World Applied Sciences Journal 7 (Special Issue for Applied Math):25-32, 2009
ISSN 1818-4952

Co-fuzzy Lie Superalgebras Over a Co-fuzzy Field

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Abstract: The concept of a co-fuzzy field and some properties are introduced. We then introduce the notion of a co-fuzzy Lie sub-superalgebra of a Lie superalgebra over a co-fuzzy field. Some examples are developed to demonstrate the definition of a co-fuzzy Lie sub-superalgebra over a co-fuzzy field. Some fundamental properties of a co-fuzzy Lie sub-superalgebra of a Lie superalgebra over a co-fuzzy field are investigated. The concept of co-fuzzy Lie sub-superalgebras of Lie superalgebras over a co-fuzzy field under homomorphisms is also discussed.

Key words: Lie superalgebras, co-fuzzy field, co-fuzzy subspaces, Co-fuzzy Lie sub-superalgebras, Homomorphism.

INTRODUCTION

The theory of Lie superalgebras was constructed by V.G. Kac [1] in 1977 as a generalization of the theory of Lie algebras. This theory had played an important role in both mathematics and physics. In particular, Lie superalgebras are important in theoretical physics where they are used to describe the mathematics of supersymmetry. Furthermore, Lie superalgebras had found many applications in computer science such as unimodal polynomials [2].

Fuzzy set theory [3] has been developed in many directions by many scholars and has evoked great interest among mathematicians working in different fields of mathematics. There have been wide range applications of the theory of fuzzy sets, from the design of robots and computer simulation to engineering and water resources planning. Algebraic structures also play a prominent role in mathematics with wide range of applications in many disciplines such as theoretical physics, computer sciences, control engineering, information sciences, coding theory and topological spaces. This provides sufficient motivation to the researchers to review various concepts and results from the realm of abstract algebra in the broader framework of fuzzy setting. In this connection, A. Rosenfeld [4] first introduced the fuzzy sets in the realm of group theory. Since then many mathematicians have been involved in extending the concepts and results of abstract algebra to the broader frame work of the fuzzy setting. The notions of fuzzy ideals and fuzzy subalgebras of Lie algebras over a field were introduced first in [5] by S.E. Yehia and the notion of a new fuzzy Lie subalgebra of a Lie algebra over a fuzzy field was introduced by M. Akram in [6]. It is now natural to study fuzzy Lie superalgebras which are the generalization of Lie algebras. With this objectives in view, W. Chen considered the notion of fuzzy quotient Lie superalgebras over a field in [7]. In this paper we introduce the notion of co-fuzzy Lie sub-superalgebras of Lie superalgebras over a co-fuzzy field and investigate some useful properties. The definitions and terminologies that we used in this paper are standard. For other notations, terminologies and applications, the readers are referred to [8-14].

PRELIMINARIES

In this Section, we review some elementary aspects that are necessary for this paper.

Definition 1 [1] Suppose that V is a vector space and $V_{\bar{0}}, V_{\bar{1}}$ are its (vector) subspaces. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be the direct sum of the subspaces. Then V (with this decomposition) is called a $\mathbb{Z}_2$*-graded vector space* if each element v of a $\mathbb{Z}_2$-graded vector space has a unique expression of the form $v = v_{\bar{0}} + v_{\bar{1}}$ ($v_{\bar{0}} \in V_{\bar{0}}$, $v_{\bar{1}} \in V_{\bar{1}}$). The subspaces $V_{\bar{0}}$ and $V_{\bar{1}}$ are called the even part and odd part of V, respectively. In particular, if v is an element of either $V_{\bar{0}}$ or $V_{\bar{1}}$, v is said to be homogeneous.

Definition 2 [1] A $\mathbb{Z}_2$-graded vector space $\mathbb{L} = \mathbb{L}_{\bar{0}} \oplus \mathbb{L}_{\bar{1}}$ with a Lie bracket

$$[\ ,\ ] : \mathbb{L} \times \mathbb{L} \xrightarrow{\text{bilinear}} \mathbb{L}$$

is called a *Lie superalgebra*, if it satisfies the following conditions:

(1) $[\mathbb{L}_i, \mathbb{L}_j] \subseteq \mathbb{L}_{i+j}$ for $i, j \in \mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$,

(2) $[x, y] = -(-1)^{|x||y|}[y, x]$ (antisymmetry),

(3) $[x,[y,z]] = [[x,y],z] + (-1)^{|x||y|}[[x,z],y]$(Jacobi identity),

where for any homogeneous element $a \in \mathbb{L}_{\overline{n}}$, $n = 0, 1$. The subspaces $\mathbb{L}_{\overline{0}}$ and $\mathbb{L}_{\overline{1}}$ are called the even and odd parts of $\mathbb{L}$, respectively. Therefore, a Lie algebra is a Lie superalgebra with trivial odd part.

Definition 3 [1] If $\varphi : \mathbb{L} \to \acute{\mathbb{L}}$ is a linear map between Lie superalgebras $\mathbb{L} = \mathbb{L}_{\overline{0}} \oplus \mathbb{L}_{\overline{1}}$ and $\acute{\mathbb{L}} = \acute{\mathbb{L}}_{\overline{0}} \oplus \acute{\mathbb{L}}_{\overline{1}}$ such that

(4) $\varphi(\mathbb{L}_i) \subseteq \acute{\mathbb{L}}_i$ $(i \in \mathbb{Z}_2)$ (preserving the grading),

(5) $\varphi([x,y]) = [\varphi(x), \varphi(y)]$ (preserving the Lie bracket).

Then φ is called a *homomorphism* of Lie superalgebras.

Throughout this paper, we denote V is a vector space, $\mathbb{L}$ is a Lie superalgebra and F is a field.
Let μ be a *fuzzy set* on V, i.e., a map $\mu : V \to [0,1]$. Let V be a complete lattice whose minimum and maximum we denote by 0 and 1, respectively. In this paper, we use the notations $x \vee y = \max\{x,y\}$ and $x \wedge y = \min\{x,y\}$.

Definition 4 [14] An mapping $f : V \to V$ is called a *closure* if, for every x, $y \in V$ (i) $x \geq y \Rightarrow f(x) \geq f(y)$(monotony) (ii) $x \leq f(x)$ (inclusion) (iii) $f(f(x)) = f(x)$ (idempotence).

Definition 5 Let μ and ν be co-fuzzy sets of V. We define the *sum* of μ and ν by $(\mu + \nu)(x) = \inf\limits_{x=a+b}\{\mu(a) \wedge \nu(b)\}$.

Definition 6 [11] A fuzzy set λ of a field F is called a *fuzzy field* if the following conditions are satisfied:

(6) $(\forall\, x, y \in F)(\lambda(x-y) \geq \lambda(x) \wedge \lambda(y))$,

(7) $(\forall\, x, y \in F, x \neq 0)(\lambda(xy^{-1}) \geq \lambda(x) \wedge \lambda(y))$.

Lemma 7 [11] If λ is a fuzzy field of F, then
$\lambda(0) \geq \lambda(1) \geq \lambda(x) = \lambda(-x)$ for all $x \in F$ and
$\lambda(x) = \lambda(x^{-1})$ for $x \in F - \{0\}$.

Definition 8 [15] Let V be a vector space. A fuzzy subset μ of V is called a *fuzzy subspace* of V if for all x, $y \in V$ and $\alpha \in F$ the following axioms are satisfied:

(8) $\mu(x+y) \geq \mu(x) \wedge \mu(y)$,

(9) $\mu(\alpha x) \geq \mu(x)$.

Definition 9 [7] Let $V = V_{\overline{0}} \oplus V_{\overline{1}}$ be a $\mathbb{Z}_2$-graded vector space over a field F and let $\mu_{\overline{0}}$ and $\mu_{\overline{1}}$ be fuzzy subspaces of (vector) subspaces $V_{\overline{0}}$ and $V_{\overline{1}}$, respectively. Then $\mu(x) = \mu_{\overline{0}} \oplus \mu_{\overline{1}}(x) = \mu_{\overline{0}}(x_{\overline{0}}) \wedge \mu_{\overline{1}}(x_{\overline{1}})$, where $x_{\overline{0}} \in V_{\overline{0}}$, $x_{\overline{1}} \in V_{\overline{1}}$ and $x = x_{\overline{0}} + x_{\overline{1}} \in V$, is called a $\mathbb{Z}_2$-*graded fuzzy subspace*.

Definition 10 [7] A fuzzy set $\mu : \mathbb{L} \to [0,1]$ is called a *fuzzy Lie sub-superalgebra* of $\mathbb{L}$ if

(10) μ is a $\mathbb{Z}_2$-graded fuzzy subspace,

(11) $\mu([x,y]) \geq \mu(x) \wedge \mu(y)$

hold for all $x, y \in \mathbb{L}$.

CO-FUZZY FIELD

Definition 11 A fuzzy set λ of a field F is called a *co-fuzzy subfield* if the following conditions are satisfied:

(12) $(\forall\, x, y \in F)(\lambda(x-y) \leq \lambda(x) \vee \lambda(y))$,

(13) $(\forall\, x, y \in F, x \neq 0)(\lambda(xy^{-1}) \leq \lambda(x) \vee \lambda(y))$.

Example 12 Consider a field $Z_5 = \{0, 1, 2, 3, 4\}$ with the following Cayley tables:

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

·	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Let $\mu : Z_5 \to [0,1]$ be a fuzzy set defined by

$$\mu(x) = \begin{cases} 0.1 & \text{if } x\text{=0}, \\ 0.8 & \text{otherwise.} \end{cases}$$

By routine computations, it is easy to check that λ is a co-fuzzy subfield of Z_5.

The following Lemma is trivial.

Lemma 13 If λ is a co-fuzzy field of F, then

- $(\forall\, x \in F)(\lambda(0) \leq \lambda(1) \leq \lambda(x) = \lambda(-x))$ and
- $(\forall\, x \in F - \{0\})$ $(\lambda(x) = \lambda(x^{-1}))$.

Theorem 14 Let λ be a co-fuzzy field in a field F. Then λ is a co-fuzzy field of F if and only if the set $L(\lambda; s) = \{x \in F | \lambda(x) \leq s\}$, $s \in [0,1]$, is a field of F when it is nonempty.

Proof: The proof is similar to the proof of Theorem 3.4 in [8]. □

Definition 15 A co-fuzzy field λ of a field F is said to be *co-fuzzy characteristic* if $\lambda^f(x) = \lambda(x)$ for all $x \in F$ and $f \in$Aut(F). A co-fuzzy field λ of field F is said to be *fully invariant co-fuzzy* if $\lambda(f(x)) \geq \lambda(x)$ for all $x \in F$ and $f \in$ End(F).

Theorem 16 A co-fuzzy field is characteristic if and only if each its level set is a characteristic field.

Proof: The proof is similar to the proof of Theorem 3.20 in [8]. □

As a consequence of the above Theorem we obtain the following theorem.

Theorem 17 If λ is a fully invariant co-fuzzy field of F, then it is characteristic

Theorem 18 *A field F is Northerian if and only if for any co-fuzzy field λ, $(Im(\lambda), \leq)$ is well ordered.*

Proof: $(\Rightarrow)$ Let F be Northerian. If for some co-fuzzy field λ of F, $(Im(\lambda), \leq)$ is not well ordered, then there is a strictly increasing number sequence in $Im(\lambda)$: $t_1 < t_2 < \cdots$. Denote $p = \sup\{t_i \,|\, i = 1, 2, \cdots\}$. It is easy to verify that $U = \{x \in F \,|\, \lambda(x) < p\}$ is a subfield of F. Thus there are $a_1, \cdots, a_n \in U$ such that $U = (a_1, \cdots, a_n]$, and so $\lambda(a_1) \vee \cdots \vee \lambda(a_n)$ is the greatest element of $(\{\lambda(x) \,|\, x \in U\}; \leq)$. We observe that $(\{t_i \,|\, i = 1, 2, \cdots\}; <)$ is a subset of $(\{\lambda(x) \,|\, x \in U\}; \leq)$, a contradiction. Hence $(Im(\lambda), \leq)$ is not well ordered.

$(\Leftarrow)$ Suppose that for any co-fuzzy field λ of F, $(Im(\lambda); \leq)$ are well ordered subsets of [0, 1]. If F is not Noetherian, then there is a strictly ascending chain of fields of F:

$$U_1 \subsetneqq U_2 \subsetneqq \cdots \subsetneqq U_n \subsetneqq \cdots,$$

where $U_i \subsetneqq U_j$ expresses $U_i \subseteq U_j$ but $U_i \neq U_j$. We construct the fuzzy set λ of F by

$$\lambda(x) := \begin{cases} \frac{n}{n+1} & \text{if} \quad x \in U_n \setminus U_{n-1},\ n = 1, 2, \cdots, \\ 1 & \text{if} \quad x \notin \bigcup_{n=1}^{\infty} U_n, \end{cases}$$

where $U_0 =$. We now prove that λ is a co-fuzzy field of F. We consider the following cases.

Case I. If at least one of x and y belong to $F \setminus \bigcup_{n=1}^{\infty} U_n$, then at least one of $\lambda(x)$ and $\lambda(y)$ is equal to 1, thereby $\lambda(x - y) \leq \lambda(x) \vee \lambda(y)$.

Case II. If $x \in U_i \setminus U_{i-1}$ and $y \in U_j \setminus U_{j-1}$ with $i \leq j$, then x, $y \in U_j$. Hence $x - y \in U_j$, and so

$$\lambda(x - y) \leq \frac{j}{j+1} = \frac{i}{i+1} \vee \frac{j}{j+1} = \lambda(x) \vee \lambda(y).$$

Case III. If $x \in U_i \setminus U_{i-1}$ and $y \in U_j \setminus U_{j-1}$ with $j \leq i$, then by the way similar to Case II we can prove $\lambda(x - y) \leq \lambda(x) \vee \lambda(y)$.

Therefore λ is a co-fuzzy field of F. But

$$(Im(\lambda), \leq) = \Big(\{\frac{1}{2}, \frac{2}{3}, \cdots, \frac{n}{n+1}, \cdots, 1\}, \leq\Big)$$

is not well ordered, a contradiction. The verification for other condition is obvious. Hence F is Noetherian. □

Let F/K be a field extension and let $\mathbb{I}(F/K) = \{S | S \ subfield \ of \ F, K \subseteq S\}$ be the lattice of its intermediate fields .

Definition 19 Let F/K be an extension of fields. A fuzzy set $\lambda : F \to [0, 1]$ is called a *co-fuzzy intermediate field* of F/K if the following conditions are satisfied:

(14) $(\forall\, x, y \in F)(\lambda(x - y) \leq \lambda(x) \vee \lambda(y))$,

(15) $(\forall\, x, y \in F, x \neq 0)(\lambda(xy^{-1}) \leq \lambda(x) \vee \lambda(y))$,

(16) $(\forall\, x \in F)(\forall k \in K)(\lambda(x) \geq \lambda(k))$.

Let $\mathbb{CFI}(F/K)$ denote the set of all co-fuzzy intermediate fields of F/K.

Lemma 20 λ is a co-fuzzy intermediate field if and only if for all $Im(\lambda)$, the level set $L(\lambda; s)$ is an intermediate field of F/K.

A fuzzy subset λ of F is said to have the inf property if, for every nonempty subset A of Im (λ), there exists $x \in \{y \in F | \lambda(y) \in A\}$ such that $\lambda(x) = \inf(A)$.

Definition 21 Let F/K be an extension of fields and $\lambda \in \mathbb{CFI}(F/K)$. Then λ is called a *co-fuzzy chain subfield* of F/K if for all x, $y \in F$, $\lambda(x) = \lambda(y) \Longleftrightarrow K(x) = K(y)$.

We now give characterizations without their proofs.

Theorem 22 Let F/K be a field extension. Then every co-fuzzy intermediate field of F/K has the inf property if and only if there are no infinite strictly increasing sequences of intermediate fields of F/K.

Theorem 23 The intermediate fields of F/K are chained if and only if F/K has a co-fuzzy chain subfield.

Theorem 24 Let F/K be an extension such that the intermediate fields of F/K are chained. If:

(a) F/K is algebraic,

(b) any intermediate field S of F/K with $S \neq F$ is a finite simple extension of K,

(c) $(\mathbb{CFI}(F/K), \supseteq)$ satisfies the ascending chain condition.

Then $\mathbb{CFI}(F/K)$ is well ordered.

CO-FUZZY LIE SUPERALGEBRAS OVER A CO-FUZZY FIELD

In this Section, we present the notion of co-fuzzy Lie sub-superalgebras of Lie superalgebras over a co-fuzzy field and study some fundamental properties.

Definition 25 Let V be a vector space. A fuzzy subset μ of V is called a *co-fuzzy subspace* of V over a co-fuzzy field λ if the following axioms are satisfied:

(i) $\mu(x+y) \leq \mu(x) \vee \mu(y)$,

(ii) $\mu(\alpha x) \leq \lambda(\alpha) \vee \mu(x)$

for all $x, y \in V$ and $\alpha \in F$.

Definition 26 Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a $\mathbb{Z}_2$-graded vector space and let $\mu_{\bar{0}}$ and $\mu_{\bar{1}}$ be co-fuzzy subspaces of subspaces $V_{\bar{0}}$ and $V_{\bar{1}}$ over a co-fuzzy field λ, respectively. Then $\mu(x) = (\mu_{\bar{0}} \oplus \mu_{\bar{1}})(x) = \mu_{\bar{0}}(x_{\bar{0}}) \vee \mu_{\bar{1}}(x_{\bar{1}})$, where $x_{\bar{0}} \in V_{\bar{0}}$, $x_{\bar{1}} \in V_{\bar{1}}$ and $x = x_{\bar{0}} + x_{\bar{1}} \in V$, is called a *$\mathbb{Z}_2$-graded co-fuzzy subspace over a co-fuzzy field.*

Definition 27 A fuzzy set $\mu : \mathbb{L} \to [0,1]$ is called a *co-fuzzy Lie sub-superalgebra* of $\mathbb{L}$ over a co-fuzzy field λ if

(iii) μ is a co-fuzzy subspace over a co-fuzzy field,

(iv) μ is a $\mathbb{Z}_2$-graded co-fuzzy subspace over a co-fuzzy field,

(v) $\mu([x,y]) \leq \mu(x) \vee \mu(y)$

hold for all $x, y \in \mathbb{L}$.

Example 28 Let $\mathbb{L} = \mathbb{L}_0 \oplus \mathbb{L}_1$ be a Lie superalgebra, where $\mathbb{L}_0 = \Re^2$ (the real vector space), $\mathbb{L}_1 = 0$ (null (sub)space) and $[x,y] = x \times y$($\times$ is cross product of vectors) for $x, y \in \mathbb{L}_0$ and the remaining Lie brackets are zero. In fact, $\mathbb{L}$ is a Lie algebra over a field $\mathbb{R}$.
Let $\mu_{\bar{0}} : \mathbb{L}_0 \to [0,1]$ be a fuzzy set defined by

$$\mu_{\bar{0}}(x,y) = \begin{cases} 0 & \text{if } x = y = 0, \\ 1 & \text{otherwise,} \end{cases}$$

Let $\mu_{\bar{1}} : \mathbb{L}_1 \to [0,1]$ be a fuzzy set defined by $\mu_{\bar{1}}(x) = 0$ for all $x \in \mathbb{L}_1$. Let $\lambda : \mathbb{R} \to [0,1]$ be fuzzy set defined by

$$\lambda(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}, \\ 1 & x \in \mathbb{R} - \mathbb{Q}(\sqrt{3}). \end{cases}$$

Then it is easy to see that $\mu_{\bar{0}}$ and $\mu_{\bar{1}}$ are co-fuzzy subspaces of $\mathbb{L}_0$ and $\mathbb{L}_1$ over a co-fuzzy field, respectively. Let $\mu : \mathbb{L} \to [0,1]$ be a fuzzy set defined by $\mu(x) = \mu_{\bar{0}}(x)$ for all $x \in \mathbb{L}$. By routine computations, it is easy to check that μ is co-fuzzy Lie sub-superalgebra of $\mathbb{L}$ over a co-fuzzy field.

Example 29 Let $\mathbb{L} = \mathbb{L}_0 \oplus \mathbb{L}_1$ be a Lie superalgebra, where $\mathbb{L}_0 = < e >$, $\mathbb{L}_1 = < x_1, x_2, y_1, y_2 >$, $[x_i, y_i] = e$ for $i = 1, 2$, and remaining Lie brackets are zero.
Let $\mu_{\bar{0}} : \mathbb{L}_0 \to [0,1]$ be a fuzzy set defined by

$$\mu_{\bar{0}}(x) = \begin{cases} 0.6 & \text{if } x \in \mathbb{L}_0 - \{0\}, \\ 0 & x\text{=0,} \end{cases}$$

Let $\mu_{\bar{1}} : \mathbb{L}_1 \to [0,1]$ be a fuzzy set defined by

$$\mu_{\bar{1}}(x) = \begin{cases} 0.5 & \text{if } x \in \mathbb{L}_1 - \{0\}, \\ 0 & x\text{=0.} \end{cases}$$

Let $\lambda : \mathbb{R} \to [0,1]$ be fuzzy set defined by

$$\lambda(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}, \\ 1 & x \in \mathbb{R} - \mathbb{Q}(\sqrt{3}). \end{cases}$$

Then, clearly $\mu_{\bar{0}}$ and $\mu_{\bar{1}}$ are co-fuzzy subspaces of $\mathbb{L}_0$ and $\mathbb{L}_1$ over a co-fuzzy field, respectively. Let $\mu : \mathbb{L} \to [0,1]$ be a fuzzy set defined by

$$\mu(x) = \begin{cases} 0.6 & \text{if } x \in \mathbb{L} - \{0\}, \\ 0 & x\text{=0.} \end{cases}$$

It is easy to see that:

(i) For $x \neq 0$, $(\mu_{\bar{0}} + \mu_{\bar{1}})(x) = \mu_{\bar{0}}(x) \vee \mu_{\bar{1}}(x) = 0.6 = \mu(x)$. So, μ is $\mathbb{Z}_2$ co-fuzzy subspace of $\mathbb{L}$ over a co-fuzzy field.

(ii) Let $x = a_i$, $y = b_i \in \mathbb{L}$, then $0.6 = \mu(e) = \mu([a_i, b_i]) = \mu([x,y] \leq \mu(x) \vee \mu(y) = 0.6$.

If not, $0 = \mu(0) = \mu([x,y]) \leq \mu(x) \vee \mu(y) = 0$. Hence μ is co-fuzzy Lie sub-superalgebra of $\mathbb{L}$ over a co-fuzzy field.

The proofs of the following propositions are obvious.

Proposition 30 Let μ be a fuzzy set of $\mathbb{L}$. Then μ is a co-fuzzy Lie sub-superalgebra of $\mathbb{L}$ over a co-fuzzy field if and only if μ^c is a fuzzy Lie sub-superalgebra of $\mathbb{L}$ over a fuzzy field.

Proposition 31 If μ and ν are co-fuzzy Lie sub-superalgebras over a co-fuzzy field, then $\mu + \nu$ and $\mu \cap \nu$ are co-fuzzy Lie sub-superalgebras in $\mathbb{L}$ over a co-fuzzy field.

Definition 32 For a family of fuzzy sets $\{\mu_i | i \in \Lambda\}$ in a Lie superalgebra $\mathbb{L}$, the *union* $\bigvee \mu_i$ of $\{\mu_i | i \in \Lambda\}$ is defined by

$$(\bigvee \mu_i)(x) = \sup\{\mu_i(x) | i \in \Lambda\}$$

for each $x \in \mathbb{L}$.

Proposition 33 If $\{\mu_i | i \in \Lambda\}$ is a family of co-fuzzy Lie sub-superalgebras of $\mathbb{L}$ over a co-fuzzy field λ then so is $\bigvee \mu_i$.

Proof: For $x, y \in \mathbb{L}$ and $\alpha \in F$, we have

$$\begin{aligned}
(\bigvee \mu_i)(x+y) &= \sup\{\mu_i(x+y) | i \in \Lambda\} \\
&\leq \sup\{\mu_i(x) \vee \mu_i(y) | i \in \Lambda\} \\
&= (\sup\{\mu_i(x) | i \in \wedge\}) \vee (\sup\{\mu_i(y) | i \in \Lambda\}) \\
&= ((\bigvee \mu_i)(x)) \vee ((\bigvee \mu_i)(y)),
\end{aligned}$$

$$\begin{aligned}
(\bigvee \mu_i)(\alpha x) &= \sup\{\mu_i(\alpha x) | i \in \Lambda\} \\
&\leq \sup\{\lambda_i(\alpha) \vee \mu_i(x) | i \in \Lambda\} \\
&= (\sup\{\lambda_i(\alpha) | i \in \wedge\}) \vee (\sup\{\mu_i(x) | i \in \Lambda\}) \\
&= ((\bigvee \lambda_i)(\alpha)) \vee ((\bigvee \mu_i)(x)).
\end{aligned}$$

So $\bigvee \mu_i$ is a co-fuzzy subspace of $\mathbb{L}$ over a co-fuzzy field λ.
For $x \in \mathbb{L}$, $x = x_{\overline{0}} + x_{\overline{1}}$, where $x_{\overline{0}} \in \mathbb{L}_0$ and $x_{\overline{1}} \in \mathbb{L}_1$, we have

$$\begin{aligned}&(\bigvee \mu_i)(x) = (\bigvee \mu_i)(x_{\overline{0}} + x_{\overline{1}})\\ &= \sup\{\mu_i(x_{\overline{0}} + x_{\overline{1}}) | i \in \Lambda\}\\ &= \sup\{\mu_i(x_{\overline{0}}) \vee \mu_i(x_{\overline{1}}) | i \in \Lambda\}\\ &= (\sup\{\mu_i(x_{\overline{0}}) | i \in \wedge\}) \vee (\sup\{\mu_i(x_{\overline{1}}) | i \in \Lambda\})\\ &= ((\bigvee \mu_i)(x_{\overline{0}})) \vee ((\bigvee \mu_i)(x_{\overline{1}})).\end{aligned}$$

This shows that $\bigvee \mu_i$ is a $\mathbb{Z}_2$-grading co-fuzzy subspace of $\mathbb{L}$. For $x, y \in \mathbb{L}$, we have

$$\begin{aligned}&(\bigvee \mu_i)([x,y]) = \sup\{\mu_i([x,y]) | i \in \Lambda\}\\ &\leq \sup\{\mu_i(x) \vee \mu_i(y) | i \in \Lambda\}\\ &= (\sup\{\mu_i(x) | i \in \wedge\}) \vee (\sup\{\mu_i(y) | i \in \Lambda\})\\ &= ((\bigvee \mu_i)(x)) \vee ((\bigvee \mu_i)(y)).\end{aligned}$$

Hence $\bigvee \mu_i$ is a co-fuzzy Lie sub-superalgebra of $\mathbb{L}$ over a co-fuzzy field. □

Definition 34 Let μ be a fuzzy set and $s \in [0, 1]$. Then lower level subset $L(\mu; s)$ and weak level subset ${}^{>}L(\mu; t)$ of $\mathbb{L}$ are defined by

- $L(\mu; s) = \{x \in \mathbb{L} \,|\, \mu(x) \leq s\}$ and ${}^{>}L(\mu; s) = \{x \in \mathbb{L} \,|\, \mu(x) < s\}$, respectively.

Theorem 35 Let μ be a co-fuzzy Lie sub-superalgebra of $\mathbb{L}$ over a co-fuzzy field λ and let ν be the closure of the co-image of μ. Then the following conditions are equivalent:

(a) μ is a co-fuzzy Lie sub-superalgebra of $\mathbb{L}$ over a co-fuzzy field λ,

(b) the non-empty weak level subset ${}^{>}L(\mu; s)$ of μ is a Lie sub-superalgebra of $\mathbb{L}$ for all $s \in [0, 1]$,

(c) the non-empty weak level subset ${}^{>}L(\mu; s)$ of μ is a Lie sub-superalgebra of $\mathbb{L}$ for all $s \in Im(\mu) \setminus \nu$,

(d) the nonempty level subset $L(\mu; s)$ of μ is a Lie sub-superalgebra of $\mathbb{L}$ for all $s \in Im(\mu)$,

(e) the nonempty level subset $L(\mu; s)$ of μ is a Lie sub-superalgebra of $\mathbb{L}$ for all $s \in [0, 1]$.

Proof: (a)⇒(b): Let $x \in {}^{>}L(\mu; s) \subseteq \mathbb{L} = \mathbb{L}_0 \oplus \mathbb{L}_1$. Then x=$x_{\overline{0}} + x_{\overline{1}}$, where $x_{\overline{0}} \in \mathbb{L}_0$ and $x_{\overline{1}} \in \mathbb{L}_1$. Since $\mu(x) = \max\{\mu_0(x_{\overline{0}}), \mu_1(x_{\overline{1}})\} < s$, $\mu_0(x_{\overline{0}}) < s$ and $\mu_1(x_{\overline{0}}) < s$ which imply that $x_{\overline{0}}$, $x_{\overline{1}} \in {}^{>}L(\mu; s)$. Let $x, y \in {}^{>}L(\mu; s)$. Then $\mu(x) < s$ and $\mu(y) < s$. $\mu([x,y]) \leq \mu(x) \vee \mu(y) <= s$ which imply that $[x,y] \in L(\mu; >, s)$.
(b)⇒(c) and (c)⇒(d) are obvious.
(d)⇒(e): Let $x, y \in L(\mu; s)$ for $s \in [0, 1]$. Then $\mu(x) \leq s$ and $\mu(y) \leq s$. Assume that $t = \mu(x) \vee \mu(y)$. Then $t \leq s$, $\mu(x) \leq t$ and $\mu(y) \leq t$. Since $\mu(x) \vee \mu(y) = t$, $\mu(x) = t$ and $\mu(y) = t$, i.e., $t \in Im(\mu)$. Thus x, $y \in L(\mu; t)$. Since $L(\mu; t)$ is Lie sub-superalgebra of $\mathbb{L}$, $[x,y] \in L(\mu; t)$. So $\mu(x) \leq t \leq s$. Thus $[x,y] \in L(\mu; s)$. Hence $L(\mu; s)$ is Lie sub-superalgebra of $\mathbb{L}$.
(e)⇒(a): Let x, $y \in \mathbb{L}$ and $\alpha \in F$. Suppose that $\mu(x) \vee \mu(y) = s$, then $\mu(x) \leq s$ and $\mu(y) \leq s$, so x, $y \in L(\mu; s)$. Since $L(\mu; s)$ is Lie sub-superalgebra of $\mathbb{L}$, $x + y, \alpha x \in L(\mu; s)$. Thus $\mu_{\overline{0}}(x + y) \leq \mu(x) \vee \mu(y) \leq s$ and $\mu(\alpha x) \leq F(\alpha) \vee \mu(x) \leq s$. Hence μ is co-fuzzy subspace of $\mathbb{L}$ over a co-fuzzy field.
Let $x \in \mathbb{L}$. Assume that $\mu(x) = s$. Then $x \in L(\mu; s)$. Since $L(\mu; s)$ is $\mathbb{Z}_2$-graded subspace of $\mathbb{L}$, we can express $x = x_{\overline{0}} + x_{\overline{1}}$, where $x_{\overline{0}} \in \mathbb{L}_0 \cap L(\mu; s)$ and $x_{\overline{1}} \in \mathbb{L}_1 \cap L(\mu; s)$. Thus $\mu(x) \leq \max(\mu(x_{\overline{0}}), \mu(x_{\overline{1}}))$. Define a mapping $\mu_{\overline{0}} : \mathbb{L}_0 \to [0, 1]$ by $\mu_{\overline{0}}(x) = \mu(x)$ and $\mu_{\overline{1}} : \mathbb{L}_1 \to [0, 1]$ by $\mu_{\overline{1}}(x) = \mu(x)$. Then $s = \mu(s) \leq \mu(x_{\overline{0}}) \vee \mu(x_{\overline{1}}) = \mu_{\overline{0}}(x_{\overline{1}}) \vee \mu_{\overline{0}}(x_{\overline{1}})$ and $\mu_{\overline{0}}(x_{\overline{1}}) \vee \mu_{\overline{0}}(x_{\overline{1}}) \leq s$. Thus $\mu_{\overline{0}}(x_{\overline{0}}) \vee \mu_{\overline{0}}(x_{\overline{1}}) = s = \mu(x)$. Hence μ is $\mathbb{Z}_2$-graded co-fuzzy subspace.
Let x, $y \in \mathbb{L}$ be such that $x \in L(\mu; s)$, $y \in L(\mu; s)$. Then $\mu(x) \leq s$, $\mu(y) \leq s$. It follows that $\mu([x,y]) \leq \mu(x) \vee \mu(y) \leq s$ so that $[x,y] \in L(\mu; s)$. Hence μ is a co-fuzzy Lie sub-superalgebra of $\mathbb{L}$ over a co-fuzzy field λ. □

Theorem 36 If μ is a co-fuzzy Lie sub-superalgebra of $\mathbb{L}$ over a co-fuzzy field, then for all $x \in \mathbb{L}$

$$\mu(x) = \inf\{s \in [0, 1] \mid x \in L(\mu; s)\}.$$

Proof: Let $t := \inf\{s \in [0, 1] \mid x \in L(\mu; s)\}$, and let $\epsilon > 0$. Then $t - \epsilon > s$ for some $s \in [0, 1]$ such that $x \in L(\mu; s)$, and so $t - \epsilon > \mu(x)$. Since ϵ is an arbitrary, it follows that $t \geq \mu(x)$. Now let $\mu(x) = v$, then $x \in L(\mu; v)$ and so $v \in \{s \in [0, 1] \mid x \in L(\mu; s)\}$. Thus $\mu(x) = v \geq \inf\{s \in [0, 1] \mid x \in L(\mu; s)\} = t$. Hence $\mu(x) = t$. This completes the proof. □
A co-fuzzy Lie sub-superalgebra μ of a Lie superalgebra $\mathbb{L}$ is said to be *abnormal* if $\mu(0) = 0$.

Theorem 37 Let μ be a co-fuzzy Lie sub-superalgebra of $\mathbb{L}$ over a co-fuzzy field and μ^* be a fuzzy set in $\mathbb{L}$ defined by $\mu^*(x) = \mu(x) - \mu(0)$ for all $x \in \mathbb{L}$. Then μ^* is an abnormal Lie sub-superalgebra of $\mathbb{L}$ containing μ.

Proof: For $x, y \in \mathbb{L}$ and $\alpha \in F$, we have

$$\begin{aligned}\mu^*(x + y) &= \mu(x + y) - \mu(0)\\ &\leq \mu(x) \vee \mu(y) - \mu(0)\\ &= \mu(x) - \mu(0) \vee \mu(y) - \mu(0))\\ &= \mu^*(x) \vee \mu^*(y),\\ \mu^*(\alpha x) &= \mu(\alpha x) - \mu(0)\\ &\leq \lambda(\alpha) \vee \mu(x) - \mu(0)\\ &= \lambda(\alpha) - \lambda(0) \vee \mu(x) - \mu(0))\\ &= \lambda^*(\alpha) \vee \mu^*(x).\end{aligned}$$

This shows that μ^* is a co-fuzzy subspace over a co-fuzzy field. For $x \in \mathbb{L}$, $x = x_{\bar{0}} + x_{\bar{1}}$, where $x_{\bar{0}} \in \mathbb{L}_0$ and $x_{\bar{1}} \in \mathbb{L}_1$, we have

$$\begin{aligned} \mu^*(x) &= \mu^*(x_{\bar{0}} + x_{\bar{1}}) = \mu(x_{\bar{0}} + x_{\bar{1}}) - \mu(0) \\ &= \mu(x_{\bar{0}}) \vee \mu(x_{\bar{1}}) - \mu(0) \\ &= \mu(x_{\bar{0}}) - \mu(0) \vee \mu(x_{\bar{1}}) - \mu(0)) \\ &= \mu^*(x_{\bar{0}}) \vee \mu^*(x_{\bar{1}}). \end{aligned}$$

This shows that μ^* is a $\mathbb{Z}_2$-grading co-fuzzy subspace of $\mathbb{L}$.

For any $x, y \in \mathbb{L}$,

$$\begin{aligned} \mu^*([x,y]) &= \mu([x,y]) - \mu(0) \\ &\leq \mu(x) \vee \mu(y) - \mu(0) \\ &= (\mu(x) - \mu(0)) \vee (\mu(y) - \mu(0)) \\ &= \mu^*(x) \vee \mu^*(y). \end{aligned}$$

Hence μ^* is an abnormal Lie sub-superalgebra of $\mathbb{L}$. Clearly, $\mu^*(0) = 1$ and $\mu \subset \mu^*$. This ends the proof. □

Corollary 38 If μ is a co-fuzzy Lie sub-superalgebra of $\mathbb{L}$ over a co-fuzzy field satisfying $\mu^+(x) = 1$ for some $x \in \mathbb{L}$, then $\mu(x) = 1$.

THE HOMOMORPHISMS

In this Section, we present the concept of co-fuzzy Lie sub-superalgebras of Lie superalgebras over a co-fuzzy field under homomorphisms.

Theorem 39 Let $f : \mathbb{L} \to \acute{\mathbb{L}}$ be an epimorphism of Lie superalgebras. If ν is a co-fuzzy Lie sub-superalgebras of $\acute{\mathbb{L}}$ over a co-fuzzy field and μ is the pre-image of ν under f. Then μ is a co-fuzzy Lie sub-superalgebra of $\mathbb{L}$ over a co-fuzzy field .

Proof: For any $x, y \in \mathbb{L}$ and $\alpha \in F$,

$$\begin{aligned} \mu(x+y) &= \nu(f(x+y)) = \nu(f(x) + f(y)) \\ &\leq \nu(f(x)) \vee \nu(f(y)) = \mu(x) \vee \mu(y), \\ \mu(\alpha x) &= \nu(f(\alpha x)) = \nu(f(\alpha) f(x)) \\ &\leq \lambda(f(\alpha)) \vee \nu(f(x)) = \lambda(\alpha) \vee \mu(x). \end{aligned}$$

This show that μ is a co-fuzzy subspace of $\mathbb{L}$ over a co-fuzzy field.
Let $x \in \mathbb{L}$. Then $x = x_{\bar{0}} + x_{\bar{1}}$, where $x_{\bar{0}} \in \mathbb{L}_0$ and $x_{\bar{1}} \in \mathbb{L}_1$. Since f preserves the grading, $f(x) = f(x_{\bar{0}} + x_{\bar{1}}) = f(x_{\bar{0}}) + f(x_{\bar{1}})$. So

$$\begin{aligned} \mu(x) &= \nu(f(x)) = \nu(f(x_{\bar{0}}) + f(x_{\bar{1}})) \\ &= \nu(f(x_{\bar{0}})) \vee \nu(f(x_{\bar{1}})) \\ &= \mu(x_{\bar{0}}) \vee \mu(x_{\bar{1}}). \end{aligned}$$

This shows that μ is a $\mathbb{Z}_2$-grading co-fuzzy subspace of $\mathbb{L}$. For any $x, y \in \mathbb{L}$,

$$\begin{aligned} \mu([x,y]) &= \nu(f([x,y])) = \nu([f(x), f(y)]) \\ &\leq \nu(f(x)) \vee \nu(f(y)) = \mu(x) \vee \mu(y). \end{aligned}$$

Hence μ is a co-fuzzy Lie subsperalgebra of $\mathbb{L}$ over a co-fuzzy field. □

Definition 40 Let $\mathbb{L}$ and $\acute{\mathbb{L}}$ be two Lie superalgebras and let f be a function of $\mathbb{L}$ into $\acute{\mathbb{L}}$. If μ is a fuzzy set in $\acute{\mathbb{L}}$, then the *preimage* of μ under f is the fuzzy set in $\mathbb{L}$ defined by $f^{-1}(\mu)(x) = \mu(f(x))$ for all $x \in \mathbb{L}$.

Theorem 41 Let $f : \mathbb{L} \to \acute{\mathbb{L}}$ be an onto homomorphism of Lie superalgebras. If μ is a co-fuzzy Lie sub-superalgebra of $\acute{\mathbb{L}}$ over a co-fuzzy field λ, then $f^{-1}(\mu)$ is a co-fuzzy Lie sub-superalgebra of $\mathbb{L}$ over a co-fuzzy field λ.

Proof: Let $x_1, x_2 \in \mathbb{L}$ and $\alpha \in F$, then

$$\begin{aligned} f^{-1}(\mu)(x_1 + x_2) &= \mu(f(x_1) + f(x_2)) \\ &\leq \mu(f(x_1) \vee \mu(f(x_2)) \\ &= f^{-1}(\mu)(x_1) \vee f^{-1}(\mu)(x_2), \\ f^{-1}(\mu)(\alpha x_1) &= \mu(f(\alpha) + f(x_1)) \\ &\leq \lambda(f(\alpha) \vee \mu(f(x_1)) \\ &= f^{-1}(\lambda)(\alpha) \vee f^{-1}(\mu)(x_1). \end{aligned}$$

This shows that $f^{-1}(\mu)$ is a co-fuzzy subspace of $\mathbb{L}$ over a co-fuzzy field.
Let $x \in \mathbb{L}$. Then $x = x_{\bar{0}} + x_{\bar{1}}$, where $x_{\bar{0}} \in \mathbb{L}_0$ and $x_{\bar{1}} \in \mathbb{L}_1$. Since f preserves the grading, $f(x) = f(x_{\bar{0}} + x_{\bar{1}}) = f(x_{\bar{0}}) + f(x_{\bar{1}})$.

$$\begin{aligned} f^{-1}(\mu)(x) &= \mu(f(x)) = \mu(f(x_{\bar{0}}) + f(x_{\bar{1}})) \\ &= \mu(f(x_{\bar{0}})) \vee \mu(f(x_{\bar{1}})) \\ &= f^{-1}(\mu)(x_{\bar{0}}) \vee f^{-1}(\mu)(x_{\bar{1}}). \end{aligned}$$

This shows that μ is a $\mathbb{Z}_2$-grading co-fuzzy subspace of $\mathbb{L}$.

$$\begin{aligned} f^{-1}(\mu)([x_1, x_2]) &= \mu([f(x_1), f(x_2)]) \\ &\leq \mu(f(x_1) \vee \mu(f(x_2)) \\ &= f^{-1}(\mu)(x_1) \vee f^{-1}(\mu)(x_2). \end{aligned}$$

Hence $f^{-1}(\mu)$ is a co-fuzzy Lie sub-superalgebra of $\mathbb{L}$ over a co-fuzzy field. □

Corollary 42 Let $f : \mathbb{L} \to \acute{\mathbb{L}}$ be an onto homomorphism of Lie superalgebras. If μ is a co-fuzzy Lie sub-superalgebra of $\acute{\mathbb{L}}$, then $f^{-1}(\mu^c) = (f^{-1}(\mu))^c$.

Theorem 43 Let μ be an co-fuzzy Lie sub-superalgebra of $\mathbb{L}$ over a co-fuzzy field λ and Let $f : \mathbb{L} \to \acute{\mathbb{L}}$. Then the image $f(\mu)$ is a co-fuzzy Lie sub-superalgebra of $\acute{\mathbb{L}}$ over a co-fuzzy field λ.

Proof: Consider $f(x)$, $f(y) \in$ f($\mathbb{L}$). Let x_0, $y_0 \in$ $f^{-1}(f(x))$ be such that $\mu(x_0) = \inf_{t \in f^{-1}(f(x))} \mu(t)$ and $\mu(y_0) = \inf_{t \in f^{-1}(f(y))} \mu(t)$, respectively. Then we can deduce that

$$\begin{aligned}&\mu(f(x+y))=\mu(f(x)+f(y))\\&=\inf_{t\in f^{-1}(f(x)+f(y))}\mu(t)\\&\leq\mu(x_0+y_0)\leq\mu(x_0)\vee\mu(y_0)\\&=\inf_{t\in f^{-1}(f(x))}\mu(t)\vee\inf_{t\in f^{-1}(f(y))}\mu(t)\\&=\mu(f(x))\vee\mu(f(y)),\\&\mu(f(\alpha x))=\mu(f(\alpha)f(x))=\inf_{t\in f^{-1}(f(\alpha)f(x))}\mu(t)\\&\leq\mu(\alpha_0x_0)\leq\lambda(\alpha_0)\vee\mu(x_0)\\&=\inf_{t\in f^{-1}(f(\alpha))}\mu(t)\vee\inf_{t\in f^{-1}(f(x))}\mu(t)\\&=\lambda(f(\alpha))\vee\mu(f(x)).\end{aligned}$$

So μ is a co-fuzzy subspace of $\acute{\mathbb{L}}$ over a co-fuzzy field.
Let $f(x)\in f(\mathbb{L})$. Then $f(x)=f(x_{\bar{0}})+f(x_{\bar{1}})$, where $f(x_{\bar{0}})\in f(\mathbb{L}_0)$ and $f(x_{\bar{1}})\in f(\mathbb{L}_1)$.

$$\begin{aligned}\mu(f(x))&=\mu(f(x_{\bar{0}})+f(x_{\bar{1}}))\\&=\inf_{t\in f^{-1}(f(x_{\bar{0}})+f(x_{\bar{1}}))}\mu(t)\\&=\mu(x_0+y_0)=\mu(x_0)\vee\mu(y_0)\\&=\inf_{t\in f^{-1}(f(x_{\bar{0}}))}\mu(t)\vee\inf_{t\in f^{-1}(f(x_{\bar{1}}))}\mu(t)\\&=\mu(f(x_{\bar{0}}))\vee\mu(f(x_{\bar{1}})).\end{aligned}$$

This shows that μ is a $\mathbb{Z}_2$-grading co-fuzzy subspace of $\acute{\mathbb{L}}$.

$$\begin{aligned}&\mu(f([x,y]))=\mu([f(x),f(y)])=\inf_{t\in f^{-1}([f(x),f(y)])}\mu(t)\\&\leq\mu([x_0,y_0])\leq\mu(x_0)\vee\mu(y_0)\\&=\inf_{t\in f^{-1}(f(x))}\mu(t)\vee\inf_{t\in f^{-1}(f(y))}\mu(t)\\&=\mu(f(x))\vee\mu(f(y)).\end{aligned}$$

Hence μ is a co-fuzzy Lie sub-superalgebra of $\acute{\mathbb{L}}$ over a co-fuzzy field. □

Definition 44 Let $\mathbb{L}$ and $\acute{\mathbb{L}}$ Lie superalgebras and let f be a function of μ is a fuzzy set in $\mathbb{L}$, then the *co-image* of μ under f is the fuzzy set defined by

$$f(\mu)(y)=\begin{cases}\inf\{\mu(t)\mid t\in\mathbb{L},f(t)=y\}, & \text{if } f^{-1}(y)\neq\emptyset\\ 1, & \text{otherwise.}\end{cases},$$

Definition 45 Let $\mathbb{L}$ and $\acute{\mathbb{L}}$ be any sets and let $f:\mathbb{L}\to\acute{\mathbb{L}}$ be any function. A fuzzy set μ is called f-*invariant* if and only if for $x,y\in\mathbb{L}$, $f(x)=f(y)$ implies $\mu(x)=\mu(y)$.

Theorem 46 Let $f:\mathbb{L}\to\acute{\mathbb{L}}$ be an epimorphism of Lie superalgebras. Then μ is an f-invariant co-fuzzy Lie sub-superalgebra of $\mathbb{L}$ over a co-fuzzy field λ if and only if $f(\mu)$ is co-fuzzy Lie sub-superalgebra of $\acute{\mathbb{L}}$ over a co-fuzzy field λ.

Proof: Let $x,y\in\acute{\mathbb{L}}$ and $\alpha\in F$. Then there exist $a,b\in\mathbb{L}$ and $\beta\in F$ such that $f(a)=x$, $f(b)=y$, $f(\beta)=\alpha$, $x+y=f(a+b)$ and $\alpha x=\beta f(a)$. Since μ is f-invariant, by assumption, we have

$$\begin{aligned}&f(\mu)(x+y)=\mu(a+b)\leq\mu(a)\vee\mu(b)\\&=f(\mu)(x)\vee f(\mu)(y),\\&f(\mu)(\alpha x)=\mu(\beta a)\leq\lambda(\beta)\vee\mu(b)\\&=f(\lambda)(\alpha)\vee f(\mu)(x).\end{aligned}$$

Thus $f(\mu)$ is a co-fuzzy subspace of $\acute{\mathbb{L}}$ over a co-fuzzy field.
Let $x\in\acute{\mathbb{L}}$ and $x=x_{\bar{0}}+x_{\bar{1}}$, where $x_{\bar{0}}\in\acute{\mathbb{L}}_{\bar{0}}$ and $x_{\bar{1}}\in\acute{\mathbb{L}}_{\bar{0}}$. Then there exist $a\in\mathbb{L}$ and $a=a_{\bar{0}}+a_{\bar{1}}$, where $a_{\bar{0}}\in\mathbb{L}_0$, $a_{\bar{1}}\in\mathbb{L}_1$ such that $f(a)=x$, $f(a_{\bar{0}})=x_{\bar{0}}$, $f(a_{\bar{1}})=x_{\bar{1}}$.

$$\begin{aligned}f^{-1}(\mu)(x)&=f(\mu)(x)=\mu(a)=\mu(a_{\bar{0}}+a_{\bar{1}})\\&=\mu(a_{\bar{0}})\vee\mu(a_{\bar{1}})\\&=f^{-1}(\mu)(x_{\bar{0}})\vee f^{-1}(\mu)(x_{\bar{1}}).\end{aligned}$$

This shows that $f(\mu)$ is a $\mathbb{Z}_2$-grading co-fuzzy subspace of $\acute{\mathbb{L}}$.

$$\begin{aligned}&f(\mu)([x,y])=\mu([a,b])\leq\mu(a)\vee\mu(b)\\&=f(\mu)(x)\vee f(\mu)(y).\end{aligned}$$

Hence $f(\mu)$ is a co-fuzzy Lie sub-superalgebra of $\mathbb{L}$ over a co-fuzzy field.
Conversely, if $f(\mu)$ is a co-fuzzy Lie sub-superalgebra of $\acute{\mathbb{L}}$, then for any $x\in\mathbb{L}$

$$\begin{aligned}f^{-1}(f(\mu))(x)&=f(\mu)(f(x))\\&=\inf\{\mu(t)\mid t\in\mathbb{L},f(t)=f(x)\}\\&=\inf\{\mu(t)\mid t\in\mathbb{L},\mu(t)=\mu(x)\}\\&=\mu(x).\end{aligned}$$

Hence $f^{-1}(f(\mu))=\mu$ is a co-fuzzy Lie sub-superalgebra by Theorem 41. This completes the proof. □

Theorem 47 Let $f:\mathbb{L}\to\acute{\mathbb{L}}$ be an epimorphism of Lie superalgebras. If μ and ν are co-fuzzy Lie sub-superalgebras of $\mathbb{L}$ over a co-fuzzy field, then $f(\mu+\nu)=f(\mu)+f(\nu)$.

Proof: The proof is trivial and hence it is omitted. □

CONCLUSIONS

We discussed the concept of co-fuzzy Lie sub-superalgebras of Lie superalgebras over a co-fuzzy field and looked some fundamental properties. It is clear that the most of these results can be simply extended to fuzzy Lie sub-superalgebras of Lie superalgebras with respect to an s-norm. We shall study:(1) Redefined fuzzy Lie superalgebras over a fuzzy field; (2) Intuitionistic fuzzy Lie superalgebras over an intuitionistic field; (3)Vague Lie superalgebras over a vague field; (4) Soft Lie superalgebras over a soft field. Our obtained results probably can be applied in various fields such as artificial intelligence, signal processing, multiagent systems, pattern recognition, robotics, expert systems, medical diagnosis and engineering.

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