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Modified Variational Iteration Method for Systems of partial Differential Equations

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Abstract: In this paper, we apply the Modified Variational Iteration Method (MVIM) for solving systems of partial differential equations. The proposed modification is made by introducing He's polynomials in the correction functional of Variational Iteration Method (VIM). Several examples are given to verify the reliability and efficiency of the method. The fact that the MVIM solves nonlinear problems without using Adomian's polynomials is an advantage of this algorithm over the decomposition method.

Key words: Variational iteration method . He's polynomials . systems of partial differential equations

INTRODUCTION

The partial differential equations are of great significance in the diversified physical problems related to physics, astrophysics, magnetic dynamics, water surface, gravity waves, ion acoustic waves in plasma, electromagnetic radiation reactions, engineering and applied sciences [1-55]. Several techniques including decomposition, homotopy perturbation, exp-function and variational iteration have been employed to solve such equations analytically and numerically, see [8, 13, 28, 30-33, 38, 39, 51-54] and the reference therein. Most of these used schemes are coupled with the inbuilt deficiencies like calculation of the so-called Adomian's polynomials and non compatibility with the physical nature of the problems. He [14-26] developed the variational iteration and homotopy perturbation methods which proved applicable for a wide class of physical problems [1-12, 14-26, 29, 34, 35, 37, 39-46] and the references therein. In a later work Ghorbani *et al*. [10, 11] introduced He's polynomials which are compatible with Adomian's polynomials but are easier to calculate and are more user friendly. Recently, Noor and Mohyud-Din [40-43] made the elegant coupling of He's polynomials and the correction functional of VIM. It is worth mentioning that He's polynomials are calculated by applying He's HPM. This very reliable modified version (MVIM) has been proved useful in coping with the physical nature of the nonlinear problems and hence absorbs all the positive features of the coupled techniques [40-43]. Inspired and motivated by the ongoing research in this area, we applied the Modified Variational Iteration Method (MVIM) for solving system of partial differential equations. The obtained results are very encouraging.

MODIFIED VARIATIONAL ITERATION METHOD (MVIM)

To illustrate the basic concept of the MVIM, we consider the following general differential equation:

$$Lu + Nu = g(x) \tag{1}$$

where L is a linear operator, N a nonlinear operator and g (x) is the forcing term. According to VIM [1-7, 9, 12, 14, 16, 21-27, 29, 34, 35, 40-44, 46], we can construct a correction functional as follows

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi)\left(\begin{matrix} L\,u_n(\xi) \\ +N\,\tilde{u}_n(\xi) \end{matrix} - g(\xi)\right)d\xi \tag{2}$$

where λ is a Lagrange multiplier [12, 14, 16-21], $\tilde{u}_n$ is a restricted variation; (2) is called a correction functional. Now, we apply He's polynomials [10, 11]

$$\sum_{n=0}^{\infty} p^{(n)} u_n = u_0(x) + p\int_0^x \lambda(\xi)\left(\begin{matrix}\sum_{n=0}^{\infty} p^{(n)} L(u_n) + \\ \sum_{n=0}^{\infty} p^{(n)} N(\tilde{u}_n)\end{matrix}\right)d\xi - \int_0^x \lambda(\xi)\, g(\xi)\,d\xi \tag{3}$$

which is the MVIM [40-43] and is formulated by the coupling of VIM and He's polynomials. The comparison of like powers of p gives solutions of various orders.

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NUMERICAL APPLICATIONS

In this section, we apply the MVIM for solving systems of partial differential equations. Numerical results are very encouraging.

Example 3.1: Consider the following linear system of partial differential equations

$$u_t + v_x = 0, \qquad v_t + u_x = 0$$

with initial conditions

$$u(x,0) = e^x, \qquad v(x,0) = e^{-x}$$

The correction functional for the above system is given as

$$u_{n+1}(x,t) = e^x + \int_0^t \lambda(\xi)\left(\frac{\partial u_n(x,\xi)}{\partial \xi} + \frac{\partial \tilde{v}_n(x,\xi)}{\partial x}\right)d\xi$$

$$v_{n+1}(x,t) = e^{-x} + \int_0^t \lambda(\xi)\left(\frac{\partial v_n(x,\xi)}{\partial \xi} + \frac{\partial \tilde{u}_n(x,\xi)}{\partial x}\right)d\xi$$

Making the correction functional stationary, the Lagrange multipliers can easily be identified as $\lambda_1 = \lambda_2 = -1$ consequently,

$$u_{n+1}(x,t) = e^x - \int_0^t \left(\frac{\partial u_n(x,\xi)}{\partial \xi} + \frac{\partial v_n(x,\xi)}{\partial x}\right)d\xi$$

$$v_{n+1}(x,t) = e^{-x} - \int_0^t \left(\frac{\partial v_n(x,\xi)}{\partial \xi} + \frac{\partial u_n(x,\xi)}{\partial x}\right)d\xi$$

Applying the Modified Variational Iteration Method (MVIM)

$$u_0 + pu_1 + p^2u_2 + \cdots = e^x - p\int_0^t \left(\frac{\partial u_0(x,\xi)}{\partial \xi} + p\frac{\partial u_1(x,\xi)}{\partial \xi} + p^2\frac{\partial u_2(x,\xi)}{\partial \xi} + \cdots\right)d\xi - p\int_0^t \left(\frac{\partial v_0(x,\xi)}{\partial x} + p\frac{\partial v_1(x,\xi)}{\partial x} + p^2\frac{\partial v_2(x,\xi)}{\partial x} + \cdots\right)d\xi$$

$$v_0 + pv_1 + p^2v_2 + \cdots = e^{-x} - p\int_0^t \left(\frac{\partial v_0(x,\xi)}{\partial \xi} + p\frac{\partial v_1(x,\xi)}{\partial \xi} + p^2\frac{\partial v_2(x,\xi)}{\partial \xi} + \cdots\right)d\xi - p\int_0^t \left(\frac{\partial u_0(x,\xi)}{\partial x} + p\frac{\partial \tilde{u}_1(x,\xi)}{\partial x} + p^2\frac{\partial u_2(x,\xi)}{\partial x} + \cdots\right)d\xi$$

Comparing the co-efficient of like powers of p, following approximants are obtained

$$p^{(0)}: \begin{cases} u_0(x,t) = e^x \\ v_0(x,t) = e^{-x} \end{cases}$$

$$p^{(1)}: \begin{cases} u_1(x,t) = e^x + te^{-x} \\ v_1(x,t) = e^{-x} - te^x \end{cases}$$

$$p^{(2)}: \begin{cases} u_2(x,t) = e^x + te^{-x} + \frac{t^2}{2!}e^x \\ v_2(x,t) = e^{-x} - te^x + \frac{t^2}{2!}e^{-x} \end{cases}$$

$$p^{(3)}: \begin{cases} u_3(x,t) = e^x + te^{-x} + \frac{t^2}{2!}e^x + \frac{t^3}{3!}e^{-x} \\ v_3(x,t) = e^{-x} - te^x + \frac{t^2}{2!}e^{-x} - \frac{t^3}{3!}e^x \end{cases}$$

$$\vdots$$

The series solution is given as

$$u(x,t) = e^x\left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots\right) + e^{-x}\left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right)$$

$$v(x,t) = e^{-x}\left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots\right) - e^x\left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right)$$

and the closed form solution is given by

$$(u,v) = (e^x \cosh t + e^{-x} \sinh t,\ e^{-x} \cosh t - e^x \sinh t)$$

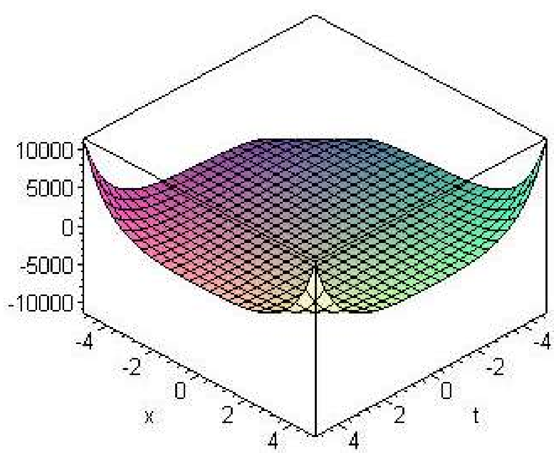

Fig. 1: (U)

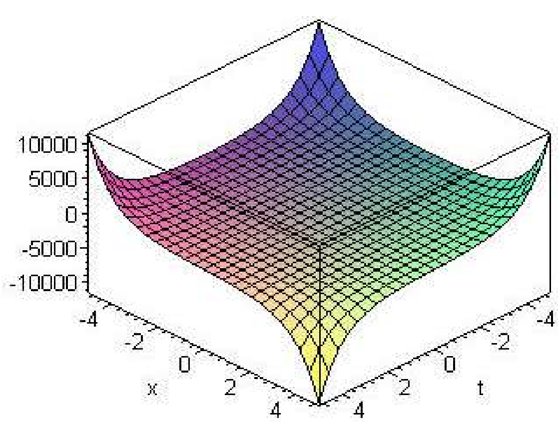

Fig. 2: (V)

Example 3.2: Consider the following linear system of partial differential equations

$$u_t + u_x - 2v = 0, \qquad v_t + v_x + 2u = 0$$

with initial conditions

$$u(x,0) = \sin x, \qquad v(x,0) = \cos x$$

The correction functional for the above system is given as

$$u_{n+1}(x,t) = \sin x + \int_0^t \lambda(\xi)\left(\frac{\partial u_n(x,\xi)}{\partial \xi} + \frac{\partial \tilde{u}_n(x,\xi)}{\partial x} - 2\tilde{v}_n\right)d\xi$$

$$v_{n+1}(x,t) = \cos x + \int_0^t \lambda(\xi)\left(\frac{\partial v_n(x,\xi)}{\partial \xi} + \frac{\partial \tilde{v}_n(x,\xi)}{\partial x} + 2\tilde{u}_n\right)d\xi$$

Making the correction functional stationary, the Lagrange multipliers can easily be identified as $\lambda_1 = \lambda_2 = -1$ consequently,

$$u_{n+1}(x,t) = \sin x - \int_0^t \left(\frac{\partial u_n(x,\xi)}{\partial \xi} + \frac{\partial u_n(x,\xi)}{\partial x} - 2v_n\right)d\xi$$

$$v_{n+1}(x,t) = \cos x - \int_0^t \left(\frac{\partial v_n(x,\xi)}{\partial \xi} + \frac{\partial v_n(x,\xi)}{\partial x} + 2u_n\right)d\xi$$

Applying the modified variational iteration method (MVIM)

$$u_0 + pu_1 + p^2u_2 + \cdots = \sin x - p\int_0^t \left(\frac{\partial u_0(x,\xi)}{\partial \xi} + p\frac{\partial u_1(x,\xi)}{\partial \xi} + p^2\frac{\partial u_2(x,\xi)}{\partial \xi} + \cdots\right)d\xi - p\int_0^t \begin{pmatrix} \left(\frac{\partial u_0(x,\xi)}{\partial x} + p\frac{\partial u_1(x,\xi)}{\partial x} + p^2\frac{\partial u_2(x,\xi)}{\partial x} + \cdots\right) \\ -2\left(v_0 + pv_1 + p^2v_2 + \cdots\right) \end{pmatrix} d\xi$$

$$v_0 + pv_1 + p^2v_2 + \cdots = \cos x - p\int_0^t \left(\frac{\partial v_0(x,\xi)}{\partial \xi} + p\frac{\partial v_1(x,\xi)}{\partial \xi} + p^2\frac{\partial v_2(x,\xi)}{\partial \xi} + \cdots\right)d\xi - p\int_0^t \begin{pmatrix} \left(\frac{\partial v_0(x,\xi)}{\partial x} + p\frac{\partial v_1(x,\xi)}{\partial x} + p^2\frac{\partial v_2(x,\xi)}{\partial x} + \cdots\right) \\ +2\left(u_0 + pu_1 + p^2u_2 + \cdots\right) \end{pmatrix} d\xi$$

Comparing the co-efficient of like powers of p, following approximants are obtained

$$p^{(0)}: \begin{cases} u_0(x,t) = \sin x \\ v_0(x,t) = \cos x \end{cases}$$

$$p^{(1)}: \begin{cases} u_1(x,t) = \sin x + t\cos x \\ v_1(x,t) = \cos x - t\sin x \end{cases}$$

$$p^{(2)}: \begin{cases} u_2(x,t) = \sin x + t\cos x - \frac{t^2}{2!}\sin x \\ v_2(x,t) = \cos x - t\sin x - \frac{t^2}{2!}\cos x \end{cases}$$

$$p^{(3)}: \begin{cases} u_3(x,t) = \sin x + t\cos x - \frac{t^2}{2!}\sin x - \frac{t^3}{3!}\cos x \\ v_3(x,t) = \cos x - t\sin x - \frac{t^2}{2!}\cos x - \frac{t^3}{3!}\sin x \end{cases}$$

$$\vdots$$

The series solution is given by

$$u(x,t) = \sin x\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots\right) + \cos x\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots\right)$$

$$v(x,t) = \cos x\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots\right) - \sin x\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots\right)$$

and the closed form solution is obtained as

$$(u,v) = (\sin(x+t), \cos(x+t))$$

Example 3.3: Consider the following nonlinear system of partial differential equations

$$u_t + vu_x + u = 1, \qquad v_t - uv_x - v = 1$$

with initial conditions

$$u(x,0) = e^x, \qquad v(x,0) = e^{-x}$$

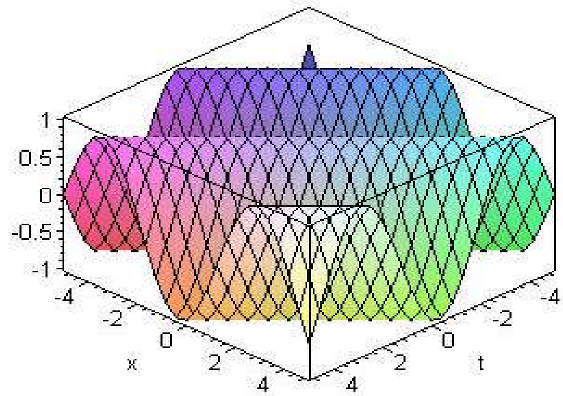

Fig. 3: (U)

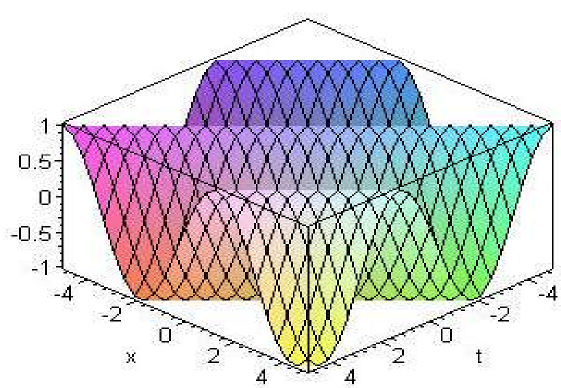

Fig. 4: (V)

The correction functional for the above system is given as

$$u_{n+1}(x,t) = e^{x} + \int_0^t \lambda_1(\xi)\left(\frac{\partial u_n(x,\xi)}{\partial \xi} + \tilde{v}_n(x,\xi)\left(\frac{\partial \tilde{u}_n(x,\xi)}{\partial x}\right) + \tilde{u}_n(x,\xi) - 1\right)d\xi$$
$$v_{n+1}(x,t) = e^{-x} + \int_0^t \lambda_2(\xi)\left(\frac{\partial v_n(x,\xi)}{\partial \xi} - u_n(x,\xi)\left(\frac{\partial \tilde{v}_n(x,\xi)}{\partial x}\right) - \tilde{v}_n(x,\xi) - 1\right)d\xi$$

Making the correct functional stationary, the Lagrange multipliers can easily be identified as $\lambda_1 = \lambda_2 = -1$ consequently,

$$u_{n+1}(x,t) = e^{x} - \int_0^t \left(\frac{\partial u_n(x,\xi)}{\partial \xi} + v_n(x,\xi)\left(\frac{\partial u_n(x,\xi)}{\partial x}\right) + u_n(x,\xi) - 1\right)d\xi$$
$$v_{n+1}(x,t) = e^{-x} - \int_0^t \left(\frac{\partial v_n(x,\xi)}{\partial \xi} - u_n(x,\xi)\left(\frac{\partial v_n(x,\xi)}{\partial x}\right) - v_n(x,\xi) - 1\right)d\xi$$

Applying the modified variational iteration method (MVIM)

$$u_0 + pu_1 + p^2u_2 + \cdots = e^{x} - p\int_0^t \left(\left(\frac{\partial u_0(x,\xi)}{\partial \xi} + p\frac{\partial u_1(x,\xi)}{\partial \xi} + p^2\frac{\partial u_2(x,\xi)}{\partial \xi} + \cdots\right) + \left(u_0 + pu_1 + p^2u_2 + \cdots\right) - 1\right)d\xi$$
$$-p\int_0^t \left(\left(v_0 + pv_1 + p^2v_2 + \cdots\right)\left(\frac{\partial u_0(x,\xi)}{\partial x} + p\frac{\partial u_1(x,\xi)}{\partial x} + p^2\frac{\partial u_2(x,\xi)}{\partial x} + \cdots\right)\right)d\xi$$
$$v_0 + pv_1 + p^2v_2 + \cdots = e^{-x} - p\int_0^t \left(\left(\frac{\partial v_0(x,\xi)}{\partial \xi} + p\frac{\partial v_1(x,\xi)}{\partial \xi} + p^2\frac{\partial v_2(x,\xi)}{\partial \xi} + \cdots\right) - \left(v_0 + pv_1 + p^2v_2 + \cdots\right) - 1\right)d\xi$$
$$+p\int_0^t \left(\left(u_0 + pu_1 + p^2u_2 + \cdots\right)\left(\frac{\partial v_0(x,\xi)}{\partial x} + p\frac{\partial v_1(x,\xi)}{\partial x} + p^2\frac{\partial v_2(x,\xi)}{\partial x} + \cdots\right)\right)d\xi$$

Comparing the co-efficient of like powers of p, consequently, following approximants are obtained

$$p^{(0)}:\begin{cases} u_0(x,t) = e^{x} \\ v_0(x,t) = e^{-x} \end{cases}$$

Proceeding as before, the series solution is given as

$$(u,v) = \left(e^{x}\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right), e^{-x}\left(1 + t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right)\right)$$

and in the closed form solution is obtained as

$$(u,v) = (e^{x-t}, e^{-x+t})$$

Example 3.4: Consider the following nonlinear system of partial differential equations

$$u_t + v_x w_y - v_y w_x = -u$$
$$v_t + w_x u_y + w_y u_x = v$$
$$w_t + u_x v_y + u_y v_x = -w$$

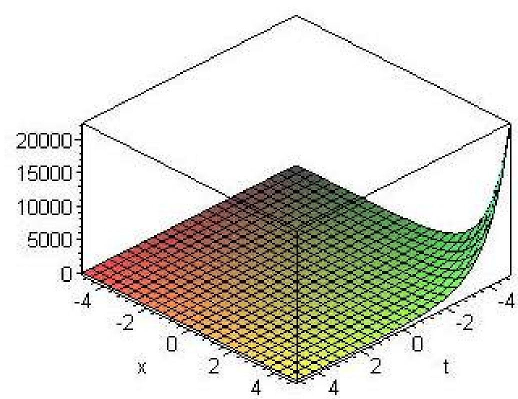

Fig. 5: (U)

with initial conditions

$$u(x,y,0) = e^{x+y}$$
$$v(x,y,0) = e^{x-y}$$
$$w(x,y,0) = e^{-x+y}.$$

The correction functional for the above system is given as

$$u_{n+1}(x,y,t) = e^{x+y} + \int_0^t \lambda_1(\xi)\left(\frac{\partial u_n(x,y,\xi)}{\partial \xi} + \left(\frac{\partial \tilde{v}_n(x,y,\xi)}{\partial x}\right)\left(\frac{\partial \tilde{w}_n(x,y,\xi)}{\partial y}\right)\right)d\xi - \int_0^t \lambda_1(\xi)\left(\left(\frac{\partial \tilde{v}_n(x,y,\xi)}{\partial y}\right)\left(\frac{\partial \tilde{w}_n(x,y,\xi)}{\partial x}\right) + \tilde{u}_n\right)d\xi$$

$$v_{n+1}(x,y,t) = e^{x-y} + \int_0^t \lambda_2(\xi)\left(\frac{\partial v_n(x,y,\xi)}{\partial \xi} + \left(\frac{\partial \tilde{u}_n(x,y,\xi)}{\partial y}\right)\left(\frac{\partial \tilde{w}_n(x,y,\xi)}{\partial x}\right)\right)d\xi + \int_0^t \lambda_2(\xi)\left(\left(\frac{\partial \tilde{u}_n(x,y,\xi)}{\partial x}\right)\left(\frac{\partial \tilde{w}_n(x,y,\xi)}{\partial y}\right) - \tilde{v}_n\right)d\xi$$

$$w_{n+1}(x,y,t) = e^{-x+y} + \int_0^t \lambda_3(\xi)\left(\frac{\partial w_n(x,y,\xi)}{\partial \xi} + \left(\frac{\partial \tilde{u}_n(x,y,\xi)}{\partial x}\right)\left(\frac{\partial \tilde{v}_n(x,y,\xi)}{\partial y}\right)\right)d\xi + \int_0^t \lambda_3(\xi)\left(\left(\frac{\partial \tilde{v}_n(x,y,\xi)}{\partial x}\right)\left(\frac{\partial \tilde{u}_n(x,y,\xi)}{\partial y}\right) + \tilde{w}_n\right)d\xi$$

Making the correction functional stationary, the Lagrange multipliers can easily be identified as $\lambda_1 = \lambda_2 = \lambda_3 = -1$, consequently,

$$u_{n+1}(x,y,t) = e^{x+y} - \int_0^t \left(\frac{\partial u_n(x,y,\xi)}{\partial \xi} + \left(\frac{\partial v_n(x,y,\xi)}{\partial x}\right)\left(\frac{\partial w_n(x,y,\xi)}{\partial y}\right)\right)d\xi + \int_0^t \left(\left(\frac{\partial v_n(x,y,\xi)}{\partial y}\right)\left(\frac{\partial w_n(x,y,\xi)}{\partial x}\right) + u_n\right)d\xi$$

$$v_{n+1}(x,y,t) = e^{x-y} - \int_0^t \left(\frac{\partial v_n(x,y,\xi)}{\partial \xi} + \left(\frac{\partial u_n(x,y,\xi)}{\partial y}\right)\left(\frac{\partial w_n(x,y,\xi)}{\partial x}\right)\right)d\xi - \int_0^t \left(\left(\frac{\partial u_n(x,y,\xi)}{\partial x}\right)\left(\frac{\partial w_n(x,y,\xi)}{\partial y}\right) - v_n\right)d\xi$$

$$w_{n+1}(x,y,t) = e^{-x+y} - \int_0^t \left(\frac{\partial w_n(x,y,\xi)}{\partial \xi} + \left(\frac{\partial u_n(x,y,\xi)}{\partial x}\right)\left(\frac{\partial v_n(x,y,\xi)}{\partial y}\right)\right)d\xi - \int_0^t \left(\left(\frac{\partial v_n(x,y,\xi)}{\partial x}\right)\left(\frac{\partial u_n(x,y,\xi)}{\partial y}\right) + w_n\right)d\xi$$

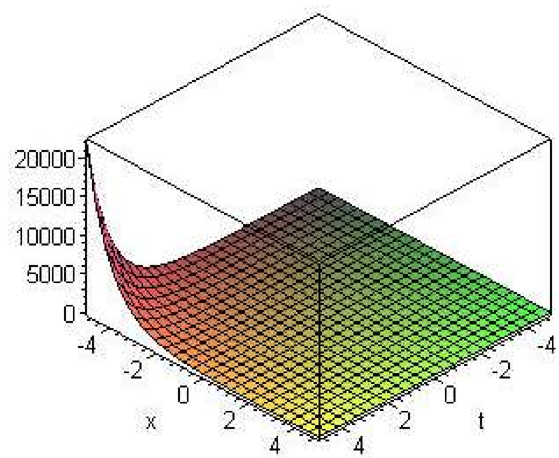

Fig. 6: (V)

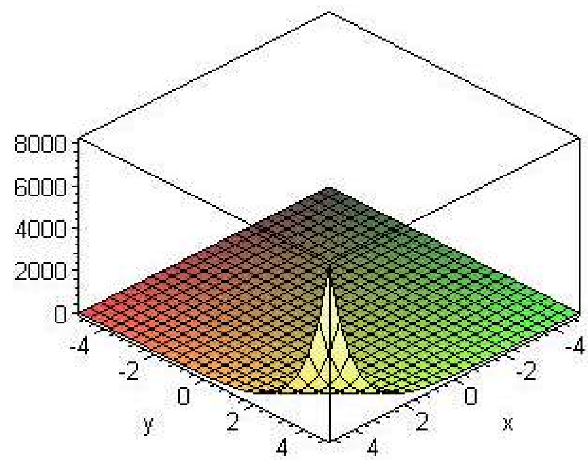

Fig. 7: (U, t=1)

Comparing the co-efficient of like powers of p, consequently, following approximants are obtained

$$p^{(0)}: \begin{cases} u_0(x,y,t) = e^{x+y} \\ v_0(x,y,t) = e^{x-y} \\ w_0(x,y,t) = e^{-x+y} \end{cases}$$

$$p^{(1)}: \begin{cases} u_1(x,y,t) = e^{x+y}(1-t) \\ v_1(x,y,t) = e^{x-y}(1+t) \\ w_1(x,y,t) = e^{-x+y}(1+t) \end{cases}$$

$$p^{(2)}: \begin{cases} u_2(x,y,t) = e^{x+y}(1-t+t^2/2!) \\ v_2(x,y,t) = e^{x-y}(1+t+t^2/2!) \\ w_2(x,y,t) = e^{-x+y}(1+t+t^2/2!) \end{cases}$$

$$p^{(3)}: \begin{cases} u_3(x,y,t) = e^{x+y}(1-t+t^2/2!-t^3/3!) \\ v_3(x,y,t) = e^{x-y}(1+t+t^2/2!+t^3/3!) \\ w_3(x,y,t) = e^{-x+y}(1+t+t^2/2!-t^3/3!) \end{cases}$$

$$\vdots$$

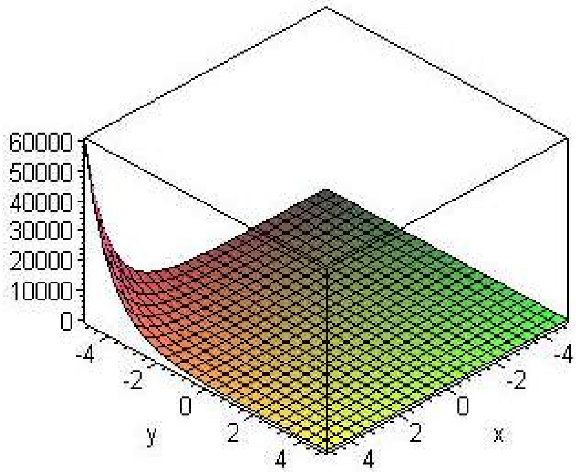

Fig. 8: (V, t=1)

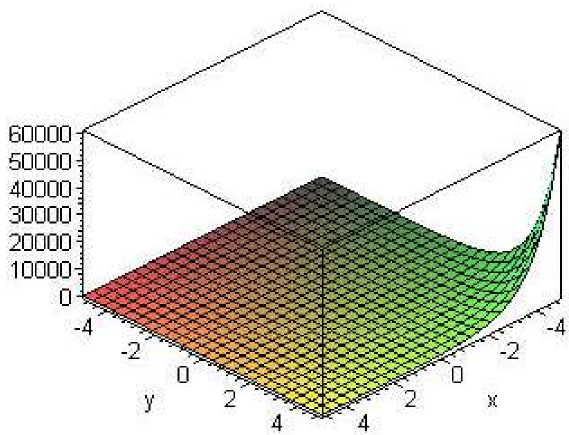

Fig. 9: (W, t=1)

The closed form solution is given as

$$(u,v,w) = (e^{x+y-t}, e^{x-y+t}, e^{-x+y+t})$$

CONCLUSION

In this paper, we applied the modified variational iteration method (MVIM) for solving systems of partial differential equations. The method is applied in a direct way without using linearization, transformation, discretization or restrictive assumptions. It may be concluded that the MVIM is very powerful and efficient in finding the analytical solutions for a wide class of boundary value problems. The method gives more realistic series solutions that converge very rapidly in physical problems. It is worth mentioning that the method is capable of reducing the volume of the computational work as compare to the classical methods while still maintaining the high accuracy of the numerical result. The fact that the MVIM solves nonlinear problems without using the Adomian's polynomials is a clear advantage of this technique over the decomposition method.

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