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The Category of Semitopological S-Acts

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Abstract: Let S-Top be the category of topological S-acts over a topological semigroup S and S-SemiTop be the category of semitopological S-acts over a semitopological semigroup S. It is obvious that any topological S-act is a semitopological S-act, however we will see that the converse is not true in general. In this note, we study the universal objects in the category of semitopological S-acts and introduce them completely. Furthermore, we study the left and right adjoint situation between the category of topological S-acts and the category of semitopological S-acts over a topological semigroup S. Similarly, we present the left adjoint to the inclusion functor from the category of topological semigroups (groups) to the category of semitopological semigroups (groups). Finally as a result of this study, we partially answer to this question that "when the point convergence topology is admissible?"

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INTRODUCTION

Topological groups and their actions have been studied by many mathematicians and have been used in many fields such as geometry, Lie groups and analysis. Also many works have been done on topological semigroups and their representations [1-5]. Semitopological S-acts, which sometimes are also called flows [2, 3, 6], are studied by many authors in different branches of Mathematics (Normak in [7] mentions some of these works).

Category theory has many applications in different branches of Mathematics and is studied by many mathematicians. One of the main part of these studies is devoted to the study of universal objects in different categories. These studies are usually one of the first step in using categorical tools for studying dassical problems [8, 9]. In this note, we study the category of semitopological S-acts and then use it as a tool for studying a classical problem in function spaces.

It is a famous problem that for which space X there exists an admissible topology on C(X,Y) for any space Y. Arens and Dugundji were the first topologists who studied and answered to this question for a Hausdorff space X, by investigating this question that for which Hausdorff space X, the compact-open topology on C(X,Y) is admissible [10, 11]. In fact, they showed that if X is a locally compact space, then the compact-open topology on C(X,Y) is admissible for any space Y [12-15]. In this note, after we introduce the universal objects in S-SemiTop, as an application of our results, we show that if X is the underlying space of an Alexanderoff topological monoid, then the point convergence topology is admissible and it is equal to the compact-open topology.

We now briefly recall some definitions about S-acts needed in the sequel [16, 17].

The definitions of a subact A of B, written as A=B and a homomorphism between S-acts are clear. In fact S-homomorphisms, or S-maps, are action-preserving maps: $f: A\to B$ with $f(sa) = sf(a)$, for $s\in S$, $a\in A$.

Recall that, for a semigroup S, a set A is a left S-act (or S-set) if there is, so called, an action $\mu: S\times A\to A$ such that, denoting $\mu(s, a) := sa$, $(st)a = s(ta)$ and, if S is a monoid with 1, $1a = a$. Right S-acts are defined similarly.

Each semigroup S can be considered as an S-act with the action given by its multiplication.

Note that the free S-act, for a monoid S, on a set X is the set $S\times X$ with the action defined by $t(s, x) = (ts, x)$ and $\psi: X\to S\times X$ is given by $\psi(x) = (1, x)$.

A topological congruence on a semitopological S-act (A, t) is an S-act congruence θ (that is, if $a\theta a'$ for a, $a'\in A$, then $as\theta a's$, for all $s\in S$) with the property that the S-act A/θ with the quotient topology can be made into a semitopological S-act. Recall that a congruence on a semigroup S is an equivalence relation θ on it such that for any s, s' and $t\in S$, if $(s, s')\in \theta$, then $(st, s't)$ and (ts, ts') belong to θ.

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SEMITOPOLOGICAL S-ACTS AND TOPOLOGICAL S-ACTS

In this section we briefly state the notions we need about semitopological S-acts. For more information see [2, 3, 7]. First recall the following

Definition 2.1: A semigroup S is a topological semigroup if there is a topology t_S on S such that the multiplication $S\times S\to S$ is (jointly) continuous, where $S\times S$ has the product topology. Equivalently, if $st\in V$ and $V\in t_S$, there exist open sets V_s and V_t in t_S containing s and t, respectively, such that $V_s\, V_t\subseteq V$.

In the language of category theory, topological semigroups are semigroup objects in the category of topological spaces, in the same way that ordinary semigroups are the semigroup objects in the category of sets.

Definition 2.2: For a topological semigroup S, a (left) S-topological act or a topological S-act is a left S-act A with a topology t_A such that the action $S\times A\to A$ is (jointly) continuous. Equivalently, if for $sa\in U$ and $U\in t_A$ there exist open sets $V_s\in t_S$ and $W_a\in t_A$ containing s and a, respectively, such that $V_s\, W_a\subseteq U$. The topological S-maps between topological S-acts are continuous S-maps.

Again, in the language of category theory, topological S-acts are S-act objects in the category of topological spaces, in the same way that ordinary S-acts are the S-act objects in the category of sets.

Notation 2.3: For any set H, we denote H with the discrete topology by (H, t_{dis}). For any S-act A, by $|A|$ we mean the underlying set of A.

Remark 2.4: Recall that for a semigroup S and an S-act A, the functions λ_s and ρ_a are defined for any $s\in S$ and $a\in A$ as follows

$$\begin{array}{lcl} \lambda_s : A\to A & \text{and} & \rho_a : S\to A \\ y\mapsto sy & & t\mapsto ta \end{array}$$

In the special case A=S, we use the notation $\lambda_s^{(S)}:S\to S$, to prevent misunderstanding.

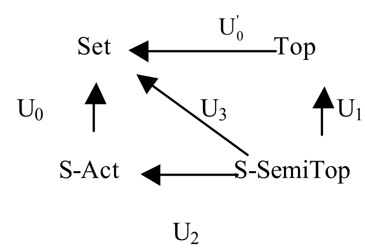
Now if S has a topology τ_S for which its multiplication $S\times S\to S$ is (separately) continuous, that is, $\lambda_s^{(S)}$ and ρ_s are continuous for any $s\in S$, then S is called a semitopological semigroup.

Similarly, one can define a semitopological S-act on a semitopological semigroup by taking λ_s: $A\to A$ and ρ_a: $S\to A$ to be continuous for each $s\in S$ and $a\in A$.

- As the first example for this definition, let S and T be semitopological semigroups such that S is a subsemigroup and also a subspace of T. Let λ be the restriction of the multiplication of T to $S\times T$. Then λ is an action of S on $|T|$ and T is a semitopological S-act with action λ.
- Let S be a semitopological semigroup and X be a topological space. For simplicity, we denote the free S-act over the underlying set of X by $F_0'(X)$ (we will use this terminology in the rest of the paper). Consider the product topology on $F_0'(X)$. Then $F_0'(X)$ with this topology is a semitopological S-act.

Clearly, any topological S-act is a semitopological S-act, because every jointly continuous function is separately continuous, however as we will see in Example 2.16, the converse is not true in general. Note that for some topological semigroups, these definitions are equivalent. We will see an example of these topological semigroups in Corollary 2.14 and we will use this fact to characterize the cofree topological S-act over a topological space X.

We denote the categories of all sets, all left S-acts, all topological spaces, all left topological S-acts, all left semitopological S-acts, with their appropriate maps, by Set, S-Act, Top, S-Top and S-SemiTop, respectively. We have the following diagram (a) of forgetful functors between these categories



Remark 2.5: (i) Let S be a semitopological monoid and H be a set. Let $\lambda:S\times H^S\to H^S$ be defined by $\lambda(s,f):=s\cdot f$, where $(s\cdot f)(t):=f(st)$ for any function $f\in H^S$ and $s,t\in S$. It is a known fact that H^S with action λ is the cofree S-act over H [17]. We will denote this S-act by $K_0(H)$. (ii) Let Y and Z be two topological spaces. A topology t on the set C (Y, Z) is called splitting if for every space X, the continuity of a map g: $X\times Y\to Z$ implies the continuity of the map $\tilde{g}:X\to C_\tau(Y,Z)$, defined by $\tilde{g}(x)(y) = g(x, y)$, for every $x\in X$ and $y\in Y$. This topology is also called proper [12, 13]. A topology τ on C (Y, Z) is admissible if the mapping $\omega(y, f) := f(y)$ from $Y\times C(Y, Z)$ into Z is continuous in y and f. Equivalently, a topology t on $C(Y,Z)$ is admissible if for every

topological space X, the continuity of f: $X \to C_t(Y, Z)$ implies the continuity of the map $\tilde{f}: X \times Y \to Z$, where $\tilde{f}(x, y) := f(x)(y)$ for every $(x, y) \in X \times Y$ [12-15] (the latter definition is usually used as the definition of admissible topology, but for our purpose, we prefer to use the former).

First, we study the universal objects in the category of semitopological S-acts with continuous S-homomorphisms. Then, we use these results to get some conclusion about the category of topological spaces, topological semigroups and topological S-acts.

We begin our work with the free semitopological S-acts and we introduce the free semitopological S-act over a topological space and then over an S-act.

Remark 2.6: (The free semitopological S-act over a set) Note that the free semitopological S-act over a set Z is the free semitopological S-act over the discrete space (Z, t_{dis}).

Proposition 2.7: (The free semitopological S-act over a space) Let S be a semitopological monoid and (X, t_X) be a space. The free semitopological S-act over the space X is the free S-act $F_0'(X)$ with topology t*, where t* is the generated topology by the subbasis $\{O \in t \mid t \in \mathbf{A}\}$ where $\mathbf{A} = \{t \mid (F_0'(X), t) \in$ S-SemiTop and $\psi: (X, t_X) \to (F_0'(X), t)$ is continuous$\}$.

Proof: First note that the product topology on $F_0'(X)$ belongs to **A**. Let $(s, x) \in F_0'(X)$, $t \in S$ and O be an open set in τ_j where $\tau_j \in \mathbf{A}$. Since $(F_0'(X), \tau_j)$ is a semitopological S-act, $?^{-1}_t(O)$ and $\rho^{-1}_{(s,x)}(O)$ are open sets in $(F_0'(X), \tau_j)$ and S, respectively. So they are open in $(F_0'(X), \tau^*)$ and S, respectively. Hence $(F_0'(X), \tau^*)$ is a semitopological S-act. Since $\psi: (X, \tau_X) \to (F_0'(X), \tau_j)$ is continuous, $\psi^{-1}(O)$ is open in X. Hence $\psi: (X, \tau_X) \to (F_0'(X), \tau^*)$ is continuous. Now suppose that we are given a semitopological S-act (A, τ_A) and a continuous function $f: X \to (A, \tau_A)$. Since $F_0'(X)$ is the free S-act over X, there exists an S-homomorphism $\tilde{f}: F_0'(X) \to A$ such that $\tilde{f} \circ \psi = f$. We show that $\tilde{f}$ is continuous. For this purpose, we show that $\{\tilde{f}^{-1}(O): O \in \tau_A\} \in \mathbf{A}$. Since $\{\tilde{f}^{-1}(O): O \in \tau_A\}$ is a topology on $F_0'(X)$ which makes it a semitopological S-act, $\tilde{f} \circ \psi = f$ and furthermore f and ψ are continuous, it follows that $\{\tilde{f}^{-1}(O): O \in \tau_A\} \in \mathbf{A}$. Hence, $\{\tilde{f}^{-1}(O): O \in \tau_A\} \subset \tau^*$. Therefore $\tilde{f}$ is continuous and $(F_0'(X), \tau^*)$ is the free semitopological S-act over the space (X, τ_X).

Proposition 2.8: Let S be a semitopological monoid and A be an S-act. The free semitopological S-act over the S-act A is (A, t^o), where t^o is the generated topology by the subbasis $\{O \in t \mid t \in \mathbf{A}\}$ where $\mathbf{A} = \{t \mid (A, t) \in$ S-SemiTop$\}$.

Proof: Let $f: A \to B$ be an S-homomorphism where A is an S-act, (B, t_B) is a semitopological S-act. Consider the topology $t = \{O \subseteq A \mid O = f^{-1}(V)$ for some open set V in $(B, t_B)\}$ on A. We claim that (A, t) is a semitopological S-act. Since f is an S-homomorphism, for any $s \in S$, $f?\lambda_s: A \to A \to B$ is equal to $\lambda_s?f: A \to B \to B$. Since B is a semitopological S-act, $f?\lambda_s$ is continuous for any $s \in S$. Therefore $\lambda_s^{-1}(O)$ is open in (A, t), when O is open in (A, t). Hence λ_s is continuous. On the other hand, for any $a \in A$, $f?\rho_a = \rho_{f(a)}$. Therefore, since (B, t_B) is a semitopological S-act and since for any open set O in (A, t), there exists an open set V in (B, t_B) such that $O = f^{-1}(V)$, the set $\rho_a^{-1}(O)$ is open in S. Now define

$$\mathbf{A} = \{t \mid (A, t) \in \text{S-SemiTop}\}$$

and let τ^o be the generated topology by the subbasis $\{O \in t \mid t \in \mathbf{A}\}$. It is straightforward to see that (A, τ^o) is a semitopological S-act. Furthermore, by the above discussion it has the universal property, too. So (A, τ^o) is the free semitopological S-act.

The following facts can be proved straightforward, so we state them without proofs.

Remark 2.9

(i) (Product) For any number of semitopological S-acts $\{(A_i, t_i)\}_{i \in I}$, the product semitopological S-act of (A_i, t_i) is the product S-act $\Pi_{i \in I} A_i$ with the product topology.

(ii) (Equalizer) Suppose that (A, t_A) and (B, t_B) are semitopological S-acts and f, g: $(A, t_A) \to (B, t_B)$ are two continuous S-homomorphisms. The equalizer of f and g in S-SemiTop is $E = \{x \in A \mid f(x) = g(x)\}$ with the subspace topology which is inherited from (A, t_A) and with inclusion.

(iii) (Pull back) Let (A, t_A), (B, t_B) and (C, t_C) be semitopological S-acts and f: $(A, t_A) \to (B, t_B)$ and g: $(C, t_C) \to (B, t_B)$ be continuous S-maps. The pull back of this diagram is $P = \{(x, y) \in A \times C \mid f(x) = g(y)\} = \{(x, y) \in A \times C \mid f?\pi_1((x, y)) = g?\pi_2((x, y))\}$ with the subspace topology and with inclusion.

(iv) (Coproduct) For any family of semitopological S-acts $\{(A_i, t_i)\}_{i \in I}$, the coproduct semitopological S-act of (A_i, t_i) is the coproduct S-act $\sqcup_{i \in I} A_i$ with the topology of the discrete sum of $(|A_i|, t_i)$.

Before we continue our study about pushout and coequalizer in the category S-SemiTop, we need the following lemma which clarifies the concept of semitopological S-act congruences.

Lemma 2.10: For a semitopological S-act (A, t_A) and a congruence θ on it, the quotient S-act A/θ with the quotient topology is a semitopological S-act.

Proof: Let (A, t_A) be a semitopological S-act and θ be a congruence on it. Consider an arbitrary $[a]\in A/\theta$ and $s\in S$ and fix them. Obviously we have

$$(\rho_{[a]}: S\to A/\theta) = (p?\rho_a: S\to A/\theta).$$

Since A/θ has the quotient topology, p is continuous and (A, t_A) is a semitopological S-act, it follows that $\rho_{[a]}$ is continuous.

Now suppose that U^θ is an open set in A/θ with the quotient topology. Since $p?\lambda_s: A\to A/\theta$ is a continuous function, $p^{-1}(\lambda_s^{-1}(U^\theta))$ is an open set in (A, t_A). On the other hand $\pi(\pi^{-1}(\lambda_s^{-1}(U^\theta)))$ is an open set in A/θ with the quotient topology, since $p^{-1}(\lambda_s^{-1}(U^\theta))$ is an open set in A. Furthermore, since

$$\tilde{\lambda}_s^{-1}(U^\theta) = \pi(\pi^{-1}(\lambda_s^{-1}(U^\theta)))$$

it follows that $\tilde{\lambda}_s$ is continuous.

By the above discussion, A/θ with the quotient topology is a semitopological S-act. So, any congruence on a semitopological S-act is topological. Similarly we can prove that for a semitopological semigroup S and a semigroup congruence θ on it, the quotient semigroup S/θ with the quotient topology is a semitopological semigroup. Recall that for an S-act A and a set $H\subseteq A\times A$, the generated S-act congruence on A exists over the set H ([17] for more information about the structure of the generated congruence). Therefore, by the above lemma, the generated topological congruence exists over any subset $H\subseteq A\times A$, where (A, t_A) is a semitopological S-act. So we have

Remark 2.11

(i) (Coequalizer) For any given semitopological S-acts (A, t_A) and (B, t_B) and continuous S-homomorphisms $f, g : (A, t_A)\to(B, t_B)$, the coequalizer of f and g is $C = (A \sqcup B)/\theta$ with the quotient topology, where θ is the generated congruence by the following set $\{(f(y), g(y))|y\in A\}$.

(ii) (Push out) For any given semitopological S-acts (A, t_A), (B, t_B), (C, t_C) and continuous S-homomorphisms $f: (A, t_A)\to(B, t_B)$ and $g: (A, t_A)\to(C, t_C)$, the push out of this diagram is the following S-act with the quotient topology $((B\sqcup C)\sqcup(B\sqcup C))/\theta$ where θ is the generated congruence on $B\sqcup C$ by the following set $\{(q_1?g(x), q_2?f(x))|x\in A\}$ and q_1 and q_2 are the embeddings from B and C to $B\sqcup C$, respectively.

The following remark presents the cofree semitopological S-act over an arbitrary set and an S-act.

Remark 2.12

(i) For a semitopological monoid S and an S-act A, the cofree semitopological S-act over A is A with the trivial topology.
(ii) For a semitopological monoid S and a set H, the cofree semitopological S-act over H is $K_0(H)$ with the trivial topology.

The cofree semitopological over a topological space is clarified in the following proposition.

Proposition 2.13: For a semitopological monoid S and a topological space X with topology t_X, the cofree semitopological S-act over X is C (S, X) with the point convergence topology which is generated by $\{(s, U)\}_{U\in tS, s\in S}$.

Proof: It is easy to see that ψ is continuous. Furthermore, for any $s\in S$, $f\in C(S, X)$ and an open set $(\{t\}, U)$, we have

$$\begin{aligned}\lambda_s^{-1}((\{t\},U)) &= \{g\in C(S,X)|g(st) = s\cdot g(t)\in U\}\\ &= (\{st\},U)\\ \rho_t^{-1}((\{t\},U)) &= \{s'\in S|s'\cdot f(t)\in U\}\\ &= \{s'\in S|f(s't)\in U\}\\ &= \{s'\in S|f\rho_t(s')\in U\} = \rho_t^{-1}f^{-1}(U)\}\end{aligned}$$

Clearly, the first set is open in S and the second set is open in C (S, X) with the point convergence topology.

Now suppose that we are given a semitopological S-act (A, t_A) and a continuous function $h : (A, t_A)\to X$. Since X^S is the cofree S-act over the underlying set of X, there exists an S-homomorphism $\tilde{h}: A\to X^S$ such that $\psi?\tilde{h} = h$ and we have $\tilde{h}(a)(t) = h(ta)$. Now, since

$$\tilde{h}^{-1}(\{t\},U) = \{a\in A|\tilde{h}(a)(t) = h(ta) = h\circ\lambda_t(a)\in U\} = \lambda_t^{-1}h^{-1}(U)$$

and we know that h and λ_t are continuous, it follows that $\tilde{h}$ is a continuous S-map. Clearly $\tilde{h}$ is unique and commutes the diagram by the proof of the universal property of the cofree object in the category S-Act [17].

As a quick consequence of the above proposition, we have

Corollary 2.14: Let X be a topological space and S be a semitopological monoid which is an Alexandroff space. Then S is a topological monoid and C (S, X) with the point convergence topology is the cofree topological S-act over the space X.

Proof: It is straightforward to see that the separately continuity of the multiplication S is equivalent to jointly continuity of it. So S is a topological monoid. Similarly, one can easily conclude that any semitopological S-act is a topological S-act for an Alexandroff topological semigroup. Therefore, SSemiTop=S-Top. The rest of the proof is obvious by Proposition 2.13.

Georgiou and Illiadis studied the admissible and splitting topologies on the function space C(X,Y) with the compact-open topology, when Y is an Alexandroff space [14].

In the next corollary, we study admissible and splitting topologies on the function space C(X,Y) with the point convergence topology and compact-open topology, where X is the underlying space of an Alexandroff topological monoid. As a quick consequence of the above corollary we have

Corollary 2.15: Let X be a topological space which is the underlying space of an Alexandroff topological monoid and Y be a topological space. Then the point convergence topology on C (X, Y) is admissible and splitting.

We know that if the point convergence topology is admissible, then the compact-open topology is admissible. Also it is a well known fact that for topological spaces X and Y, the splitting admissible topology on C(X,Y), if it exists, is unique [12-15]. Therefore by the above corollary, we know that the compact-open topology is equal to the point convergence topology and it is admissible and splitting on the function space C(X,Y), when X is the underlying space of an Alexandroff topological monoid.

By Proposition 2.13, the cofree semitopological S-act over a space X, always exists. Furthermore, by Corollary 2.14, when S is an Alexandroff topological monoid, it is the cofree topological S-act over the space X. However the next example shows that the cofree semitopological S-act does not necessarily belong to S-Top. Therefore it is not the cofree topological S-act over the space X in general.

Example 2.16: Consider N with the discrete topology and its usual multiplication. Let S be the product semigroup $(N^8,\cdot)$ with the product topology and let X be the Sirpiskey space. Consider the point convergence topology on C (S, X). By Proposition 2.13. C(S,X) with this topology is a semitopological S-act. We claim that C(S,X) with this topology is not a topological S-act. For this purpose, we introduce a continuous function g: $S\to X$, a point $s\in S$ and an open set U in C (S, X) such that $sg\in U$, but there is no open neighborhoods of g and s, respectively such that $W_s\cdot V_g \subseteq U$.

First note that since S is not connected, there exists a nonconstant continuous function g from S to {0, 1} with the discrete topology. So there exists a continuous function g from S to the Sirpinskey space X and a point (n_i) in N^8 such that $g((n_i)) = 1$. Obviously, (1) $g\in((n_i), \{1\})$, where (1) denotes the element (c_i) in N^8, where c_i =1 for any i. Now we show that C (S, X) with the point convergence topology is not a topological S-act. On the contrary, suppose that C (S, X) with this topology is a topological S-act. Therefore without lose of generality, we can assume that there exists an open set $\cap_{k=1}^{m}((y_i^k),U_k)$ around g and an open set W_1 around (1) such that

$$W_1 \cdot \cap_{k=1}^{m} \left((y_i^k), U_k\right) \subseteq ((n_i),[1])".$$

There is two possibilities for the open set $\cap_{k=1}^{m}((y_i^k),\{1\})$. First, suppose that for all k, $U_k = \{0,1\}$, then the constant zero function c_0 belongs to $\cap_{k=1}^{m}((y_i^k),U_k)$ and obviously $c_0((n_i))\neq 1$ which is a contradiction. The second possibility is that there exists some k such that $U_k = \{1\}$. Since $((y_i^k), \{0, 1\}) = C$ (S, X), without lose of generality, we can assume that for each k, $U_k = \{1\}$. Since $(1)\in W_1$ and for every function $f\in C$ (S, X), (1). $f\in((y_i^k),\{1\})$, we have $f((y_i^k))=1$ for all k. We claim that there exists a continuous function h from S to X such that for all k, $h((y_i^k))=1$ but $h((n_i)) = 0$. Now choose an open set U in S which does not contain $\{(n_i)\}$ (note that S is Hausdorff) and define

$$h((x_i)) = \begin{cases} 1; & (x_i)\in U \\ 0; & \text{o.w.} \end{cases}$$

Obviously since the non-empty open sets in S are {1} and {0,1}, h is continuous and for all k, $h((y_i^k)) = 1$ but $h((n_i)) = 0$. So $h \in \cap_{k=1}^{m}((y_i^k),\{1\})$ but $h((n_i))\neq 1$ which is a contradiction, since

$$h = h(1) \in W_1 \cdot \cap_{k=1}^{m} \left((y_i^k),\{1\}\right) \subset ((n_i),\{1\})$$

Therefore C(S,X) is a semitopological S-act, but it is not a topological S-act. Hence C(S,X) with the point convergence topology is not the cofree topological S-act.

Finally, in this note, we study the adjoint situation between S-SemiTop and S-Top.

Proposition 2.17: Let S be a semitopological semigroup and A be an S-act. The free semitopological S-act over the S-act A is (A, t_*), where t_* is the generated topology on |A| by the subasis B = $\{\tau | (A,\tau) \in$ S-SemiTop$\}$.

Proof: First note that, similar to the proof of Proposition 2.7, we can easily show that (A,τ_*) is a semitopological S-act. Now let $f:A \to (B, \tau_B)$ be an S-homomorphism. Again, similar to the proof of Proposition 2.7, we can easily show that $\{f^{-1}(O)\}_{O\in\tau B}$ belongs to B and hence $\{f^{-1}(O)\}_{O\in\tau B} \subseteq \tau_*$. Therefore f is a continuous S-homomorphism and (A, t_*) is the free semitopological S-act over S-act A.

Proposition 2.18: Let S be a topological semigroup. For every semitopological S-act (A, t_A), there exists a topology $t_F \subseteq t$ such that (A, t_F) is the finest topology on A such that (i) (A, t_F) is a topological S-act. (ii) t_F is the finest topology on A which is contained in t_A and satisfies (i). In fact, if F: S-SemiTop→S-Top is defined as follows

$$F((A, t_A)) := (A, t_F),$$

then F is a left adjoint for inclusion I : S-Top→S-SemiTop.

Proof: Let (A, t_A) be a semitopological S-act. Define

$$\mathbf{A} := \{t' \subseteq t_A \mid (A, t') \in \text{S-Top}\}.$$

Obviously $t_{tri} \in A$ and so A is not empty. Let t_F be the generated topology by the subbasis $\bigcup_{\tau\in A}\tau$. We claim that (A, t_F) is a topological S-act. So suppose that we are given $a\in A$, $s\in S$ and $U \in t_F$ such that $sa \in U$ where $U \in \bigcup_{\tau\in A}\tau$. Clearly $U \in t_1$, for some $t_1 \in A$. Since $(A, t_1) \in$ S-Top, there exist open sets V_a and W_s which contain a and s, respectively such that $sa \in W_s \cdot V_a \subseteq U$. Since $t_1 \in t_F$, we have the result. Therefore t_F satisfies condition (i). By the definition of **A**, t_F satisfies condition (ii), too. The rest of the proof is obvious.

Similar to the proof of the above proposition, we can show that

Proposition 2.19: Let TopSgr be the category of topological semigroups and SemiTopSgr be the category of semitopological semigroups. The inclusion functor from TopSgr to SemiTopSgr has a left adjoint. More specially, there is a left adjoint for the inclusion functor from the category of topological groups to the category of semitopological groups.

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