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Convergence Analysis of Spline Solutions for Special Nonlinear Two-Order Boundary Value Problems

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Abstract: The smooth approximate solution of nonlinear second order boundary value problems are developed by using non-polynomial quintic spline function. A new approach convergence analysis of the presented methods is discussed. Some examples are considered in our references. By considering the maximum absolute errors in the solution at grid points and tabulated in tables for different choices of step size. We conclude that our presented method produces the accurate results in comparison with those obtained by the existing methods.

Key words: Two-point boundary value problem . non-polynomial quintic spline . convergence analysis

INTRODUCTION

We consider the nonlinear two-point boundary value problem

$$u^{"} = f(x,u), u(a) = \lambda, u(b) = \mu, a \le x \le b \qquad (1)$$

where f (x, u) is sufficiently differentiable on [a,b] and a, b, λ and μ are arbitrary real finite constants. For existence and uniqueness of a solution of (1) see [2]. Special differential equations of the second order and in particular systems of such equations, occur frequently, for instance in mechanical problems without dissipation (see [2]). These special boundary-value problems also occur in other engineering contexts, for example in Troeshs problem relating to the confinement of a plasma column by radiation pressure. A more commonly used finite difference method for solving (1) numerically is discussed by many authors and we refer the reader in particular to Fox [l], Henrici [2], Aziz *et al*. [3], Bramble *et al*. [4], Fischer *et al*. [5] and Usmani [6]. The possibility of using spline functions for obtaining a smooth approximate solution of (1) is briefly discussed by Ahlberg *et al*. [7]. Since then Albasiny and Hoskins [8], Bickley [9], Fyfe [10] and Sakai and Usmani [11] have used the cubic spline for obtaining approximations. Bhatta *et al*. [12] have used the spline functions of degree seven and eight, Usmani *et al* [13] used the Quintic spline, Also Usmani and Sakai [14] used cubic and quartic spline. Khan [15] used the parametric cubic spline. Rashidinia *et al*. [16] developed non-polynomial spline methods for the solution of a system of obstacle problems. Tirmizi and Twizell [17] have been developed finite difference methods of orders six and eight for numerical solutions (1).

Several techniques including decomposition, homotopy perturbation, polynomial and non polynomial spline, Sink Galerkin, perturbation, homotopy analysis, finite difference and modified variational iteration have been employed to solve such problems [20-30]

In this paper, we will use the consistency relation of our non-polynomial quintic spline in [16] for solution of (1) and we obtained sixth-order convergent for arbitrary α, β, p, r and s. Section 2 is devoted to the description of the methods and development of boundary conditions. The new approach for convergence analysis discussed in Section 3. Finally, in section 4, some numerical evidences are included to show the practical applicability and superiority of our methods and compare with the other methods.

DESCRIPTION OF THE METHOD AND DEVELOPMENT OF BOUNDARY CONDITIONS

Let us consider a mesh with nodal points x_i i = 1(1) n on [a,b] such that:

$$\Delta: \quad a = x_0 \prec x_1 \prec \ldots \prec x_n = b, \qquad h = \frac{b-a}{n}$$

We also denote the function value u (x_i) by u_i. For each segment [x_i, x_{i+1}], i = 0 (1) n-1, by using the non-polynomial quintic spline relation derived in our paper [16] we have

$$pM_{i-2} + rM_{i-1} + sM_i + rM_{i+1} + pM_{i+2} =$$
$$\frac{1}{h^2}[\alpha(u_{i+2} + u_{i-2}) + 2(\beta-\alpha)(u_{i+1} + u_{i-1}) + (2\alpha - 4\beta) u_i]$$

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where

$$\begin{aligned} p &= \alpha_1 + \tfrac{\alpha}{6}, \qquad r = 2[\tfrac{1}{6}(2\alpha + \beta) - (\alpha_1 - \beta_1)], \\ s &= 2[\tfrac{1}{6}(\alpha + 4\beta) - (\alpha_1 - 2\beta_1)], \\ \alpha &= (\tfrac{1}{\theta^2})(\theta \csc(\theta) - 1), \beta = (\tfrac{1}{\theta^2})(1 - \theta\cot(\theta)), \\ \alpha_1 &= (\tfrac{1}{\theta^2})(\tfrac{1}{6} - \alpha), \quad \beta_1 = (\tfrac{1}{\theta^2})(\tfrac{1}{3} - \beta), \end{aligned} \tag{2}$$

At the mesh point x_i the proposed differential equation (1) may be discretized by

$$M_i = f(x_i, u_i) \tag{3}$$

where

$$M_i = s_i''(x_i), u_i = u(x_i)$$

Substituting (3) in the spline relation (2), we obtain

$$\begin{aligned} &(\alpha u_{i-2} - ph^2 f(x_{i-2}, u_{i-2})) + (2(\beta - \alpha)u_{i-1} - \\ &h^2 r f(x_{i-1}, u_{i-1})) + ((2\alpha - 4\beta)u_i - s h^2 f(x_i, u_i)) \\ &+(2(\beta - \alpha) - h^2 r f(x_{i+1}, u_{i+1})) + (\alpha u_{i+2} - \\ &p h^2 f(x_{i+2}, u_{i+2})) = 0, \qquad i = 2(1)n-2 \end{aligned} \tag{4}$$

To obtain unique solution for the nonlinear system (4) we need two more equation to be associate, so that we use the following boundary conditions. Following [16] in order to obtain the sixth-order boundary formula we define the following identities:

$$\left.\begin{aligned} &\sum_{k=0}^{3} a_k u_k + h^2 \sum_{k=0}^{5} b_k u_k'' + t_1 h^8 u_0^{(8)}), i = 1, \\ &\sum_{k=0}^{3} a_k u_{n-k} + h^2 \sum_{k=0}^{5} b_k u_{n-k}'' + t_{n-1} h^8 u_n^{(8)}), i = n-1, \end{aligned}\right\} \tag{5}$$

In order to obtain unknown coefficients a's and b's in (5) by Taylor's expansion we obtain:

$$(a_0, a_1, a_2, a_3) = (-10, 19, -8, 1), t_1 = t_{n-1} = (\tfrac{2179}{60480})$$
$$(b_0, b_1, b_2, b_3, b_4, b_5) = (\tfrac{179}{240}, \tfrac{1057}{120}, \tfrac{39}{40}, \tfrac{41}{60}, \tfrac{-61}{240}, \tfrac{1}{24})$$

CONVERGENCE ANALYSIS

In this section, we investigate the convergence analysis of the method. The equation (4) along with boundary condition (5) yields nine diagonal nonlinear systems of equations and may be written in matrix form as:

$$A_0 U^{(1)} + h^2 B f^{(1)}(U^{(1)}) = R^{(1)} \tag{6}$$

in (6) the matrices A_0 and B are order (n- 1) and are given by

$$A_0 = \begin{bmatrix} 19 & -8 & -1 & & & & \\ -8 & 18 & -8 & -1 & & & \\ -1 & -8 & 18 & -8 & -1 & & \\ & & & & & & \\ & & -1 & -8 & 18 & -8 & -1 \\ & & & -1 & -8 & 18 & -8 \\ & & & & -1 & -8 & 19 \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{1057}{120} & \frac{39}{40} & \frac{41}{60} & \frac{-61}{240} & \frac{1}{24} & & & & \\ \frac{-672}{360} & \frac{-2952}{360} & \frac{-672}{360} & \frac{-12}{360} & & & & & \\ \frac{-12}{360} & \frac{-672}{360} & \frac{-2952}{360} & \frac{-672}{360} & \frac{-12}{360} & & & & \\ & & & & & & & & \\ & & & \frac{-12}{360} & \frac{-672}{360} & \frac{-2952}{360} & \frac{-672}{360} & \frac{-12}{360} & \\ & & & & \frac{-12}{360} & \frac{-672}{360} & \frac{-2952}{360} & \frac{-672}{360} & \\ & & & \frac{1}{24} & \frac{-61}{240} & \frac{41}{60} & \frac{39}{40} & \frac{1057}{120} & \end{bmatrix} \tag{7}$$

where A_0, is a monotone five band matrix of order n-1, with $A_0^* = (a_{ij})$ is a tri-diagonal matrix defined by

$$a_{ij} = \begin{cases} 2, & i = j = 1,2,\ldots,n-1 \\ -1, & |i-j| = 1 \\ 0, & \text{otherwise} \end{cases} \tag{8}$$

And $A_1 = (a_{ij}^\bullet)$, is a tri-diagonal matrix defined by

$$a_{ij}^\bullet = \begin{cases} 4, & i = j = 1,2,\ldots,n-1 \\ 1, & |i-j| = 1 \\ 0, & \text{otherwise} \end{cases} \tag{9}$$

The matrixes $f^{(1)}$ and $R^{(1)}$ each have (n-1) components and are given by

$$f^{(1)} = (f_1^{(1)}, \ldots, f_{n-1}^{(1)}) \tag{10}$$

where

$$f_l^{(1)}(U^{(1)}) = f(x_l, u_l^{(1)}),^{(1)} l = 1(1)n-1$$

and

$$R^{(1)} = \begin{bmatrix} 10\lambda - \frac{179}{240}h^2 f(x_0, \lambda) \\ -\alpha\lambda + h^2 f(x_0, \lambda) \\ 0 \\ \vdots \\ 0 \\ -\alpha\mu + h^2 f(x_n, \lambda) \\ 10\mu - \frac{179}{240}h^2 f(x_n, \lambda) \end{bmatrix} \tag{11}$$

$$A_0 \overline{U}^{(1)} + h^2 B f^{(1)}(\overline{U}^{(1)}) = R^{(1)} + t^{(1)} \tag{12}$$

where the vector

$$\overline{U}^{(1)} = u(x_l), l = 1(1)n-1$$

is the exact solution and $t^{(1)} = [t_1, t_2, \ldots, t_{n-1}]^T$ is the vector of order n-1 of local truncation errors. From (6) and (12) we have:

$$AE^{(1)} = [A_0 + h^2 BF_k\ (U^{(1)})]E^{(1)} = t^{(1)} \quad (13)$$

where

$$E^{(1)} = \overline{U}^{(1)} - U^{(1)} = [e_1, e_2, \ldots, e_{n-1}]^T$$

$$f^{(1)}(\overline{U}^{(1)}) - f^{(1)}(U^{(1)}) = F_k(U^{(1)})E^{(1)} \quad (14)$$

and

$$F_k(U^{(1)}) = \operatorname{diag}\{\frac{\partial f_l}{\partial u_l^{(1)}}\}, l = 1(1)n-1 \quad (15)$$

is a diagonal matrix of order n-1.

Lemma 3.1 If M is a square matrix of order n and $\|M\| \prec 1$, then $(I+M)^{-1}$, exists and

$$\|(I+M)^{-1}\| \prec \frac{1}{1-\|M\|}$$

To explain the existence of A^{-1} since

$$A = A_0 + h^2 BF_k\ (U^{(1)})$$

we have to show

$$A_0 = A_0^* A_1 + 6A_0^*$$

is nonsingular.

Lemma 3.2 The matrix A_0 is nonsingular and

$$\|A_0^{-1}\| \le \frac{(b-a)^2}{44h^2} \quad (16)$$

Proof: by using lemma 3.1 and Henrici [2] we shall first require bounds for the element of $(A_0^*)^{-1} = (a_{ij}^*)$, where

$$a_{ij}^* = \begin{cases} \frac{j(n-i)}{n}, & i \ge j \\ \frac{i(n-j)}{n}, & i \le j \end{cases} \quad (17)$$

$$\sum_{j=1}^{n} a_{ij}^* = \sum_{j=1}^{i} \frac{j(n-i)}{n} + \sum_{j=i+1}^{n} \frac{i(n-j)}{n} \le \frac{n^2}{8} \quad (18)$$

where the equality holds only if n is odd. Inequality can be written as

$$\|(A_0^*)^{-1}\| \le \frac{(b-a)^2}{8h^2} \quad (19)$$

Also by using [13] we have

$$\|A_1^{-1}\| \le \frac{1}{2}, (A_0^* A_1 + 6A_0^*)^{-1} = (I + \tfrac{1}{6}A_1)^{-1}(6A_0^*)^{-1}$$

By using lemma 3.1 $(I + \frac{1}{6}A_1)^{-1}$ exists and we get

$$\|(A_0^* A_1 + 6A_0^*)^{-1}\| \le \frac{\|(6A_0^*)^{-1}\|}{1 - \|\frac{1}{6}A_1\|} = \frac{(b-a)^2}{44h^2} \quad (20)$$

where the norm referred to is the L_∞ norm.

Lemma 3.3: The matrix

$$A = A_0 + h^2 BF_k\ (U^{(1)})$$

is nonsingular, provided

$$\|Y\| \le \frac{11}{3(b-a)^2}$$

where

$$Y = \max\left|\frac{\partial f_l^{(1)}}{\partial u_l^{(1)}}\right|, l = 1(1)n-1$$

Proof: We know that

$$[A_0 + h^2 BF_k\ (U^{(1)})] = A_0[I + h^2 A_0^{-1} BF_k\ (U^{(1)})]$$

we need to show that inverse of

$$[I + h^2 A_0^{-1} BF_k\ (U^{(1)})]$$

exist. By using lemma 3.1, we have

$$h^2 \|A_0^{-1} BF_k\ (U^{(1)})\| \le h^2 \|A_0^{-1}\| \|B\| \|F_k\ (U^{(1)})\| \prec 1 \quad (21)$$

by using (20), (21) and

$$\|B\| \le 12, \|F_k(U^{(1)})\| \le Y = \max\left|\frac{\partial f_l^{(1)}}{\partial u_l^{(1)}}\right|, l = 1(1)n-1$$

we obtain

$$\|Y\| \le \frac{11}{3(b-a)^2}$$

As a consequence of Lemmas 3.1, 3.2 and 3.3 the nonlinear system (6) has a unique solution if

$$\|Y\| \le \frac{11}{3(b-a)^2}$$

We give alternative method in the theorem 3.1 to show that the matrix

$$A = A_0 + h^2 BF_k\ (U^{(1)})$$

is monotone.

Theorem 3.1: If

$$\|Y\| \le \frac{11}{3(b-a)^2}$$

then the matrix A. given by (13) is monotone.

Proof: From (13) we have

$$A = A_0 + h^2 BF_k\ (U^{(1)}) = P^2 + h^2 BF_k\ (U^{(1)})$$

Hence

$$\begin{aligned} &AP^{-2} = I + h^2 BF_k\ (U^{(1)})P^{-2} \\ &P^2 A^{-1} = (I + h^2 BF_k\ (U^{(1)})P^{-2})^{-1} = \\ &I - (h^2 BF_k\ (U^{(1)})P^{-2}) + (h^2 BF_k\ (U^{(1)})P^{-2})^2 - \\ &(h^2 BF_k\ (U^{(1)})P^{-2})^3 + \ldots = [I - (h^2 BF_k\ (U^{(1)})P^{-2})] \\ &[I + (h^2 B F_k\ (U^{(1)})P^{-2})^2 + (h^2 BF_k\ (U^{(1)})P^{-2})^4 + \ldots], \end{aligned}$$

Also if then, the two infinite series convergence. Let

$$\rho(h^2 BF_k\ (U^{(1)})P^{-2}) \prec 1$$

then, the two infinite series convergence. We obtain

$$\begin{aligned} &[P^{-2} - P^{-2}(h^2 BF_k\ (U^{(1)})P^{-2})] \\ &[I + (h^2 B F_k\ (U^{(1)})P^{-2})^2 + (h^2 BF_k\ (U^{(1)})P^{-2})^4 + \ldots] \end{aligned}$$

where the infinite series is nonnegative. Hence to show that A is monotone, it sufficient to show that

$$[P^{-2} - P^{-2}(h^2 BF_k\ (U^{(1)})P^{-2})] \succ 0$$

Here we have

$$\begin{aligned} &P^{-2} \succ P^{-2} h^2 BF_k\ (U^{(1)})P^{-2} \Rightarrow \\ &I \succ h^2 BF_k\ (U^{(1)})P^{-2} \Rightarrow \left\| h^2 BF_k\ (U^{(1)})P^{-2} \right\| \\ &\le h^2 \left\| A_0^{-1} \right\| \left\| B \right\| \left\| F_k\ (U^{(1)}) \right\| < 1 \end{aligned} \tag{22}$$

By substituting $\|A_0^{-1}\|, \|B\|$ and $\|F_k\ (U^{(1)})\|$ into (22) we get

$$\|Y\| \le \frac{11}{3(b-a)^2}$$

Theorem 3.2 Let u (x), be the exact solution of the boundary value Problem (1) and we assume u_i, i = 1 (1) n-1 be the numerical solution obtained by solving the system (13) then we have E = O (h^6), provided

$$\|y\| \le \frac{11}{3(b-a)^2}, \alpha = \frac{1}{12}, \beta = \frac{5}{12}, p = \frac{1}{360}, r = \frac{56}{360}, s = \frac{246}{360}$$

Proof: We can write the error equation (13) in the following form

$$\begin{aligned} &E^{(1)} = [A_0 + h^2 B\, F_k\ (U^{(1)})]^{-1} t^{(1)} = \\ &[I + h^2 A_0^{-1} BF_k\ (U^{(1)})]^{-1} A_0^{-1} t^{(1)}, \\ &\left\| E^{(1)} \right\| \le \left\| [I + h^2 A_0^{-1} BF_k\ (U^{(1)})]^{-1} \right\| \left\| A_0^{-1} \right\| \left\| t^{(1)} \right\| \end{aligned}$$

It follows that

$$\|E\| \le \frac{\left\| A_0^{-1} \right\| \left\| t^{(1)} \right\|}{1 - h^2 \left\| A_0^{-1} \right\| \left\| B \right\| \left\| F_k\ (U^{(1)}) \right\|}, \tag{23}$$

provided that

$$h^2 \left\| A_0^{-1} \right\| \left\| B \right\| \left\| F_k\ (U^{(1)}) \right\| < 1,$$

following [16] we have

$$\left\| t^{(1)} \right\| \le \frac{2179\, h^8 M_8}{60480} \tag{24}$$

where

$$M_8 = \max \left| u^{(8)}(\xi) \right|, a \le \xi \le b \tag{25}$$

Substituting

$$\left\| A_0^{-1} \right\|, \left\| B \right\|, \left\| F_k\ (U^{(1)}) \right\|$$

and $\|t^{(1)}\|$, from above relations in (24) and simplifying we obtain

$$\|E\| \le \frac{2179(b-a)^2 h^6 M_8}{60480(44 - 12(b-a)^2 \|Y\|)} \equiv O(h^6) \tag{26}$$

Provided that

$$\|Y\| \le \frac{11}{3(b-a)^2}$$

NUMERICAL ILLUSTRATIONS

In order to test the viability of the proposed method based on non-polynomial spline and to demonstrate its convergence computationally, we consider the following three test boundary-value problems.

Example 1: We consider the following boundary-value problem

$$-u''+\pi u = 2\pi^2 \sin(\pi x), u(0) = u(1) = 0$$

with the exact solution, $u(x) = \sin(\pi x)$

This problem has been solved using our method with different values of n = 8, 16, 32, 64, 128 and also the maximum absolute errors in solutions are tabulated in Table 1. The maximum absolute errors in solutions of this problem are compared with method in [18] for n = 10. and tabulated in Table 2.

Example 2: We consider the following boundary-value problem

$$u'' = \frac{(1+x+u)^2}{2}, \quad u(0)=u(1)=0$$

with the exact solution,

$$u(x) = \frac{2}{2-x} - x - 1$$

We applied our method to solve this problem with n = 8, 16, 32, 64, 128 and the computed solutions are compared with the exact solution at grid points. The maximum absolute errors at the nodal points, max = $|u(x_i)-u_i|$ are given to comparison with references [17, 19]. The observed maximum absolute errors are tabulated in Table 3.

Example 3: We consider the following example in [12] and

$$x^2 u'' = 2u - x, \quad u(2)=u(3)=0$$

With the exact solution,

$$u(x) = \frac{19x^2 - 5x^3 - 36}{38x}$$

We applied our method to solve this problem for n = 8, 16, 32, 64 and the computed solutions are compared with the exact solution at grid points. The observed maximum absolute errors are tabulated in Table 4. In this table we compared our results with the results given [12]. This shows that our results are more accurate.

Table 1: Maximum absolute errors for example 1

n	Our method
8	1.47×10^{-7}
16	6.76×10^{-9}
32	1.42×10^{-10}
64	2.26×10^{-12}
128	1.05×10^{-14}

Table 2: Maximum absolute errors for example 1

x	Our method	In [18]
0.1	1.47×10^{-7}	1.51×10^{-4}
0.2	8.24×10^{-8}	2.25×10^{-4}
0.3	4.87×10^{-8}	2.27×10^{-4}
0.4	2.57×10^{-8}	1.97×10^{-4}
0.5	1.78×10^{-8}	1.00×10^{-6}
0.6	2.57×10^{-8}	2.95×10^{-4}
0.7	4.87×10^{-8}	6.55×10^{-4}
0.8	8.24×10^{-8}	1.03×10^{-4}
0.9	1.47×10^{-7}	1.36×10^{-3}

Table 3: Maximum absolute errors for example 2

n	Our method	In [17]	In [19]
8	1.66×10^{-6}	6.32×10^{-7}	7.72×10^{-6}
16	1.68×10^{-8}	6.33×10^{-9}	2.01×10^{-7}
32	1.05×10^{-10}	1.57×10^{-10}	4.15×10^{-9}
64	1.11×10^{-12}	2.72×10^{-12}	7.50×10^{-11}
128	3.42×10^{-14}		

Table 4: Maximum absolute errors for example 3

n	Our method	In [17]
8	1.78×10^{-9}	1.31×10^{-8}
16	1.62×10^{-11}	1.56×10^{-10}
32	7.51×10^{-13}	1.53×10^{-12}
64	3.68×10^{-15}	1.53×10^{-14}

CONCLUSION

The approximate solutions of the second-order nonlinear boundary-value problems by using non-polynomial spline, show that our method are better in the sense of accuracy and applicability. These have been verified by the maximum absolute errors max $|e_i|$ given in tables. A new approach convergence analysis of the presented method is discussed.

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