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New Fuzzy Lie Subalgebras over a Fuzzy Field

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Abstract: In this paper, we introduce a new generalized fuzzy Lie subalgebra of a Lie algebra over a fuzzy field called, an $(\in, \in \vee q_m)-$ fuzzy Lie subalgebra and present some important properties. We characterize it by their level subsets. Some characterization of the generalized fuzzy Lie subalgebras over a fuzzy field are also established.

Key words: Lie algebras, fuzzy field, $(\in, \in \vee q_m)-$ fuzzy Lie subalgebra

INTRODUCTION

Lie algebras were discovered by famous Norwegian mathematician Sophus Lie (1842-1899) while he was attempting to classify certain smooth subgroups of general linear groups. The groups he considered are now called Lie groups. He found that by taking the tangent space at the identity element of such a group, one obtained a Lie algebra. A Lie algebra has many applications to the spectroscopy of molecules, atoms, nuclei and hadrons. One of the key concepts in the application of Lie algebraic methods in physics is that of spectrum generating algebras and their associated dynamic symmetries. Now a days, a Lie algebra has also been used by electrical engineers, mainly in the mobile robot control [1].

Murali [2] proposed a definition of a fuzzy point belonging to a fuzzy subset under a natural equivalence on a fuzzy set. The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [3] played a vital role to generate some different types of fuzzy subgroups. A new type of fuzzy subgroups, $(\in, \in \vee q)$-fuzzy subgroups, was introduced in earlier paper of Bhakat and Das [4] by using the combined notions of *belongingness* and *quasi-coincidence* of fuzzy point and fuzzy set. In fact, $(\in, \in \vee q)$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. On the other hand, The notions of fuzzy ideals and fuzzy subalgebras of Lie algebras over a field were considered in [5] by Yehia. Presently, Akram and Shum [6] introduced the notion of vague Lie subalgebras over a vague field and studied some properties. In this paper we introduce a new generalized fuzzy Lie subalgebras over a fuzzy field and present some important properties. We characterize it by their level subsets. Some characterization of the generalized fuzzy Lie subalgebras over a fuzzy field are also established. The definitions and terminologies that we used in this paper are standard. For other notations, terminologies and applications, the readers are referred to [7-13].

PRELIMINARIES

Throughout this paper, L is a Lie algebra and X is a field. It is clear that the multiplication of a Lie algebra is not necessary associative, that is, $[[x,y],z] = [x,[y,z]]$ does not hold in general, however it is *anti- commutative*, that is, $[x,y] = -[y,x]$. $[y,z]]$ and is anti commutative, that is, $[x,y] = -[y,x]$. Let μ be a *fuzzy set* on L, that is, a map $\mu : L \to [0,1]$. For any fuzzy set μ in L and any $t \in [0,1]$, we define set $U(\mu;t) = \{x \in L \mid \mu(x) \geq t\}$, which is called *upper t-level cut* of μ.

Definition 1 [14] *A fuzzy set F of X is called a fuzzy field if the following conditions are satisfied:*

- $(\forall\, m, n \in X)(F(m-n) \geq \min\{F(m), F(n)\})$,
- $(\forall m, n \in X, n \neq 0)(F(mn^{-1}) \geq \min\{F(m), F(n)\})$.

Lemma 2 [14] If λ is a fuzzy subfield of X, then $\lambda(0) \geq \lambda(1) \geq \lambda(m) = \lambda(-m)$ for all $m \in X$ and $\lambda(-m) = \lambda(m^{-1})$ for all $m \in X - \{0\}$.

Lemma 3 [14] Let λ be a fuzzy subfield of X. Then for $t \in [0,1]$, the fuzzy-cut $U(\lambda;t)$ is a crisp subfield of X.

Definition 4 [3] A fuzzy set μ in a set L of the form

$$\mu(y) = \begin{cases} t \in (0,1] & \text{if } y = x, \\ 0 & \text{y} \neq x. \end{cases}$$

is said to be a *fuzzy point* with support x and value t and is denoted by $F(x,t)$.

Definition 5 [3] A fuzzy point $F(x,t)$ is called *belong to* a fuzzy set μ, written as $F(x,t) \in \mu$, if $\mu(x) \geq t$. A fuzzy point $F(x,t)$ is said to be *quasicoincident with* a fuzzy set μ, written as $F(x,t) q \mu$, if $\mu(x) + t > 1$. For brevity, we use the following notations:

- $F(x,t) \in \mu$ or $F(x,t)q\mu$ is written as $F(x,t) \in \vee q\mu$.
- $F(x,t) \in \mu$ and $F(x,t)q\mu$ is written as $F(x,t) \in \wedge q\mu$.

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Let m be an element of [0, 1) unless otherwise specified. By $F(x,t)\ q_m\ \mu$, we mean $\mu(x) + t + m > 1$. The notation $F(x,t) \in \vee\, q_m\ \mu$ means $F(x,t) \in \mu$ or $F(x,t)\ q_m\ \mu$.
We formulate a technical Lemma.

Lemma 6 Let μ be a fuzzy set of a Lie algebra L. Then the non-empty level set $U(\mu;t)$ is a Lie subalgebra of L for all $t \in (\frac{1-m}{2}, 1]$ if and only if

(a) $\max(\mu(x+y), \frac{1-m}{2}) \geq \min(\mu(x), \mu(y))$,

(b) $\max(\mu(\alpha x), \frac{1-m}{2}) \geq \min(\lambda(\alpha), \mu(x))$,

(c) $\max(\mu([x,y]), \frac{1-m}{2}) \geq \min(\mu(x), \mu(y))$

for all $x, y \in L$, $\alpha \in X$.

Proof: Let $t \in (\frac{1-m}{2}, 1]$ be such that $U(\mu;t)(\neq \emptyset)$ and $U(\mu;t)$ is a Lie subalgebra of L.

(a) Assume that $\max(\mu(x+y), \frac{1-m}{2}) < \min(\mu(x), \mu(y)) = t$ for some $x, y \in L$, then $t \in (\frac{1-m}{2}, 1]$, $\mu(x+y) < t$, $x \in U(\mu;t)$ and $y \in U(\mu;t)$. Since $x, y \in U(\mu;t)$, $U(\mu;t)$ is a Lie subalgebra of L, so $x + y \in U(\mu;t)$, a contradiction. Hence (a) holds.

(b) Assume that $\max(\mu(\alpha x), \frac{1-m}{2}) < \min(\lambda(\alpha), \mu(x)) = t$ for some $\alpha \in X$, $x \in L$, then $t \in (\frac{1-m}{2}, 1]$, $\mu(\alpha x) < t$, $x \in U(\mu;t)$ and $\alpha \in U(\mu;t)$. Since α, $x \in U(\mu;t)$, $U(\mu;t)$ is a Lie subalgebra of L, so $\alpha x \in U(\mu;t)$, a contradiction. Hence (b) holds.

(c) Assume that $\max(\mu([x,y]), \frac{1-m}{2}) < \min(\mu(x), \mu(y)) = t$ for some $x, y \in L$, then $t \in (\frac{1-m}{2}, 1]$, $\mu([x,y]) < t$, $x \in U(\mu;t)$ and $y \in U(\mu;t)$. Since $x, y \in U(\mu;t)$, $U(\mu;t)$ is a Lie subalgebra of L, so $[x,y] \in U(\mu;t)$, a contradiction. Hence (c) holds.

The proof of the sufficiency part is straightforward and is hence omitted. This completes the proof. □

Corollary 7 Let μ be a fuzzy set of a Lie algebra L. Then the non-empty level set $U(\mu;t)$ is a Lie subalgebra of L for all $t \in (0.5, 1]$ if and only if

(a) $\max(\mu(x+y), 0.5) \geq \min(\mu(x), \mu(y))$,

(b) $\max(\mu(\alpha x), 0.5) \geq \min(\lambda(\alpha), \mu(x))$,

(c) $\max(\mu([x,y]), 0.5) \geq \min(\mu(x), \mu(y))$

for all $x, y \in L$, $\alpha \in X$.

Definition 8 A fuzzy set μ in L is called an $(\in, \in \vee\, q_m)$-*fuzzy Lie subalgebras* over a fuzzy field λ of L, if it satisfies the following conditions:

(1) $F(x,t_1)$, $F(y,t_2) \in \mu \Longrightarrow F(x+y, \min(t_1,t_2)) \in \vee\, q_m\ \mu$,

(2) $F(\alpha,t_1) \in \lambda$, $F(x,t_2) \in \mu \Longrightarrow F(\alpha x, \min(t_1,t_2)) \in \vee\, q_m \mu$,

(3) $F(x,t_1)$, $F(y,t_2) \in \mu \Longrightarrow F([x,y], \min(t_1,t_2)) \in \vee\, q_m\ \mu$

for all $x, y \in L$, $\alpha \in X$, $t_1, t_2 \in (0,1]$.

From (2), it follows that:

(4) $F(-1,t_1) \in \lambda$, $F(x,t_2) \in \mu \Longrightarrow F(-x, \min(t_1,t_2)) \in \vee\, q_m\ \mu$.

(5) $F(0,t_1) \in \lambda$, $F(x,t_2) \in \mu \Longrightarrow F(0, \min(t_1,t_2)) \in \vee\, q_m\ \mu$.

From Definition 8, it follows that we can develop different types of fuzzy Lie subalgebras for different values of $m \in [0,1)$. Hence an $(\in, \in \vee\, q_m)$-*fuzzy Lie subalgebras* over a fuzzy field λ with $m = 0$ is called an $(\in, \in \vee\, q)$-fuzzy Lie subalgebras over a fuzzy field.

Example 9 Let $\Re^3 = \{(x,y,z) : x,y,z \in \mathbb{R}\}$ be the set of all 3-dimensional real vectors. Then $\Re^3$ with $[x,y] = x \times y$ form a real Lie algebra. Define a fuzzy set $\mu : \Re^3 \to [0,1]$ by

$$\mu(x,y,z) = \begin{cases} 0.4 & \text{if } x = y = z = 0, \\ 0.5 & \text{otherwise.} \end{cases}$$

and define fuzzy set $\lambda : R \to [0,1]$ for all $n \in \mathbb{R}$ by

$$\lambda(n) = \begin{cases} 0.4 & \text{if } n \in Q, \\ 0.5 & \text{if } n \in R - Q(\sqrt{3}). \end{cases}$$

By routine computations, we can easily check that μ forms $(\in, \in \vee q)-$, $(\in, \in \vee q_{0.2})-$, $(\in, \in \vee q_{0.4})$-fuzzy Lie subalgebras of L over the fuzzy field for $m = 0, 0.2, 0.4$, respectively.

The proof of the following Proposition is obvious.

Proposition 10 Every $(\in, \in)$-fuzzy Lie subalgebra is an $(\in, \in \vee q_m)$-fuzzy Lie subalgebra.

Corollary 11 Every $(\in, \in)$-fuzzy Lie subalgebra is an $(\in, \in \vee q)$-fuzzy Lie subalgebra.

Theorem 12 A fuzzy set μ in L is an $(\in, \in \vee q_m)$-fuzzy Lie subalgebra of L over fuzzy field λ if and only if it satisfies

(I) $\mu(x+y) \geq \min\{\mu(x), \mu(y), \frac{1-m}{2}\}$,

(II) $\mu(\alpha x) \geq \min\{\lambda(\alpha), \mu(x), \frac{1-m}{2}\}$,

(III) $\mu([x,y]) \geq \min\{\mu(x), \mu(y), \frac{1-m}{2}\}$

for all $x, y \in L$, $\alpha \in X$.

Proof: Let μ be an $(\in, \in \vee q_m)$-fuzzy Lie subalgebra of L over fuzzy field. Assume that (I) is not valid. Then there exist $x_0, y_0 \in L$ such that

$$\mu(x_0 + y_0) < \min\{\mu(x_0), \mu(y_0), \frac{1-m}{2}\}.$$

If $\min(\mu(x_0), \mu(y)_0) < \frac{1-m}{2}$, then $\mu(x_0 + y_0) < \min(\mu(x_0), \mu(y_0))$. Thus

$\mu(x_0+y_0) < t \leq \min\{\mu(x_0), \mu(y_0)\}$ for some $t \in (0,1]$.

It follows that $F(x_0, t) \in \mu$ and $F(y_0, t) \in \mu$, but $F(x_0 + y_0, t) \overline{\in} \mu$, a contradiction. Moreover, $\mu(x_0 + y_0) + t < 2t < 1 - m$, and so $F(x_0 + y_0, t)\overline{q_m}\mu$. Consequently $F(x_0 + y_0, t)\overline{\in \vee q_m}\ \mu$, a contradiction. On the other hand, if $\min(\mu(x_0), \mu(y_0) \geq \frac{1-m}{2}$, then $\mu(x_0) \geq \frac{1-m}{2}$, $\mu(y_0) \geq \frac{1-m}{2}$ and $\mu(x_0 + y_0) < \frac{1-m}{2}$. Thus $F(x_0, \frac{1-m}{2}) \in \mu$, $F(y_0, \frac{1-m}{2}) \in \mu$, but $F(x_0 + y_0, \frac{1-m}{2})\overline{\in}\mu$. Also

$$\mu(x_0 + y_0) + \frac{1-m}{2} < \frac{1-m}{2} + \frac{1-m}{2} = 1 - m,$$

i.e., $F(x_0 + y_0, \frac{1-m}{2})\overline{q_m}\mu$. Hence $F(x_0 + y_0, \frac{1-m}{2})\overline{\in \vee q_m}\ \mu$, a contradiction. So (I) is valid. By using a very similar argumentation as in the proof of the (I), we can easily prove that (II) and (III) also valid.

Conversely, assume that μ satisfies (I). Let x, $y \in L$ and t_1, $t_2 \in (0,1]$ be such that $F(x, t_1) \in \mu$ and $F(y, t_2) \in \mu$. Then

$$\begin{aligned} \mu(x+y) &\geq \min(\mu(x), \mu(y), \frac{1-m}{2}) \\ &\geq \min(t_1, t_2, \frac{1-m}{2}). \end{aligned}$$

Assume that $t_1 \leq \frac{1-m}{2}$ or $t_2 \leq \frac{1-m}{2}$. Then $\mu(x+y) \geq \min(t_1, t_2)$, which implies that $F(x+y, \min(t_1, t_2) \in \mu$. Now suppose that $t_1 > \frac{1-m}{2}$ and $t_2 > \frac{1-m}{2}$. Then $\mu(x+y) \geq \frac{1-m}{2}$, and thus

$$\mu(x+y) + \min(t_1, t_2) > \frac{1-m}{2} + \frac{1-m}{2} = 1 - m,$$

i.e., $F(x + y, \min(t_1, t_2))q_m\mu$. Hence $F(x + y, \min(t_1, t_2)) \in \vee q_m\mu$. By using a very similar argumentation, it is easy to see $F(\alpha x, \min(t_1, t_2)) \in \vee q_m\mu$ and $F([x,y], \min(t_1, t_2)) \in \vee q_m\mu$. Consequently, μ is an $(\in, \in \vee q_m)$-fuzzy Lie subalgebra of L over a fuzzy field. This ends the proof. □

Corollary 13 A fuzzy set μ in L is an $(\in, \in \vee q)$-fuzzy Lie subalgebra of L over fuzzy field λ if and only if it satisfies

(I) $\mu(x+y) \geq \min\{\mu(x), \mu(y), 0.5\}$,

(II) $\mu(\alpha x) \geq \min\{\lambda(\alpha), \mu(x), 0.5\}$,

(III) $\mu([x,y]) \geq \min\{\mu(x), \mu(y), 0.5\}$

Theorem 14 Let μ be a fuzzy set of fuzzy Lie subalgebra of L. Then μ is an $(\in, \in \vee q_m)$-fuzzy Lie subalgebra of L over a fuzzy field if and only if the level set $U(\mu; t)$, $t \in (0, \frac{1-m}{2}]$, is a Lie subalgebra of L.

Proof: Assume that μ is an $(\in, \in \vee q)$-fuzzy Lie subalgebra of L over a fuzzy field λ. Let $t \in (0, \frac{1-m}{2}]$ and $x, y, \alpha \in U(\mu; t)$. Then $\mu(x) \geq t$, $\mu(y) \geq t$ and $\lambda(\alpha) \geq t$. It follows from Definition 3.3 that:

$$\begin{aligned} \mu(x+y) &\geq \min(\mu(x), \mu(y), \frac{1-m}{2}) \\ &\geq \min(t, \frac{1-m}{2}) = t, \\ \mu(\alpha x) &\geq \min(\lambda(\alpha), \mu(x), \frac{1-m}{2}) \\ &\geq \min(t, \frac{1-m}{2}) = t, \\ \mu([x,y]) &\geq \min(\mu(x), \mu(y), \frac{1-m}{2}) \\ &\geq \min(t, \frac{1-m}{2}) = t, \end{aligned}$$

so that $x + y, \alpha x, [x,y] \in U(\mu; t)$. Hence $U(\mu; t)$ is a Lie subalgebra of L.

Conversely, suppose that the nonempty set $U(\mu; t)$ is a Lie subalgebra of L for all $t \in (0, \frac{1-m}{2}]$. If the condition (1) is not true, then there exists a, $b \in L$ such that $\mu(a+b) < \min(\mu(a), \mu(b), \frac{1-m}{2})$. Hence we can take $t \in (0,1]$ such that $\mu(a+b) < t_1 < \min(\mu(a), \mu(b), \frac{1-m}{2})$. Then $t \in \frac{1-m}{2}$ and a , $b \in U(\mu; t)$. Since $U(\mu; t)$ is a Lie subalgebra of L, it follows that $a + b \in U(\mu; t)$ so that $\mu(a+b) \geq t$. This is a contradiction. Therefore the condition (1) is valid. By using a very similar argumentation as in the proof of the (1), we can easily prove (2) and (3) are also valid. Hence μ is an $(\in, \in \vee q_m)$-fuzzy Lie subalgebra of L over a fuzzy field. □

Corollary 15 Let μ be a fuzzy set of fuzzy Lie subalgebra of L. Then μ is an $(\in, \in \vee q)$-fuzzy Lie subalgebra of L over a fuzzy field if and only if the level set $U(\mu; t)$, $t \in (0, 0.5]$, is a Lie subalgebra of L.

Theorem 16 Let μ be an $(\in, \in \vee q_m)$-fuzzy Lie subalgebra of L over a fuzzy field such that $\mu(x) < \frac{1-m}{2}$ for all $x \in L$, then μ is a fuzzy Lie subalgebra of L over a fuzzy field.

Proof: Let $x, y \in L$, $\alpha \in X$ and $t_1, t_2 \in (0, 1]$ be such that $F(x, t_1) \in \mu$, $F(y, t_2) \in \mu$, $F(\alpha, t_3) \in \lambda$. Then $\mu(x) \geq t_1$, $\mu(y) \geq t_2$ and $\lambda(\alpha) \geq t_3$. It follows from Theorem 3.5 that

$$\begin{aligned}
\mu(x+y) &\geq \min(\mu(x), \mu(y), \frac{1-m}{2}) \\
&= \min(\mu(x), \mu(y)) = \min(t_1, t_2), \\
\mu(\alpha x) &\geq \min(\lambda(\alpha), \mu(x), \frac{1-m}{2}) \\
&= \min(\mu(x), \mu(y)) = \min(t_3, t_1), \\
\mu([x,y]) &\geq \min(\mu(x), \mu(y), \frac{1-m}{2}) \\
&= \min(\mu(x), \mu(y)) = \min(t_1, t_2)
\end{aligned}$$

so that $F(x + y, \min(t_1, t_2)) \in \mu$, $F(\alpha x, \min(t_1, t_3)) \in \mu$, $F([x, y], \min(t_1, t_2)) \in \mu$. Hence μ is a fuzzy Lie subalgebra of L over a fuzzy field. □

Corollary 17 Let μ be an $(\in, \in \vee q)$-fuzzy Lie subalgebra of L over a fuzzy field such that $\mu(x) < 0.5$ for all $x \in L$, then μ is a fuzzy Lie subalgebra of L over a fuzzy field.

Theorem 18 Let μ be an $(\in, \in \vee q_m)$-fuzzy Lie subalgebra of L over a fuzzy field.

(i) If there exists $x \in L$ such that $\mu(x) \geq \frac{1-m}{2}$, then $\mu(0) \geq \frac{1-m}{2}$.

(ii) If $\mu(0) < \frac{1-m}{2}$, then μ is a fuzzy Lie subalgebra of L.

Proof:

(i) Let $x \in L$ such that $\mu(x) \geq \frac{1-m}{2}$.

$$\mu(-x) = \min(\mu(x), \frac{1-m}{2}) = \frac{1-m}{2}.$$

Hence

$$\begin{aligned}
\mu(0) &= \mu(x - x) \geq \min(\mu(x), \mu(-x), \frac{1-m}{2}) \\
&= \frac{1-m}{2}.
\end{aligned}$$

(ii) $\mu(0) < \frac{1-m}{2} \Rightarrow \mu(x) < \frac{1-m}{2}$ for all $x \in L$. Thus we conclude that:

$$\begin{aligned}
\mu(x+y) &\geq \min(\mu(x), \mu(y)), \mu(\alpha x) \\
&\geq \min(\lambda(\alpha), \mu(x)), \mu([x,y]) \\
&\geq \min(\mu(x), \mu(y))
\end{aligned}$$

for all $x, y \in L$, $\alpha \in X$. Hence μ is a fuzzy Lie subalgebra of L over a fuzzy field.

□

Corollary 19 Let μ be an $(\in, \in \vee q)$-fuzzy Lie subalgebra of L over a fuzzy field.

(i) If there exists $x \in L$ such that $\mu(x) \geq 0.5$, then $\mu(0) \geq 0.5$.

(ii) If $\mu(0) < 0.5$, then μ is a fuzzy Lie subalgebra of L.

Definition 20 An $(\in, \in \vee q_m)$-fuzzy Lie subalgebra of L over a fuzzy field is said to be *proper* if Imμ has at least two elements. Two $(\in, \in \vee q_m)$- fuzzy Lie subalgebras μ and λ are said to be *equivalent* if they have same family of level Lie subalgebras. Otherwise, they are said to be non-equivalent.

Theorem 21 A proper $(\in, \in \vee q_m)$-fuzzy Lie subalgebra of L over a fuzzy field such that cardinality of $\{\mu(x); \mu(x) < \frac{1-m}{2}\} \geq 2$. Then there exist two proper non-equivalent $(\in, \in \vee q_m)$-fuzzy Lie subalgebra of L over a fuzzy field such that μ can be expressed as the union of them.

Proof: Let μ be a proper $(\in, \in \vee q_m)$-fuzzy Lie subalgebra of L over a fuzzy field with $\{\mu(x); \mu(x) < \frac{1-m}{2}\} = \{t_1, t_2, \cdots, t_n\}$ where $t_1 > t_2 > \cdots > t_n$ and $n \geq 2$. Then

$$[\mu]_{\frac{1-m}{2}} \subseteq [\mu]_{t_1} \subseteq \cdots \subseteq [\mu]_{t_n} = L$$

is the chain of $(\in, \in \vee q_m)$-Lie subalgebras of μ. Define two fuzzy sets λ_1, $\lambda_2 \leq \mu$ defined by

$$\lambda_1(x) = \begin{cases} t_1, & \text{if x} \in [\mu]_{t_1}, \\ t_2, & \text{if x} \in [\mu]_{t_2} \setminus [\mu]_{t_1}, \\ \vdots & \\ t_n, & \text{if x} \in [\mu]_{t_n} \setminus [\mu]_{t_{n-1}}, \end{cases}$$

$$\lambda_2(x) = \begin{cases} \mu(x), & \text{if x} \in [\mu]_{\frac{1-m}{2}}, \\ n, & \text{if x} \in [\mu]_{t_2} \setminus [\mu]_{\frac{1-m}{2}}, \\ t_3, & \text{if x} \in [\mu]_{t_3} \setminus [\mu]_{t_2}, \\ \vdots & \\ t_n, & \text{if x} \in [\mu]_{t_n} \setminus [\mu]_{t_{n-1}}, \end{cases}$$

respectively, where $t_3 < n < t_2$. Then λ_1 and λ_2 are $(\in, \in \vee q_m)$-fuzzy Lie subalgebra of L over a fuzzy field with

$$[\mu]_{t_1} \subseteq [\mu]_{t_2} \subseteq \cdots \subseteq [\mu]_{t_n}$$

and

$$[\mu]_{\frac{1-m}{2}} \subseteq [\mu]_{t_2} \subseteq \cdots \subseteq [\mu]_{t_n}$$

being respectively chains of $(\in, \in \vee q_m)$-fuzzy Lie subalgebras over a fuzzy field. Hence λ_1 and λ_2 are non-equivalent and $\mu = \lambda_1 \cup \lambda_2$. □

Corollary 22 A proper $(\in, \in \vee q)$-fuzzy Lie subalgebra of L over a fuzzy field such that cardinality of $\{\mu(x); \mu(x) < \frac{1-m}{2}\} \geq 2$. Then there exist two proper non-equivalent $(\in, \in \vee q)$-fuzzy Lie subalgebra of L over a fuzzy field such that μ can be expressed as the union of them.

Theorem 23 Let $\{\mu_i : i \in I\}$ be a family of $(\in, \in \vee q_m)$-fuzzy Lie subalgebras of L over a fuzzy field. Then $\mu = \cap_i \mu_i$ is an $(\in, \in \vee q_m)$-fuzzy Lie subalgebra of L over a fuzzy field.

Proof: By Theorem 12, we have $\mu(x+y) \geq \min(\mu(x), \mu(y), \frac{1-m}{2})$, and hence

$$\begin{aligned}\mu(x+y) &= \sup_{i\in\Lambda} \mu_i(x+y) \\ &\geq \sup_{i\in\Lambda} \min\{\mu_i(x), \mu_i(y), \frac{1-m}{2}\} \\ &= \min\{\sup_{i\in\Lambda} \mu_i(x), \sup_{i\in\Lambda} \mu_i(y), \frac{1-m}{2}\} \\ &= \min\{\bigcap_{i\in\Lambda} \mu_i(x), \bigcap_{i\in\Lambda} \mu_i(y), \frac{1-m}{2}\} \\ &= \min\{\mu(x), \mu(y), \frac{1-m}{2}\}.\end{aligned}$$

For other conditions the verification is analogous. By Theorem 12, it follows that μ is an $(\in, \in \vee q_m)$-fuzzy Lie subalgebra of L over a fuzzy field. □
Taking $m = 0$ in Theorem 23, we obtain the following Corollary.

Corollary 24 Let $\{\mu_i : i \in I\}$ be a family of $(\in, \in \vee q)$-fuzzy Lie subalgebras of L over a fuzzy field. Then $\mu = \cap_i \mu_i$ is an $(\in, \in \vee q)$-fuzzy Lie subalgebra of L over a fuzzy field.

Theorem 25 Let $\{\mu_i : i \in \Lambda\}$ be a family of $(\in, \in \vee q_m)$-fuzzy Lie subalgebras of L over a fuzzy field. such that $\mu_i \subseteq \mu_j$ or $\mu_j \subseteq \mu_i$ for all $i, j \in \Lambda$. Then $\nu := \bigcup_{i\in\Lambda} \nu_i$ is an $(\in, \in \vee q_m)$-fuzzy Lie subalgebra of L over a fuzzy field.

Proof: By Theorem 12, we have $\mu(x+y) \geq \min(\mu(x), \mu(y), \frac{1-m}{2})$, and hence

$$\begin{aligned}\mu(x+y) &= \inf_{i\in\Lambda} \mu_i(x+y) \\ &\geq \inf_{i\in\Lambda} \min\{\mu_i(x)), \mu_i(y), \tfrac{1-m}{2}\} \\ &= \min\{\inf_{i\in\Lambda} \mu_i(x), \inf_{i\in\Lambda} \mu_i(y), \tfrac{1-m}{2}\} \\ &= \min\{\textstyle\bigcup_{i\in\Lambda} \mu_i(x), \bigcup_{i\in\Lambda} \mu_i(y), \tfrac{1-m}{2}\} \\ &= \min\{\mu(x), \mu(y), \tfrac{1-m}{2}\}.\end{aligned}$$

It is easy to see that

$$\inf_{i\in\Lambda} \min\{\mu_i(x), \mu_i(y), \frac{1-m}{2}\} \leq \bigcup_{i\in\Lambda} \min(\mu_i(x), \mu_i(y), \frac{1-m}{2}).$$

Suppose that

$$\inf_{i\in\Lambda} \min\{\mu_i(x), \mu_i(y), \frac{1-m}{2}\} \neq \bigcup_{i\in\Lambda} \min(\mu_i(x), \mu_i(y), \frac{1-m}{2}),$$

then there exists s such that

$$\inf_{i\in\Lambda} \min\{\mu_i(x), \mu_i(y), \frac{1-m}{2}\} < s < \bigcup_{i\in\Lambda} \min(\mu_i(x), \mu_i(y), \frac{1-m}{2}).$$

Since $\mu_i \subseteq \mu_j$ or $\mu_j \subseteq \mu_i$ for all $i, j \in \Lambda$, there exists $k \in \Lambda$ such that $s < \min(\mu_k(x), \mu_k(y), \frac{1-m}{2})$. On the other hand, $\min(\mu_i(x), \mu_i(y), \frac{1-m}{2}) > s$ for all $i \in \Lambda$, a contradiction. Hence

$$\begin{aligned}&\inf_{i\in\Lambda} \min\{\mu_i(x), \mu_i(y), \tfrac{1-m}{2}\} \\ &= \min(\textstyle\bigcup_{i\in\Lambda} \mu_i(x), \bigcup_{i\in\Lambda} \mu_i(y), \tfrac{1-m}{2}) \\ &= \min\{\mu(x), \mu(y), \tfrac{1-m}{2}\}.\end{aligned}$$

For other conditions the verification is analogous. By Theorem 12, it follows that μ is an $(\in, \in \vee q_m)$-fuzzy Lie subalgebra of L over a fuzzy field. □
Taking $m = 0$ in Theorem 25, we obtain the following Corollary.

Corollary 26 Let $\{\mu_i : i \in \Lambda\}$ be a family of $(\in, \in \vee q)$-fuzzy Lie subalgebras of L over a fuzzy field. such that $\mu_i \subseteq \mu_j$ or $\mu_j \subseteq \mu_i$ for all $i, j \in \Lambda$. Then $\nu := \bigcup_{i\in\Lambda} \nu_i$ is an $(\in, \in \vee q)$-fuzzy Lie subalgebra of L over a fuzzy field.

CONCLUSIONS

Algebraic structures play a prominent role in mathematics with wide rang of applications in many disciplines such as theoretical physics, computer sciences, control engineering, information sciences, coding theory and topological spaces. This provides sufficient motivation to the researchers to review various concepts and results from the realm of abstract algebra in the broader framework of fuzzy setting. In this connection, we have introduced a new sort of fuzzy Lie subalgebra of a Lie algebra over a fuzzy field called an $(\in, \in \vee q_m)-$ fuzzy Lie subalgebra, and have investigated some important properties. The obtained results probably can be applied in various fields such as (1)Engineering (2) Computer Sciences (3) Medical diagnosis.

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