$$

The order quantity is λ_5P in the last cycle, that will occur at time *(n-1)T.* Based on in which ordering cycle the inflationary changes occur, the *EPV* of the purchase cost in the last cycle will be one of the following phrases

$$ECP_{31} = cPE\left[\frac{1}{\tau}Ln\frac{P-R(i_1)(1-e^{\tau T})}{P}e^{-(n-1)R_1T}\right] \tag{28}$$

$$ECP_{32} = cPE\left[\frac{1}{\tau}Ln\frac{P-R(i_2)(1-e^{\tau T})}{P}e^{-(n-1)R_2T}\right] \tag{29}$$

The total expected purchase cost over the time horizon would be

$$ECP = ECP_1 + ECP_2 + \left[\frac{n_1}{n}\right]ECP_{31} + \left(1-\left[\frac{n_1}{n}\right]\right)ECP_{32} \tag{30}$$

Expected present value of holding cost (ECH): Consider ECH_1 and ECH_2 as the *EPV* of the holding cost during the first and the second inflationary periods respectively. The *EPV* of the holding cost during the last cycle can be defined with ECH_{31} (ECH_{32}) if the inflationary changes occur during the last cycle (before the last cycle). In the first period, the holding costs for j-th cycle is

$$CH_j = c_3\left[\int_0^{\lambda_1} I_1(t_1)e^{-R_1t_1}dt_1 + \int_0^{kT-\lambda_1} I_2(t_2)e^{-R_1t_2}dt_2\,e^{-\lambda_1R_1}\right]e^{-(j-1)R_1T}, \quad j=1,2,\ldots,y \tag{31}$$

After some complex calculations and taking the expected value we have

$$ECH_1 = c_3 \sum_{j=1}^{y} E\left\{ e^{-R_1(j-1)T} \left[\frac{(P-R(i_1))\left\{ e^{-\lambda_1 R_1}\left[-R_1(1-e^{-\tau\lambda_1})-\tau\right]+\tau \right\}}{\tau R_1(R_1+\tau)} + \frac{R(i_1)\left[-(R_1+\tau)e^{-\lambda_1 R_1}+\tau e^{-kTR_1}+R_1 e^{\tau kT} e^{-\lambda_1(R_1+\tau)}\right]}{\tau R_1(R_1+\tau)} \right] \right\} \quad (32)$$

Similarly, ECH_2 can be calculated as follow

$$ECH_2 = c_3 \sum_{j=y+1}^{n-1} E\left\{ e^{-R_2(j-1)T} \left[\frac{(P-R(i_2))\left\{ e^{-\lambda_3 R_2}\left[-R_2(1-e^{-\tau\lambda_3})-\tau\right]+\tau \right\}}{\tau R_2(R_2+\tau)} + \frac{R(i_2)\left[-(R_2+\tau)e^{-\lambda_3 R_2}+\tau e^{-k_1 T R_2}+R_2 e^{\tau k_1 T} e^{-\lambda_3(R_2+\tau)}\right]}{\tau R_2(R_2+\tau)} \right] \right\} \quad (33)$$

For the last cycle, if the inflation changes occur during this cycle, holding cost will be

$$CH_n = c_3\left[\int_0^{\lambda_6} I_5(t_5)e^{-R_1 t_5}dt_5 e^{-R_1(n-1)T} + \int_0^{T-\lambda_6} I_6(t_6)e^{-R_1 t_6}dt_6 e^{-R_1((n-1)T+\lambda_6)}\right] \quad (34)$$

After some complex calculations and taking the expected value we have

$$ECH_{31} = -c_3 E\left\{ e^{-R_1[\lambda_6+(n-1)T]} \left[\frac{(P-R(i_1))\left[R_1(1-e^{-\tau\lambda_6})+\tau(1-e^{\lambda_6 R_1})\right]}{-\tau R_1(R_1+\tau)} + \frac{R(i_1)\left[\tau e^{-R_1(T-\lambda_6)}-R_1(1-e^{\tau(T-\lambda_6)})-\tau\right]}{-\tau R_1(R_1+\tau)} \right] \right\} \quad (35)$$

In the similar way, if the inflation changes occur before the last cycle, the *EPV* of the holding cost is

$$ECH_{32} = -c_3 E\left\{ e^{-R_2[\lambda_5+(n-1)T]} \left[\frac{(P-R(i_2))\left[R_2(1-e^{-\tau\lambda_5})+\tau(1-e^{\lambda_5 R_2})\right]}{-\tau R_1(R_1+\tau)} + \frac{R(i_2)\left[\tau e^{-R_2(T-\lambda_5)}-R_2(1-e^{\tau(T-\lambda_5)})-\tau\right]}{-\tau R_2(R_2+\tau)} \right] \right\} \quad (36)$$

So, the total *EPV* of the holding costs over the time horizon is

$$ECH = ECH_1 + ECH_2 + \left[\frac{n_1}{n}\right]ECH_{31} + \left(1-\left[\frac{n_1}{n}\right]\right)ECH_{32} \quad (37)$$

The expected present value of shortages cost (ECS): ECS_1 and ECS_2 show the *EPV* of the shortages cost during the first and the second inflationary periods respectively. Shortages are not allowed in the last cycle. In the first inflationary period, the shortages cost, including backorder and lost sales are

$$ECS_1 = \sum_{j=1}^{y} E\left\{ \left[\begin{array}{l} \int_0^{\lambda_2-kT}\left[c_4 e^{-R_1 t_3}\sigma(t_3)+c_5(1-\sigma(t_3))e^{-(\lambda_2-kT)R_1}\right]\left[-I_3(t_3)\right]dt_3 e^{-kTR_1} \\ +\int_0^{T-\lambda_2}\left[c_4 e^{-R_1 t_4}\sigma(t_4)+c_5(1-\sigma(t_4))e^{-(T-\lambda_2)R_1}\right]\left[-I_4(t_4)\right]dt_4 e^{-\lambda_2 R_1} \end{array} \right] e^{-(j-1)R_1 T} \right\} \quad (38)$$

ECS_2 can be calculated with similar machinations

$$ECS_2 = \sum_{j=y+1}^{n-1} E\left\{\left[\begin{array}{l}\int_0^{\lambda_4 - k_1 T}\left[c_4 e^{-R_2 t_3}\sigma(t_3) + c_5\left(1-\sigma(t_3)\right)e^{-(\lambda_4 - k_1 T)R_2}\right]\left[-I_3(t_3)\right]dt_3 e^{-k_1 T R_2} \\ +\int_0^{T-\lambda_4}\left[c_4 e^{-R_2 t_4}\sigma(t_4) + c_5\left(1-\sigma(t_4)\right)e^{-(T-\lambda_4)R_2}\right]\left[-I_4(t_4)\right]dt_4 e^{-\lambda_4 R_2}\end{array}\right] e^{-(j-1)R_2 T}\right\} \tag{39}$$

Therefore, the total *EPV* of the shortages cost over the time horizon will be

$$ECS = ECS_1 + ECS_2 \tag{40}$$

The expected present value of deteriorating cost (ECD): Denote DI_1 the inventory items deteriorated per cycle in the first inflationary period:

$$DI_1 = \tau\left[\int_0^{\lambda_1} I_1(t_1)\,dt_1 + \int_0^{kT-\lambda_1} I_2(t_2)\,dt_2\right] = \frac{\left(P-a-bi_1\right)\left(\tau\lambda_1 - 1 - e^{-\tau\lambda_1}\right) - \left(a+bi_1\right)\left(1+\tau\left(kT-\lambda_1\right) - e^{\tau(kT-\lambda_1)}\right)}{\tau} \tag{41}$$

Now, assume ECD_1 and ECD_2 as the *EPV* of the deterioration cost during the first and the second inflationary periods respectively. Also, ECD_{31} (ECD_{32}) is defined the *EPV* of the deterioration cost during the last cycle if the inflationary changes occur during the last cycle (before the last cycle). ECD_1 after taking the expected value will be

$$ECD_1 = \frac{c_6}{\tau}\sum_{j=1}^{y} E\left[\left(P-a-bi_1\right)\left(\tau\lambda_1 - 1 - e^{-\tau\lambda_1}\right)e^{-(j-1)TR_1} - \left(a+bi_1\right)\left(1+\tau\left(kT-\lambda_1\right) - e^{\tau(kT-\lambda_1)}\right)e^{-(j-1+\lambda_1)TR_1}\right] \tag{42}$$

Similarly, ECD_2 is

$$ECD_2 = \frac{c_6}{\tau}\sum_{j=y+1}^{n-1} E\left[\left(P-a-bi_2\right)\left(\tau\lambda_3 - 1 - e^{-\tau\lambda_3}\right)e^{-(j-1)TR_2} - \left(a+bi_2\right)\left(1+\tau\left(k_1 T-\lambda_3\right) - e^{\tau(k_1 T-\lambda_3)}\right)e^{-(j-1+\lambda_3)TR_2}\right] \tag{43}$$

For the last cycle, if the inflation changes occur during this cycle, deterioration cost will be

$$\begin{aligned} ECD_{31} &= c_6 E\left\{\tau\left[\int_0^{\lambda_6} I_1(t_1)\,dt_1 e^{-(n-1)R_1} + \int_0^{kT-\lambda_6} I_2(t_2)dt_2 e^{-(n+\lambda_6-1)R_1}\right]\right\} \\ &= \frac{c_6}{\tau}E\left[\left(P-a-bi_1\right)\left(\tau\lambda_6 - 1 + e^{-\tau\lambda_6}\right)e^{-(n-1)R_1} - \left(a+bi_1\right)\left(1+\left(T-\lambda_6\right)\tau - e^{\tau(T-\lambda_6)}\right)e^{-(n+\lambda_6-1)R_1}\right] \end{aligned} \tag{44}$$

Similarly, ECD_{32} can be calculated as follow

$$ECD_{32} = \frac{c_6}{\tau}E\left[\left(P-a-bi_2\right)\left(\tau\lambda_5 - 1 + e^{-\tau\lambda_5}\right)e^{-(n-1)R_2} - \left(a+bi_2\right)\left(1+\left(T-\lambda_5\right)\tau - e^{\tau(T-\lambda_5)}\right)e^{-(n+\lambda_5-1)R_2}\right] \tag{45}$$

Therefore, the total *EPV* of the deterioration cost over the time horizon is

$$ECD = ECD_1 + ECD_2 + \left[\frac{n_1}{n}\right]ECD_{31} + \left(1-\left[\frac{n_1}{n}\right]\right)ECD_{32} \tag{46}$$

So, the objective function will be obtained using equations (24), (30), (37), (40) and (46) that is shown in equation (20).

THE SOLUTION PROCEDURE

Our problem is to determine the optimal values of n, k and k_1, that minimize the system cost *ETVC*. For a given value of n, taking the first derivative of *ETVC* with respect to k and k_1 and equating the partial derivatives to zero gives

$$\frac{dETVC(n,k,k_1)}{dk} = 0, \frac{dETVC(n,k,k_1)}{dk_1} = 0 \qquad (47)$$

For a given value of n, derive k^* and k_1^* from Equation (47). $ETVC(n,k^*,k_1^*)$ derives by substituting (n,k^*,k_1^*) into equation (20). Then n increase by the increment of one continually and $ETVC(n,k^*,k_1^*)$ drive again. The above stages repeat until the minimum $ETVC(n,k^*,k_1^*)$ be found. The (n^*,k^*,k_1^*) and $ETVC(n^*,k^*,k_1^*)$ values constitute the optimal solution and satisfy the following conditions

$$\Delta ETVC\left(n^*-1,k^*,k_1^*\right) \langle 0 \langle \Delta ETVC\left(n^*,k^*,k_1^*\right) \qquad (48)$$

Where

$$\Delta ETVC\left(n^*,k^*,k_1^*\right) = ETVC\left(n^*+1,k^*,k_1^*\right) - ETVC\left(n^*,k^*,k_1^*\right) \qquad (49)$$

To ensure convexity of the objective function, the derived values of (n^*,k^*,k_1^*) must satisfy the following sufficient conditions

$$\frac{d^2ETVC(n,k,k_1)}{dk^2} \rangle 0, \quad \frac{d^2ETVC(n,k,k_1)}{dk_1^2} \rangle 0 \qquad (50)$$

Note that,

$$\frac{\partial^2 ETVC(n,k,k_1)}{\partial k \partial k_1} = \frac{\partial^2 ETVC(n,k,k_1)}{\partial k_1 \partial k} = 0 \qquad (51)$$

NUMERICAL EXAMPLE

The numerical example is devised here to illustrate the effects of the general model developed in this paper with the following data:

The constant annual production rate is 10000units, the demand parametric values are a=4200units/year and b=-2000, the time horizon, H, is 10, the company interest rate is 20% and the deterioration rate of the on-hand inventory per unit time is 0.05:

P=10000units/year; a=4200units/year; b=-2000; H=10years; r=\$0.2/\$/year; t=0.05.

The backlogging rate is $d(t)=e^{-0.5t}$ and the inflation rate at time zero is stochastic with Uniform distribution: i_1~U(0.08, 0.15). After six years, the inflation rates will change as follows: i_2~Normal $(0.17, 0.08^2)$. The ordering, production, holding, backordering, lost sales and deterioration costs at the beginning of the time horizon are

c_1 = \$100/order;
c_2 = \$8/unit;
c_3 = \$0.4/unit/year;
c_4 = \$3/unit/year;
c_5 = \$10/unit and
c_6=\$13/unit.

Table 1: Optimal solution of $ETVC(n,k,k_1)$ for the numerical example

n	y	k	k_1	$ETVC(n,k,k_1)$
5	3	0.64	0.74	151,912.36
10	6	0.64	0.75	149,548.23
11	7	0.64	0.76	148,763.42
12	8	0.65	0.76	148,146.55
13	8	0.65	0.76	147,633.89
14*	9*	0.65*	0.76*	147,209.92*
15	9	0.65	0.75	147,864.62
20	12	0.65	0.75	149,667.61
50	30	0.64	0.73	157,618.79
100	60	0.62	0.68	178,917.38

Using the solution procedure described in the previous section, the results are presented in Table 1. From this table, we see that the number of replenishments n=14, the total cost TC becomes minimum. In the first inflationary period, the shortages occur after elapsing 65% of the cycle time ($k^*=0.65$). The second inflationary period occurs at the start of the tenth cycle to the end of the time horizon. In this period, shortages occur after spending 76% of the cycle time ($k_1^*=0.76$). The minimum total expected cost $ETVC(n,k_1,k_2)$=\$147209.92. We then have the time interval between replenishments is $T=H/n$=0.714year.

SPECIAL CASES

Two special cases, which followed from the main model, will be discussed in this section.

Case of no-shortages: If shortages are not allowed, $k=k_1=1$ can be substituted in expression (20) and the expected present worth of the total cost, $ETVC(n)$, can be obtained. The minimum solution of $ETVC(n)$ for discrete variable of n must satisfy the following equation:

$$\Delta ETVC(n) \le 0 \le \Delta ETVC(n+1) \qquad (52)$$

where $ETVC(n)=ETVC(n)-ETVC(n-1)$. Using recent inequality and considering the above mentioned numerical data, the following solution was obtained: n^*=17, y=11, $ETVC(n)$= 149,023.43 and T^*=0.59year. It shows that the n and $ETVC$ increase in the without shortages case.

Case of no-inflationary changes: Now, consider the inflation rate has Uniform distribution: i_1~U(0.08, 0.15) which pdf has no changes over the time horizon. The optimal solution of the problem in this case will be following: n=12, k=0.65 and $ETVC$=133,212.44. Number of replenishments and inventory system cost decrease in this case in comparison with the main model.

CONCLUDING REMARKS

In this paper, a mathematical inventory model is considered for determining the optimal replenishment schedule under stochastic and variable conditions and shortages are partially backlogged. Also, the proposed model incorporates other realistic and practical features that are likely to be associated with the inventory of certain types of goods, such as: demand rate is Inflation-dependent, the inventory deteriorates at a constant rate over time and production rate is finite.

It can be seen in the literature review that the inflation rate, usually, has been assumed constant over the time horizon. But, many economic factors may also affect the future changes of costs; such as changes in the world inflation rate, rate of investment, demand level, labor costs, cost of raw materials, rates of exchange, rate of unemployment, productivity level, tax, liquidity, etc. Therefore, the constant inflation rate assumption is not valid in the real world situation. From the inflation point of view, the developed model will be useful to the stochastic and variable inflationary conditions as it gives a better and more general inventory control system. The numerical example has been given to illustrate the theoretical results and the special cases have been discussed. These special cases are compared with the main model through the numerical example.

The proposed model can be extended in numerous ways. For example, we may extend the inflation-dependent demand to a more generalized demand pattern that fluctuates with inflation and stock-dependent demand rate. Also, we could extend the model to incorporate some more features, such as quantity discount, non constant deterioration rate, two warehouse and permissible delay in payment.

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