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On a Common Fixed Point for Compatible Mappings of Type (A)

M.H.M. Rashid, Rateb Al-Btoush and Basem S. Masaedeh

Department of Mathematics and Statistics, Faculty of Science, Mu'tah University, P.O. Box (7), Mu'tah, Jordan

Abstract: The aim of this paper is to prove the existence problem of the common fixed point for mapping of type (A) and we introduced new conditions for that type. These results are generalizes, improves and extends some results on compatible continuous maps.

Key words: Compatible mappings of type (A) . Common fixed point . Cauchy sequence . non-Archimedean menger PM-space of type (D) . Linear map . Common fixed point theorems

INTRODUCTION

Finding necessary and sufficient conditions for the existence of fixed points is an interesting aspect. Chang [1] found some relations between mapping of type (A) and the Cauchy sequences. In 1986, Fisher and Sessa [2], established common fixed points for a pair of self maps in which one map is linear and no expansive. Jungch [3] showed that compatible mapping of type (A) equivalent to compatible mapping, under some conditions, also showed that this leads to fixed point mapping. On the other hand, Khan *et al*. [4] obtained some common fixed point theorems for compatible of type (A-1) and type (A-2) on fuzzy metric space. The common fixed point for compatible mapping of type (A) has been studied by many authors, for instance [5-10].

Our work in a generalization of their work by finding some new conditions that makes our results stronger than their results.

In this paper we prove the existence problem of the common fixed point for mapping of type (A) (Theorem 2.2). In the sequel we prove some results related to our main result (Theorem 2.1).

Let $\mathbb{R}$ denote the set of reals and $\mathbb{R}^+$ the non-negative real. A mapping Ψ: $\mathbb{R}\to\mathbb{R}^+$ is called a distribution function if it is non-decreasing and left continuous with inf $\Psi = 0$ and sup $\Psi = 1$.

A metric space is an ordered pair (X, d), where X is an abstract set and d is a mapping of X×X into $\mathbb{R}$, i.e., d associates a real number d (p, q) of elements of X. The mapping d is assumed to satisfy the following conditions:

(M-1) d (p,q) = 0 if and only if p = q, (Identity)
(M-2) d (p,q) ≥ 0, (Positively)
(M-3) d (p,q) = d (q,p), (Symmetry)
(M-4) d (p,r) ≤ d (p,q) + d (q,r). (Triangle Inequality)

Definition 1.1: [1] (X,Ψ) is called a non-archimedean probabilistic metric space (shortly, a N.A.PM-space) if (X, Ψ) is a PM-space and satisfies the following condition: If $F_{x,y}(t_1) = 1$ and $F_{x,y}(t_2) = 1$ for all x, y, z∈X and $t_1, t_2 \geq 0$ then F (max $\{t_1, t_2\}$) = 1.

Definition 1.2: [1, 8] (X, Ψ, Δ) is called a non-Archimedean Menger PM-space (shortly, a N.A Menger PM-space) If (X, Ψ, Δ) is a Menger PM-space and Δ satisfies the following condition: for all x, y, z∈X and $t_1, t_2 \geq 0$, $F_{x,z}(\max\{(t_1, t_2\}) \geq \Delta(F_{x,y}(t_1), F_{y,z}(t_2))$.

The purpose of this article is to study the existence problem of common fixed point for compatible mapping of type (A).

To achieve our goal we need the following definitions and lemmas.

Definition 1.3: [11] A PM-space (X, Ψ) is said to be of type $(C)_g$, if there exists a $g\in\Omega$ such that

$$g\left(F_{x,y}(t)\right) \leq g\left(F_{x,z}(t)\right) + g\left(F_{y,z}(t)\right))$$

for all x, y, z∈X and t≥0, where $\Omega = \{g: g:[0,1]\to[0,\infty)$ is continuous, strictly decreasing, g (1) = 0 and g $(0)<\infty\}$

Definition 1.4: [1] A non-Archimedean Menger PM-space (X, Ψ, Δ) is said to be of type $(D)_g$, if there exists a $g\in\Omega$ such that $g(\Delta(s, t)) \leq g(s)+g(t)$, for all s, t [0,1].

Corresponding Author: Dr. M.H.M. Rashid, Department of Mathematics and Statistics, Faculty of Science, Mu'tah University, P.O. Box (7), Mu'tah, Jordan

Remark A: Throughout this paper, let (X, Ψ, Δ) be a complete N.A Menger PM-space of type $(D)_g$, with a continuous strictly increasing t-norm Δ.

Lemma 1.5: [1] If a function φ: [0, ∞)→[0, ∞) satisfies the condition (Φ): φ is upper-semi continuous from the right and $\varphi(t)<t$ for all t>0, then we have for all t≥0, $\lim_{n\to\infty}\varphi_n(t)=0$ where φ_n (t) is the nth iteration of f (t).

If $\{t_n\}$ is a non-decreasing sequence of real numbers and $t_{n+1}\le\varphi(t_n)$, n = 1, 2,… then t_n = 0. In particular, if t≤f(t) for all t≥0 then t = 0.

Lemma 1.6: [12] Let $\{y_n\}$ be a sequence in X such that

$$F_{y_n,y_{n+1}}(t) = 1 \text{ for all } t > 0$$

If the sequence $\{y_n\}$ is not Cauchy sequence in X, then there exist $\varepsilon_0>0$, $t_0>0$ and two sequences $\{m_i\}$,$\{n_i\}$ of positive integers such that $m_i>n_i+1$ and $n_i\to\infty$ as $t\to\infty$

$$Fy_{m_i},y_{n_i}(t_0) < 1-\varepsilon_0$$
$$\text{and } Fy_{m_{i-1}},y_{n_i}(t_0) \ge 1-\varepsilon_0 \text{ , i = 1,2, …}$$

Definition 1.7: [13] Let H, K: X→X be mappings. H and K are said to be compatible if

$$\lim_{n\to\infty} g(F_{HKx_n,KHx_n}(t)) = 0$$

for all t > 0, whenever $\{x_n\}$ is a sequences in X such that

$$\lim_{n\to\infty} Hx_n = \lim_{n\to\infty} Kx_n = z$$

for some z∈X.

Definition 1.8: [13] Let H, K: X→X be mappings. H and K are said to be compatible of type (A) if

$$\lim_{n\to\infty} g(F_{HKx_n,KKx_n}(t)) = 0,$$
$$\lim_{n\to\infty} g(F_{HKx_n,HHx_n}(t)) = 0,$$

for all t > 0, whenever $\{x_n\}$ is asequance in X such that

$$\lim_{n\to\infty} Hx_n = \lim_{n\to\infty} Kx_n = z$$

for some z∈X.

Lemma 1.9: [4] Let H, K: X→X be continuous mappings, then H and K are compatible if and only if they are compatible of type (A).

Lemma 1.10: [4] Let H, K: X→X be mappings, if H and K are compatible of type (A) and Hz=Kz for some z∈X then KHz = HHz = HKz = KKz.

Lemma 1.11: [4] Let H, K: X→X be compatible mappings of type (A) and let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} Hx_n = \lim_{n\to\infty} Kx_n = z$ for some z∈X then we have the following:

$$\lim_{n\to\infty} HKx_n = Kz \text{ if K is continuous.}$$

KHz = HKz and Hz = Kz if H and K are continuous at z.

THE MAIN RESULT

Before proving our main theorem, we need the following:

Theorem 2.1: Let H, K: X→X be mapping satisfying the conditions:

$$H(X)\subset K(X),$$

$$g(F_{Hx,Hy}(t)) = f(\max\{ g(F_{Kx,Ky}(t)), g(F_{Kx,Hx}(t)), g(F_{Kx,Hy}(t)), ½(g(F_{Kx,Hy}(t)) + g(F_{Ky,Hx}(t))),$$

then the sequence $\{y_n\}$, defined by

$$y_{2n}=Hx_{2n}=Kx_{2n+1}, \; y_{2n+1} = Kx_{2n+2} = Hx_{2n+1} \text{ for all n = 0, 1, …}$$

such that

$$\lim_{n\to\infty} g(F_{y_n},y_{n+1}(t))=0$$

for all t > 0 is a Cauchy sequence in X.

Proof: Since g∈Ω, it follows that $\lim_{n\to\infty} F_{y_n},y_{n+1}(t)=1$ for all t > 0 iff $\lim_{n\to\infty} g(F_{y_n},y_{n+1}(t))=0$ for all t > 0.

By Lemma (1.6) if $\{y_n\}$ is not a Cauchy sequence in X, there exist ε_0> 0, t_0> 0 and two sequences $\{m_i\}$, $\{n_i\}$ of positive integers such that

$$m_i > n_{i+1} \text{ and } n_i\to 8 \text{ as } i\to 8 \text{ ,}$$

$$g(Fy_{m_i},y_{n_i}(t_0)) > g(1-\varepsilon_0)$$

and

$$g(F_{y_{m_{i-1}}, y_{n_i}}(t_0)) \le g(1-\varepsilon_0)$$

for i = 1, 2, 3,…
Thus, we have

$$\begin{aligned} g(1-\varepsilon_0) &< g(F_{y_{m_i}, y_{n_i}}(t_0)) \\ &\le g(F_{y_{m_i}, y_{m_{i-1}}}(t_0)) + g(F_{y_{m_{i-1}}, y_{n_i}}(t_0)) \quad (1) \\ &\le g(1-\varepsilon_0) + g(F_{y_{m_i}, y_{m_{i-1}}}(t_0)) \end{aligned}$$

letting $i\to\infty$ in (1), we have

$$\lim_{n\to\infty} g(F_{y_{m_i}, y_{n_i}}(t_0)) = g(1-\varepsilon_0) \quad (2)$$

On the other hand, we have

$$\begin{aligned} g(1-\varepsilon_0) &< g(F_{y_{m_i}, y_{n_i}}(t_0)) \\ &\le g(F_{y_{n_i}, y_{n_{i+1}}}(t_0)) + g(F_{y_{n_{i+1}}, y_{m_i}}(t_0)) \end{aligned} \quad (3)$$

Now, consider $g(F_{y_{n_{i+1}}, y_{m_i}}(t_0))$. Without loss of generality, assume that both n_i and m_i are even. Then, by condition in Equation (2), we have

$$\begin{aligned} & g(F_{y_{n_{i+1}}, y_{m_i}}(t_0)) = g(F_{Hx_{m_i}, Kx_{n_{i+1}}}(t_0)) \\ & \le \varphi(\max\{g(F_{Kx_{m_i}, Kx_{n_{i+1}}}(t_0)), g(F_{Kx_{m_i}, Hx_{m_i}}(t_0)), \\ & g(F_{Kx_{n_{i+1}}, Hx_{n_{i+1}}}(t_0)), \tfrac{1}{2}(g(F_{Kx_{m_{i+1}}, Hx_{n_{i+1}}}(t_0)) \\ & + g(F_{Kx_{n_{i+1}}, Hx_{m_i}}(t_0)))\}) \le \varphi(\max\{g(F_{y_{m_{i-1}}, y_{n_{i+1}}}(t_0)), \\ & g(F_{y_{m_{i-1}}, y_{m_i}}(t_0)), g(F_{y_{n_i}, y_{n_{i+1}}}(t_0)), \\ & \tfrac{1}{2}(g(F_{y_{m_{i-1}}, y_{n_{i+1}}}(t_0)) + g(F_{y_{n_i}, y_{m_i}}(t_0)))\}) \end{aligned} \quad (4)$$

By (2),(3) and (4), letting $i\to\infty$ in (4), we have

$$\begin{aligned} g(1-\varepsilon_0) &= f(\max\{g(1-\varepsilon_0), 0, 0, g(1-\varepsilon_0)\}) \\ &= f(g(1-\varepsilon_0)) < g(1-\varepsilon_0) \end{aligned}$$

which is a contradiction. Therefore, $\{y_n\}$ is a Cauchy sequence in X. This completes the proof.

Now, we are ready to give our promised theorem in this paper:

Theorem 2.2: Let H, K: $X\to X$ be mappings such that $H(X)\subset K(X)$, K is continuous, H and K are compatible of type (A),

$$\begin{aligned} g(F_{Hx,Hy}(t)) &= f(\max\{g(F_{Kx,Ky}(t)), g(F_{Kx,Hx}(t)), \\ & g(F_{Ky,Hy}(t)), \tfrac{1}{2}(g(F_{Kx,Hy}(t) + g(F_{Ky,Hx}(t)))\}) \end{aligned}$$

for all $t > 0$ and $x, y \in X$, where the function

$f: [0, +8)\to[0, +8)$ satisfies the condition (Φ). Then H and K have a unique common fixed point in X.

Proof: If we prove $\lim_{n\to\infty} g(F_{y_n}, y_{n+1}(t))=0$ for all $t > 0$, then by Theorem (2.1), the sequence $\{y_n\}$ defined in (2) of Theorem (2.1) is a Cauchy sequence in X. Now, we prove $\lim_{n\to\infty} g(F_{y_n}, y_{n+1}(t))=0$ for all $t > 0$. In fact, by condition (4), we have

$$\begin{aligned} & \lim_{n\to\infty} g(F_{y_{2n}, y_{2n+1}}(t)) = g(F_{Hx_{2n}, Hx_{2n}}(t)) \\ & = f(\max\{g(F_{Kx_{2n}, Kx_{2n+1}}(t)), g(F_{Kx_{2n}, Hx_{2n}}(t)), \\ & g(F_{Kx_{2n+1}, Hx_{2n+1}}(t)), \tfrac{1}{2}(g(F_{Kx_{2n}, Hx_{2n+1}}(t)) + \\ & g(F_{Kx_{2n+1}, Hx_{2n}}(t))\}). \\ & = f(\max\{g(F_{y_{2n-1}, y_{2n}}(t)) g(F_{y_{2n-1}, y_{2n}}(t)), \\ & g(F_{y_{2n}, y_{2n+1}}(t)), \tfrac{1}{2}(g(F_{y_{2n-1}, y_{2n+1}}(t)) + g(1))\}) \\ & = f(\max\{g(F_{y_{2n-1}, y_{2n}}(t)), g(F_{y_{2n}, y_{2n+1}}(t)), \\ & \tfrac{1}{2}(g(F_{y_{2n-1}, y_{2n}}(t)) + g(F_{y_{2n}, y_{2n+1}}(t)))\}). \end{aligned}$$

If $g(F_{y_{2n-1}, y_{2n}}(t)) = g(F_{y_{2n}, y_{2n+1}}(t))$ for all $t > 0$,
Then, by condition (4):

$$g(F_{y_{2n}, y_{2n+1}}(t)) = f(g(F_{y_{2n}, y_{2n+1}}(t))),$$

which means that, by lemma (5), $g(F_{y_{2n}, y_{2n+1}}(t)) = 0$, for all $t > 0$, similarly, we have $g(F_{y_{2n+1}, y_{2n+2}}(t)) = 0$ for all $t > 0$. Thus, for all $t > 0$, we have $\lim_{n\to\infty} g(F_{y_n, y_{n+1}}(t)) = 0$

On the other hand, if $g(F_{y_{2n-1}, y_{2n}}(t)) = g(F_{y_{2n}, y_{2n+1}}(t))$ then by condition(4), for all $t > 0$, we have

$$g(F_{y_{2n-1}, y_{2n}}(t)) \ge \varphi(g(F_{y_{2n}, y_{2n+1}}(t)))$$

similarly, we have

$$g(F_{y_{2n}, y_{2n+1}}(t)) \ge \varphi(g(F_{y_{2n+1}, y_{2n+2}}(t)))$$

for all $t > 0$. Thus we have

$$g(F_{y_n, y_{n+1}}(t)) = f(g(F_{y_{n-1}, y_n}(t)))$$

for all $t > 0$ and n = 1, 2, …... Therefore, by Lemma (1.5), for all $t > 0$,

$$\lim_{n\to\infty} g(F_{y_n, y_{n+1}}(t))=0.$$

Which implies that $\{y_n\}$ is a Cauchy sequence in X by Theorem (2.1). Since (X, Ψ, Δ) is complete, the sequence $\{y_n\}$ converges to a point $z\in X$ and so the subsequences $\{Hx_{2n}\},\{Hx_{2n+1}\}$, $\{Kx_{2n}\},\{Kx_{2n+1}\}$ of y_n also converges to the limit z.

Now, suppose that K is continuous. Since H and K are compatible of type (A), by Lemma (1.10), HKx_{2n+1}, $KKx_{2n+1}\to Kz$ as n→8. Putting $x = x_{2n}$ and $y = Kx_{2n+1}$ in condition (2) of Theorem (2.1), we have $(F_{Hx_{2n}, HKx_{2n+1}}(t))=f(\max\{g(F_{Kx_{2n}, KKx_{2n+1}}(t)), g(F_{Kx_{2n}, Hx_{2n}}(t)), (F_{KKx_{2n+1}, HKx_{2n+1}}(t)), ½(g(F_{Kx_{2n}, HKx_{2n+1}}(t))+g(F_{KKx_{2n+1}, Hx_{2n}}(t))$(5) for all t > 0, letting n→8 in (5), we have $g(F_{z,Kz}(t))=f(\max\{g(F_{z,Kz}(t)), g(F_{z,z}(t)), g(F_{Kz,Hz}(t)), ½(g(F_{z,Hz}(t)) + g(F_{Kz,z}(t))) = f(g(F_{z,Kz}(t)))$ for all t > 0, which means that $g(F_{z,Kz}(t)) = 0$, for all t > 0 by Lemma (1.5) and so we have Kz = z.

Again, replacing x by x_{2n} and y by z in condition (4), we have $g(F_{Hx_{2n},Hz}(t)) = f(\max\{g(F_{Kx_{2n},Kz}(t)), g(F_{Kx_{2n}, Hx_{2n}}(t)), g(F_{Kz,Hz}(t)), ½(g(F_{Kx_{2n},Hz}(t)) + g(F_{Kz, Hx_{2n}}(t))$ (6) for all t > 0, letting n→8 in (6), we have $g(F_{z,Hz}(t)) = f(\max\{g(F_{z,Kz}(t)), g(F_{z,z}(t)), g(F_{z,z}(t)), ½(g(F_{z,z}(t)) + g(F_{z,z}(t)))$ for all t > 0, which implies that $g(F_{z,Kz}(t)) = f(g(F_{z,Hz}(t))$ for all t > 0 and so we have Hz = z. since H(X)⊂K(X), there exists a point w∈X such that Hz = Kw = z. By using condition (4) again, we have $g(F_{Hw,z}(t)) = g(F_{Hw,Hz}(t)) = f(\max\{g(F_{Kw,Kz}(t)), g(F_{Kw,Hw}(t)), g(F_{Kz,Hz}(t)), ½(g(F_{Kw,Hz}(t)) + g(F_{Kz,Hw}(t)) = f(g(F_{Hw,z}(t))$ for all t > 0, which means that Hw = z. since H and K are compatible mappings of type (A) and Hw = Kw = z, by Lemma (1.7) we have Hz = H Kw = K Kw = Kz.

Again by using condition (4), we have Hz = z. Therefore Hz = Kz = z, that is, z is a common fixed point of the given Mappings H, K. The uniqueness of the common fixed point z follows easily from condition (4). This completes the proof.

We end this paper by illustrating the following example which satisfy theorem 13.

Example: Let X = [0,∞) be endowed with the usual metric d. Define

$$H(x)=\begin{cases} 0, & \text{if } x\in[0,1) \\ \sqrt{x}, & \text{if } x\in[1,\infty) \end{cases}$$

and

$$K(x)=\begin{cases} 1, & \text{if } x\in[0,1) \\ \frac{1}{x}, & \text{if } x\in[1,\infty) \end{cases}$$

Clearly, K (x) = (0,1]∪{1}⊂H(x) = [0,∞) and K(x) is continuous, furthermore, H and K are compatible of type A. If we take g (t) = 1-t and Φ (t) = t then the conditions of above Theorem 13 are satisfied. Hence H and K has a unique common fixed point.

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