World Applied Sciences Journal 7 (Special Issue for Applied Math): 116-147, 2009
ISSN 1818-4952

Solutions of Nonlinear Differential Equations by Exp-function Method

Syed Tauseef Mohyud-Din

HITEC University, Taxila Cantt., Pakistan

Abstract: In this paper, we apply the exp-function method to construct solutions of nonlinear differential equations which arise in various branches of physics, engineering and applied sciences. The proposed scheme proves to be very efficient and compatible to deal with the nonlinearity and versatility of the problems. Numerical results clearly indicate the reliability and efficiency of the suggested technique.

Key words: Exp-function method . nonlinear differential equations . soliton solutions . periodic solutions

INTRODUCTION

The nonlinear differential equations occur in diversified physical phenomena including propagation of shallow water waves, long wave and chemical reaction-diffusion models, fluid mechanics, physics, astrophysics, solid state physics, chemistry, various branches of biology, astronomy, hydrodynamic and hydro magnetic stability, nuclear physics, beam and plate deflection theory, applied and engineering sciences [1-56]. Several techniques including finite difference, decomposition, finite element, differential transform, polynomial and non polynomial splines, Taylor's series, sink Galerkin, perturbation, homotopy, variational iteration, homotopy perturbation and modified variational iteration have been used for solving the governing equations related to various physical problems [1-3, 7-24, 27-32, 35-42, 45-47, 51] and the references therein. He and Wu [25] developed and formulated exp-function method which has been used for solving a wide class of nonlinear differential equations [4, 6, 24-26, 33, 34, 43, 44, 48-50, 52-56]. In this paper, we have discussed in length applications of exp-function method for finding solutions of nonlinear differential equations of versatile physical nature. It is observed that the proposed technique is very convenient to implement and may be treated as an excellent and efficient alternative for the solutions of nonlinear problems which arise in all physical phenomena related to engineering, medicine and applied sciences. Moreover, solutions of such problems can be very useful for productive research. Several examples are given to test and re-confirm the efficiency of the proposed method.

EXP-FUNCTION METHOD

Consider the general nonlinear partial differential equation of the type

$$P(u, u_t, u_x, u_{tt}, u_{xx}, u_{xxxx}, \ldots) = 0 \qquad (1)$$

Using a transformation

$$\eta = kx + \omega t \qquad (2)$$

where k and ω are constants, we can rewrite Eq. (1) in the following nonlinear ODE

$$Q(u, u', u'', u''', u^{(iv)}, \ldots) = 0 \qquad (3)$$

According to the exp-function method, which was developed by He and Wu [25], we assume that the wave solutions can be expressed in the following form

$$u(\eta) = \frac{\sum_{n=-c}^{d} a_n \exp[n\eta]}{\sum_{m=-p}^{q} b_m \exp[m\eta]} \qquad (4)$$

where p, q, c and d are positive integers which are known to be further determined, a_n and b_m are unknown constants. We can rewrite equation (4) in the following equivalent form.

$$u(\eta) = \frac{a_c \exp[c\eta] + \ldots + a_{-d} \exp[-d\eta]}{b_p \exp[p\eta] + \ldots + b_{-q} \exp[-q\eta]} \qquad (5)$$

This equivalent formulation plays an important and fundamental part for finding the analytic solution of problems [4, 6, 24-26, 33, 34, 43, 44, 48-50, 52-56]. To determine the value of c and p, we balance the linear term of highest order of equation (4) with the highest order nonlinear term. Similarly, to determine the value of d and q, we balance the linear term of lowest order of equation (3) with lowest order non linear term.

Numerical Applications In this section, we apply the exp-function method developed by He and Wu [25]

Corresponding Author: Dr. Syed Tauseef Mohyud-Din, HITEC University, Taxila Cantt., Pakistan

to construct soliton, periodic and exact solutions of a wide class of nonlinear differential equations. The proposed method will be tested on diversified nonlinear problems of physical nature.

Example 3.1: Consider the generalized form of CDG equation

$$u_t + \frac{1}{5}\alpha^2 u^2 u_x + \alpha u_x u_{2x} + \alpha u u_{3x} + u_{5x} = 0 \quad (6)$$

where α is an arbitrary constant. Introducing a transformation as $\eta = kx+\omega t$ we can covert equation (6) into ordinary differential equations

$$\omega u' + \frac{1}{5}k\alpha^2 u^2 u' + \alpha k^3 u' u'' + \alpha k^3 u u''' + k^5 u^{(5)} = 0 \quad (7)$$

where the prime denotes the derivative with respect to η. The solution of the equation (7) can be expressed in the form of equation (6)

$$u(\eta) = \frac{a_c \exp[c\eta] + ... + a_{-d} \exp[-dn]}{b_p \exp[p\eta] + ... + b_{-q} \exp[-qn]}$$

To determine the value of c and p, we balance the linear term of highest order of equation (7) with the highest order nonlinear term

$$u^{(5)} = \frac{c_1 \exp[(31p + c)\eta] + \cdots}{c_2 \exp[32p\eta] + \cdots} \quad (8)$$

and

$$u^2 u' = \frac{c_3 \exp[(p + 3c)\eta] + \cdots}{c_4 \exp[4p\eta] + \cdots} = \frac{c_3 \exp[(29p + 3c)\eta] + \cdots}{c_4 \exp + [32p\eta] \cdots} \quad (9)$$

where c_i are determined coefficients only for simplicity; balancing the highest order of exp-function in (8) and (9), we have

$$31p + c = 29p + 3c \quad (10)$$

which in turn gives

$$p = c \quad (11)$$

To determine the value of d and q, we balance the linear term of lowest order of equation (7) with the lowest order non-linear term

$$u^{(5)} = \frac{\cdots + d_1 \exp[(-d - 31q)\eta]}{\cdots + d_2 \exp[-32q\eta]} \quad (12)$$

and

$$u'u'' = \frac{\cdots + d_3 \exp[(-q - 3d)\eta]}{\cdots + d_4 \exp[-4q\eta]} = \frac{\cdots + d_3 \exp[(-3d - 29q)\eta]}{\cdots + d_4 \exp[-32q\eta]} \quad (13)$$

where d_i are determined coefficients only for simplicity. Now, balancing the lowest order of exp-function in (12) and (13), we have

$$-31q - d = -29q - 3d \quad (14)$$

which in turn gives

$$q = d \quad 15)$$

Case 3.1.1: We can freely choose the values of c and d, but we will illustrate that the final solution does not strongly depend upon the choice of values of c and d. For simplicity, we set p = c = 1 and q = d = 1, then the trial solution, equation (6) reduces to

$$u(\eta) = \frac{a_1 \exp[\eta] + a_0 + a_{-1} \exp[-\eta]}{b_1 \exp[\eta] + a_0 + b_{-1} \exp[-\eta]} \quad (16)$$

Substituting equation (16) into equation (7), we have

$$\begin{aligned} &\frac{1}{A}[c_5 \exp(5\eta) + c_4 \exp(4\eta) + c_3 \exp(3\eta) \\ &+ c_2 \exp(2\eta) + c_1 \exp(\eta) + c_0 + c_{-1} \exp(-\eta) \\ &+ c_{-2} \exp(-2\eta) + c_{-3} \exp(-3\eta) \\ &+ c_{-4} \exp(-4\eta) + c_{-5} \exp(-5\eta)] = 0 \end{aligned} \quad (17)$$

where

$$A = 5(b_1 \exp(\eta) + b_0 + b_{-1} \exp(-\eta))^6$$
$$c_i (i = -5, -4, ..., 4, 5)$$

are constants obtained by Maple 11.

Equating the coefficients of exp $(n\eta)$ to be zero, we obtain

$$\begin{aligned} &\{c_{-5} = 0,\ c_{-4} = 0,\ c_{-3} = 0,\ c_{-2} = 0, \\ &c_{-1} = 0, c_0 = 0, c_1 = 0, c_2 = 0, \\ &c_3 = 0, c_4 = 0,\ c_5 = 0\} \end{aligned} \quad (18)$$

Solution of (18) will yield

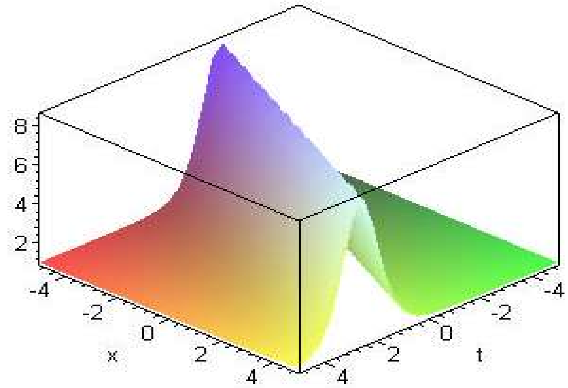

Fig. 3.1: Soliton solutions of equation (6), when $a_1 = b_0 = b_1 = k = \alpha = 1$. In case k is an imaginary number, the obtained soliton solution can be converted into periodic solution or compact-like solutions. Therefore, we write k = iK, consequently, equation (20) becomes

$$a_{-1} = \frac{1}{4}\frac{a_1 b_0^2}{b_1^2},\quad b_0 = b_0,\quad b_1 = b_1,\quad b_{-1} = \frac{1}{4}\frac{b_0^2}{b_1},$$
$$a_1 = a_1,\quad \omega = -\frac{1}{5}\frac{k\left(5k^4 b_1^2 + 5k^2\alpha a_1 b_1 + \alpha^2 a_1^2\right)}{b_1^2}, \tag{19}$$
$$a_0 = \frac{b_0\left(\alpha a_1 + 15k^2 b_1\right)}{\alpha b_1}$$

We, therefore, obtained the following generalized solitary solution u(x,t) of equation (1)

$$u(x,t) = \frac{a_1 b_0}{b_1\alpha}\left(\frac{15k^2}{b_1 e^{(kx+\omega t)} + b_0 + \frac{b_0^2}{4b_1}e^{(-kx-\omega t)}}\right) \tag{20}$$

where

$$\omega = -\frac{1}{5}\frac{k\left(5k^4 b_1^2 + 5k^2\alpha a_1 b_1 + \alpha^2 a_1^2\right)}{b_1^2}$$

and a_1, b_0, b_1, α and k are real numbers.

$$u(x,t) = \frac{a_1 b_0}{b_1\alpha}\left(\frac{-15K^2}{b_1 e^{(iKx+\omega t)} + b_0 + \frac{b_0^2}{4b_1}e^{(-iKx-\omega t)}}\right) \tag{21}$$

where

$$\omega = -\frac{1}{5}\frac{iK\left(5K^4 b_1^2 - 5K^2\alpha a_1 b_1 + \alpha^2 a_1^2\right)}{b_1^2}$$

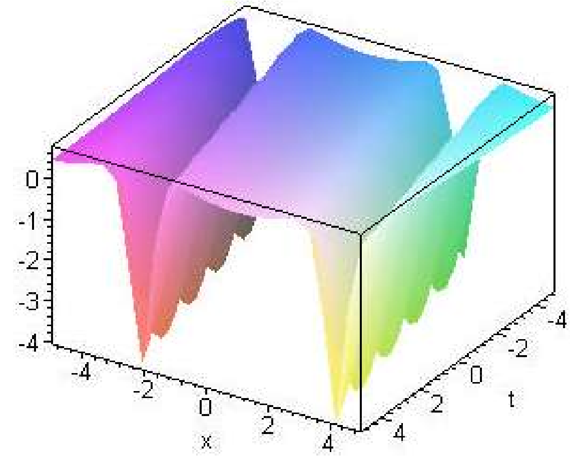

Fig. 3.2: Periodic solutions of equation (6), when $a_1 = b_0 = b_1 = k = \alpha = 1$

and a_1, b_1, α and K are real numbers. If we search for periodic solution or compact-like solution, the imaginary part in equation (21) must be zero that requires, therefore equation (21) becomes

$$u(x,t) = \frac{15 a_1 b_0 K^2}{b_1\alpha}\frac{4b_0 b_1 + \cos(-Kx+\lambda t)\left[4b_1^2 + b_0^2\right]}{8b_1^2 b_0^2 + 16b_1^4 + b_0^4 + \cos(-Kx+\lambda t)\left[32b_1^3 b_0 + 16b_1^2 b_0^2 + 8b_1 b_0^3\right]} \tag{22}$$

where

$$\lambda = \frac{1}{5}\frac{K\left(5K^4 b_1^2 - 5K^2\alpha a_1 b_1 + \alpha^2 a_1^2\right)}{b_1^2}$$

which is the periodic solution of equation (6).

Case 3.1.2: If p = c = 2 and q = d = 2, then equation (6) reduces to

$$u(\eta) = \frac{a_2\exp[2\eta] + a_1\exp[\eta] + a_0 + a_{-1}\exp[-\eta] + a_{-2}\exp[-2\eta]}{b_2\exp[2\eta] + +b_1\exp[\eta] + b_0 + +b_{-1}\exp[-\eta] + b_{-2}\exp[-2\eta]} \tag{23}$$

there are some free parameters in equation. (23), we set $b_1 = b_{-1} = 0$, for simplicity, the trial-function (23) is simplified as follows:

$$u(\eta) = \frac{a_2\exp[2\eta] + a_1\exp[\eta] + a_0 + a_{-1}\exp[-\eta] + a_{-2}\exp[-2\eta]}{b_2\exp[2\eta] + b_0 + b_{-2}\exp[-2\eta]}$$

Proceeding as before, we obtain

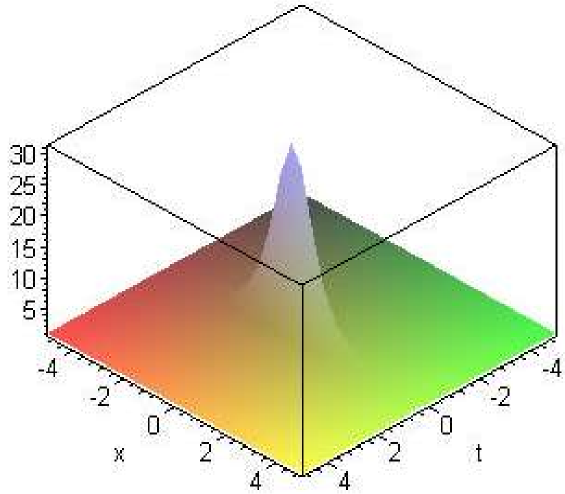

Fig. 3.3: Depicts soliton solutions of equation (6) when $a_2 = b_0 = b_2 = k = \alpha = 1$

$$a_{-1} = 0, a_2 = a_2, \; b_2 = b_2, \; b_0 = b_0,$$
$$a_{-2} = \frac{1}{4}\frac{a_2 b_0^2}{b_2^2}, \quad a_1 = 0, \; b_{-2} = \frac{1}{4}\frac{b_0^2}{b_2},$$
$$\omega = -\frac{1}{5}\frac{k\left(20k^2\alpha a_2 b_2 + \alpha^2 a_2^2 + 80k^4 b_2^2\right)}{b_2^2} \tag{24}$$
$$a_0 = \frac{b_0\left(\alpha a_2 + 60k^2 b_2\right)}{\alpha b_2}$$

Hence we get the generalized solitary wave solution of equation (1) as follows

$$u(x,t) = \frac{a_2 b_0}{b_2\alpha}\left(\frac{60K^2}{b_2 e^{2(kx+\omega t)} + b_0 + \frac{b_0^2}{4b_2}e^{2(-kx-\omega t)}}\right) \tag{25}$$

where

$$\omega = -\frac{1}{5}\frac{k\left(20k^2\alpha a_2 b_2 + \alpha^2 a_2^2 + 80k^4 b_2^2\right)}{b_2^2}$$

a_2, b_0, b_2, α and k are real numbers.

Example 3.2: [13] Consider the general form of Calogero-Degasperis-Fokas (CDF) equation

$$u_t + \frac{1}{4}u_{xxx} - \frac{3}{4}\frac{u_x u_{xx}}{u} + \frac{3}{8}\frac{u_x^3}{u^2} + \frac{3}{8}u_x\left(\alpha u^2 + \frac{\beta}{u^2} + \gamma\right) = 0 \tag{6a}$$

where α, β and γ are arbitrary constants. Introducing a transformation as $\eta = kx + \omega t$ we can covert equation (6a) into ordinary differential equations

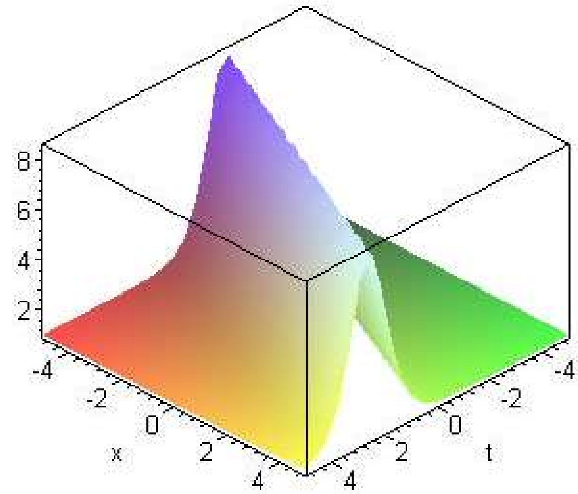

Fig. 3.4: Depicts soliton solutions of equation (6) when $a_1 = b_0 = b_1 = k = \alpha = 1$
$a_2 = b_0 = b_2 = \alpha = 1$ and $k = \frac{1}{2}$

$$8\omega u' + 2k^3 u''' - 6k^3\frac{u'u''}{u} + 3k^3\frac{u'^3}{u^2} + 3ku'\left(\alpha u^2 + \beta\frac{1}{u^2} + \gamma\right) = 0 \tag{7a}$$

where the prime denotes the derivative with respect to η. The solution of the equation (7a) can be expressed in the form, equation (6a)

$$u(\eta) = \frac{a_c \exp[c\eta] + \ldots + a_{-d}\exp[-dn]}{b_p \exp[p\eta] + \ldots + b_{-q}\exp[-qn]}$$

To determine the value of c and p, we balance the linear term of highest order of equation (7a) with the highest order nonlinear term

$$u''' = \frac{c_1 \exp[(7p+c)\eta] + \cdots}{c_2 \exp[8p\eta] + \cdots} \tag{8a}$$

and

$$u^2 u' = \frac{c_3 \exp[(p+3c)\eta] + \cdots}{c_4 \exp[4p\eta] + \cdots} = \frac{c_3 \exp[(5p+3c)\eta] + \cdots}{c_4 \exp + [8p\eta]\cdots} \tag{9a}$$

where c_i are determined coefficients only for simplicity; balancing the highest order of exp-function in (8a) and (9a), we have

$$5p+3c = 7p+c \tag{10a}$$

which in turn gives

$$p = c \tag{11a}$$

To determine the value of d and q, we balance the linear term of lowest order of equation (6a) with the lowest order non-linear term

$$u''' = \frac{\cdots + d_1 \exp[(-d-7q)\eta]}{\cdots + d_2 \exp[-8q\eta]} \tag{12a}$$

and

$$u'u'' = \frac{\cdots + d_3 \exp[(-q-3d)\eta]}{\cdots + d_4 \exp[-4q\eta]} = \frac{\cdots + d_3 \exp[(-3d-5q)\eta]}{\cdots + d_4 \exp[-8q\eta]} \tag{13a}$$

where d_i are determined coefficients only for simplicity. Now, balancing the lowest order of Exp-function in (12a) and (13a), we have

$$-7q-d = -5q-3d \tag{14a}$$

which in turn gives

$$q = d \tag{15a}$$

Case 3.2.1: We can freely choose the values of c and d, but we will illustrate that the final solution does not strongly depend upon the choice of values of c and d. For simplicity, we set p = c = 1 and q = d = 1, then the trial solution, equation (6a) reduces to

$$u(\eta) = \frac{a_1 \exp[\eta] + a_0 + a_{-1}\exp[-\eta]}{b_1 \exp[\eta] + a_0 + b_{-1}\exp[-\eta]} \tag{16a}$$

Substituting equation (16a) into equation (7a), we have

$$\frac{1}{A}[c_5\exp(5\eta) + c_4\exp(4\eta) + c_3\exp(3\eta) + c_2\exp(2\eta) + c_1\exp(\eta) + c_0 + c_{-1}\exp(-\eta) + c_{-2}\exp(-2\eta) + c_{-3}\exp(-3\eta) + c_{-4}\exp(-4\eta) + c_{-5}\exp(-5\eta)] = 0 \tag{17a}$$

where

$$A = 8(\exp(\eta) + b_0 + b_{-1}\exp(-\eta))^4 (a_1 e^{\eta} + a_0 + a_{-1}e^{\eta})^2$$

c_i (i =-5,-4,…,4,5) are constants obtained by Maple 11. Equating the coefficients of exp $(n\eta)$ to be zero, we obtain

$$\{c_{-5} = 0,\ c_{-4} = 0,\ c_{-3} = 0,\ c_{-2} = 0,\ c_{-1} = 0,\ c_0 = 0,\ c_1 = 0, c_2 = 0,\ c_3 = 0,\ c_4 = 0,\ c_5 = 0\} \tag{18a}$$

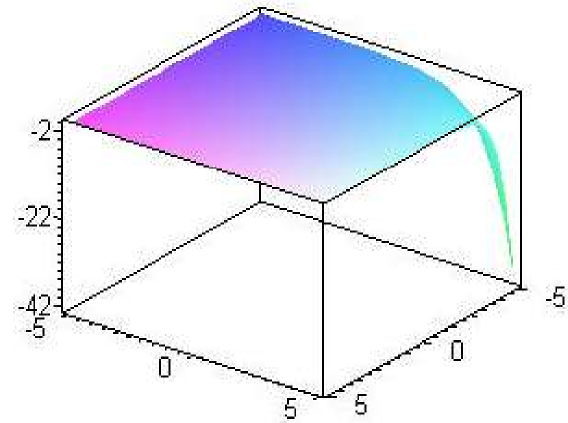

Fig. 3.5: Soliton solutions of equation (1), when $a_1 = b_0 = k = \beta = 1$ and $\alpha = \gamma = 2$. In case k is an imaginary number, the obtained soliton solution can be converted into periodic solution or compact-like solution. Therefore, we write k = iK, consequently, equation (20a) becomes

Solution of (18a) will yield

$$\{a_{-1} = \frac{1}{4}\frac{a_1 b_0^2 \begin{pmatrix} a_1^8\alpha^2 + 2a_1^6k^2\alpha - 2\beta\alpha a_1^4 \\ +a_1^4k^4 + 2a_1^2\beta k^2 + \beta^2 \end{pmatrix}}{(-\alpha a_1^4 + \beta + k^2 a_1^2)^2},\ b_0 = b_0$$

$$b_{-1} = \frac{1}{4}\frac{b_0^2 \begin{pmatrix} a_1^8\alpha^2 + 2a_1^6k^2\alpha - 2\beta\alpha a_1^4 \\ +a_1^4k^4 + 2a_1^2\beta k^2 + \beta^2 \end{pmatrix}}{(-\alpha a_1^4 + \beta + k^2 a_1^2)^2},\quad a_1 = a_1, \tag{19a}$$

$$\omega = -\frac{1}{8}\frac{k(2k^2a_1^2 + 3\gamma a_1^2 + 3\alpha a_1^4 + 3\beta)}{a_1^2},$$

$$a_0 = \frac{a_1 b_0(-\alpha a_1^4 + \beta - k^2 a_1^2)}{-\alpha a_1^4 + \beta + k^2 a_1^2}\}$$

We, therefore, obtained the following generalized solitary solution u (x,t) of equation (6a)

$$u(x,t) = a_1 + \frac{2k^2 a_1^3 b_0(\beta - \alpha)}{(-\alpha a_1^4 + \beta + k^2 a_1^2)^2 (e^{(kx+\omega t)} + b_0) + \frac{1}{4} b_0^2 \begin{pmatrix} a_1^8\alpha^2 + 2a_1^6k^2\alpha - 2\beta\alpha a_1^4 \\ +a_1^4k^4 + 2a_1^2\beta k^2 + \beta^2 \end{pmatrix} (e^{-(kx+\omega t)})} \tag{20a}$$

where

$$\omega = -\frac{1}{8}\frac{k(2k^2a_1^2 + 3\gamma a_1^2 + 3\alpha a_1^4 + 3\beta)}{a_1^2}$$

and a_1, b_0, α, β, γ and k are real numbers.

$$u(x,t)=a_1+\frac{-2K^2a_1^3b_0(\beta-\alpha)}{\left(-\alpha a_1^4+\beta-K^2a_1^2\right)^2\left(e^{(ikx+\omega t)}+b_0\right)+\frac{1}{4}b_0^2\left(a_1^8\alpha^2-2a_1^6K^2\alpha-2\beta\alpha a_1^4+a_1^4K^4-2a_1^2\beta K^2+\beta^2\right)\left(e^{-(ikx+\omega t)}\right)} \tag{21a}$$

where

$$\omega=-\frac{1}{8}\frac{iK\left(-2K^2a_1^2+3\gamma a_1^2+3\alpha a_1^4+3\beta\right)}{a_1^2}$$

and a_1, b_0, α, β, γ and K are real numbers. If we search for periodic solution or compact-like solution, the imaginary part in equation (21a) must be zero that requires, therefore equation (21a) becomes

$$u(x,t)=a_1\frac{\left[\begin{array}{l}\cos(Kx+\omega t)\begin{pmatrix}32m^4b_0+16m^2(Kx+\omega t)\,b_0^2p+8m^2b_0^3p-32K^2a_1^2b_0\beta m^2\\-8K^2a_1^2b_0^3\beta p+32K^2a_1^2b_0\alpha m^2+8K^2a_1^2b_0^3\alpha p\end{pmatrix}+16m^4b_0^2\\16m^4-8m^2b_0^2p+b_0^4p^2-32K^2a_1^2b_0^2\beta m^2+32K^2a_1^2b_0^2\alpha m^2\end{array}\right]}{(Kx+\omega t)\left(32m^4b_0+16m^2b_0^2p+8m^2b_03p\right)+16m^4b_0^2+16m^4-8m^2b_0^2p+b_0^4p^2} \tag{22a}$$

where

$$\left\{\begin{array}{l}m=-\alpha a_1^4+\beta+k^2a_1^2,\quad p=a_1^8\alpha^2+2a_1^6k^2\alpha-2\beta\alpha a_1^4+a_1^4k^4+2a_1^2\beta k^2+\beta^2\\ \omega=-\frac{1}{8}\frac{k\left(2k^2a_1^2+3\gamma a_1^2+3\alpha a_1^4+3\beta\right)}{a_1^2}\end{array}\right\}$$

which are periodic solutions of equation (6a).

Case 3.2.2: If p = c = 2 and q = d = 1, than equation (6a) reduces to

$$u(\eta)=\frac{a_2\exp[2\eta]+a_1\exp[\eta]+a_0+a_{-1}\exp[-\eta]}{b_2\exp[2\eta]+b_1\exp[\eta]+b_0+b_{-1}\exp[-\eta]} \tag{23a}$$

Proceeding as before, we obtain

$$\begin{array}{l}\{a_{-1}=a_{-1},\ a_0=a_{-1},\ a_1=a_2,\ a_2=a_2,\ \alpha=-\frac{4k^2}{(-a_2+a_{-1})^2},\quad b_2=b_0=1,\quad a_2\neq a_{-1},\ \beta=-\frac{4a_2^2k^2a_{-1}^2}{(-a_2+a_{-1})^2},\\ \omega=\frac{1}{8}\frac{k\left(4k^2a_{-1}^2-3\gamma^2a_2^2+6\gamma a_{-1}a_2-3a_{-1}^2\gamma+4k^2a_2^2+16k^2a_{-1}a_2\right)}{(-a_2+a_{-1})^2}\}\end{array} \tag{24a}$$

Hence we get the generalized solitary wave solutions of equation (6a) as follows

$$u(x,t)=\frac{a_2e^{2kx+2\omega t}+a_2e^{kx+\omega t}+a_{-1}+a_{-1}e^{-kx-\omega t}}{e^{2kx+2\omega t}+b_1e^{kx+\omega t}+1+b_{-1}e^{-kx-\omega t}} \tag{25a}$$

where

$$\omega=\frac{1}{8}\frac{k\left(4k^2a_{-1}^2-3\gamma^2a_2^2+6\gamma a_{-1}a_2-3a_{-1}^2\gamma+4k^2a_2^2+16k^2a_{-1}a_2\right)}{(-a_2+a_{-1})^2}$$

Special cases

For $\alpha=0$, equation (6a) can be written as

$$u_t+\frac{1}{4}u_{xxx}-\frac{3}{4}\frac{u_xu_{xx}}{u}+\frac{3}{8}\frac{u_x^3}{u^2}+\frac{3}{8}u_x\left(\frac{\beta}{u^2}+\gamma\right)=0 \tag{26a}$$

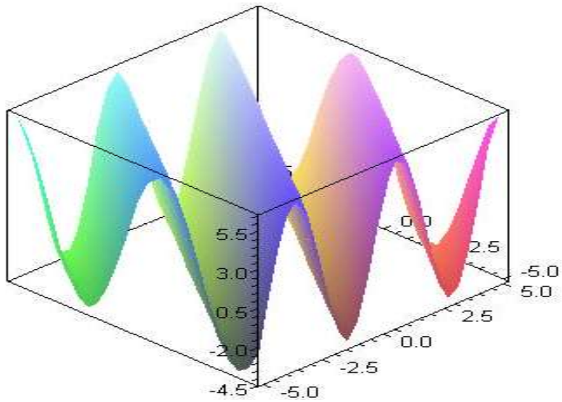

Fig. 3.6: Periodic solutions of equation (6a), when $a_1 = b_0 = K = \alpha = 1$ and $\beta = 2, \gamma = 3$

Introducing a transformation as $\eta = kx + \omega t$ we can covert equation (26a) into ordinary differential equations

$$8\omega u' + 2k^3 u''' - 6k^3 \frac{u'u''}{u} + 3k^3 \frac{u'^3}{u^2} + 3ku'\left(\beta \frac{1}{u^2} + \gamma\right) = 0 \tag{27a}$$

where the prime denotes the derivative with respect to η. The solution of the equation (27a) can be expressed in the form, as equation (6a)

$$u(\eta) = \frac{a_c \exp[c\eta] + \ldots + a_{-d} \exp[-dn]}{b_p \exp[p\eta] + \ldots + b_{-q} \exp[-qn]}$$

Proceeding as before, from equation (9a) and (10a), we have

$$p = c \tag{28a}$$

Similarly, from equation (12a) to (14a) we have

$$q = d \tag{29a}$$

Case 3.2.3: We can freely choose the values of c and d, but we will illustrate that the final solution does not strongly depend upon the choice of values of c and d. For simplicity, we set p = c = 1 and q = d = 1, then the trial solution, equation (6a) reduces to equation (17a)

$$u(\eta) = \frac{a_1 \exp[\eta] + a_0 + a_{-1} \exp[-\eta]}{b_1 \exp[\eta] + a_0 + b_{-1} \exp[-\eta]}$$

Substituting equation (17a) into equation (27a), we have

$$\begin{aligned} &\frac{1}{A}[c_5 \exp(5\eta) + c_4 \exp(4\eta) + c_3 \exp(3\eta) \\ &+ c_2 \exp(2\eta) + c_1 \exp(\eta) + c_0 + c_{-1} \exp(-\eta) \\ &+ c_{-2} \exp(-2\eta) + c_{-3} \exp(-3\eta) \\ &+ c_{-4} \exp(-4\eta) + c_{-5} \exp(-5\eta)] = 0 \end{aligned} \tag{30a}$$

where

$$A = 8\left(\exp(\eta) + b_0 + b_{-1}\exp(-\eta)\right)^4 \left(a_1 e^{\eta} + a_0 + a_{-1} e^{\eta}\right)^2$$

c_i ($i = -5, -4, \ldots, 4, 5$) are constants obtained by Maple 11. Equating the coefficients of exp ($n\eta$) to be zero, we obtain

$$\begin{aligned} &\{c_{-5} = 0,\ c_{-4} = 0,\ c_{-3} = 0,\ c_{-2} = 0, \\ &c_{-1} = 0, c_0 = 0, c_1 = 0, c_2 = 0, \\ &c_3 = 0, c_4 = 0,\ c_5 = 0\} \end{aligned} \tag{31a}$$

Solution of equations (31a) will yield

$$\begin{aligned} &\{a_{-1} = \frac{1}{4} a_1 b_0^2,\ b_0 = b_0,\ a_1 = a_1, \\ &\omega = -\frac{1}{8}\frac{k\left(2k^2 a_1^2 + 3\gamma a_1^2 + 3\beta\right)}{a_1^2}, \\ &a_0 = \frac{a_1 b_0 \left(\beta - k^2 a_1^2\right)}{\beta + k^2 a_1^2}, \quad b_{-1} = \frac{1}{4} b_0^2\} \end{aligned} \tag{32a}$$

We, therefore, obtained the following generalized soliton solution u (x, t) of equation (26a)

$$u(x,t) = a_1\left(1 - \frac{2k^2 a_1^2 b_0}{\left(k^2 a_1^2 + \beta\right)\left(e^{kx+\omega t} + b_0 + \frac{1}{4} b_0^2 e^{-kx-\omega t}\right)}\right) \tag{33a}$$

where

$$\omega = -\frac{1}{8}\frac{k\left(2k^2 a_1^2 + 3\gamma a_1^2 + 3\beta\right)}{a_1^2}$$

and a_1, b_0, β, γ and k are real numbers.

$$u(x,t) = a_1\left(1 - \frac{-2K^2 a_1^2 b_0}{\left(-K^2 a_1^2 + \beta\right)\left(e^{iKx+\omega t} + b_0 + \frac{1}{4} b_0^2 e^{-iKx-\omega t}\right)}\right) \tag{34a}$$

where

$$\omega = -\frac{1}{8}\frac{iK\left(-2K^2 a_1^2 + 3\gamma a_1^2 + 3\beta\right)}{a_1^2}$$

If we search for periodic solution or compact-like solution, the imaginary part in equation (34a) must be zero that requires, therefore equation (34a) becomes

$$u(x,t)=a_1\frac{\begin{pmatrix}\cos(Kx+\theta t)\left(-16K^2a_1^2b_0^2\cos(Kx+\theta t)+32\beta b_0+16\beta b_0^2\cos(Kx+\theta t)+8\beta b_0^3\right)\\ +24K^2a_1^2b_0^2-16K^2a_1^2-K^2a_1^2b_0^4+8\beta b_0^3+16\beta+\beta b_0^4\end{pmatrix}}{\left(-K^2a_1^2+\beta\right)\left(\cos(Kx+\theta t)\left(32b_0+16b_0^2\cos(Kx+\theta t)+8b_0^3\right)+16+b_0^4+8b_0^2\right)} \tag{35a}$$

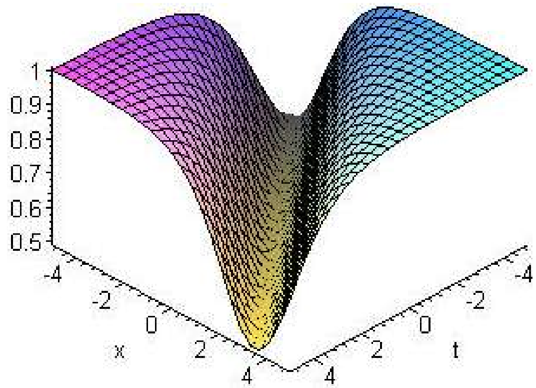

Fig. 3.7: Soliton solutions of equation (26a), when $a_1 = b_0 = k = \gamma = \beta = 1$. In case k is an imaginary number, the obtained soliton solution can be converted into periodic solution or compact-like solution. Therefore, we write $k = iK$, consequently, equation (33a) becomes

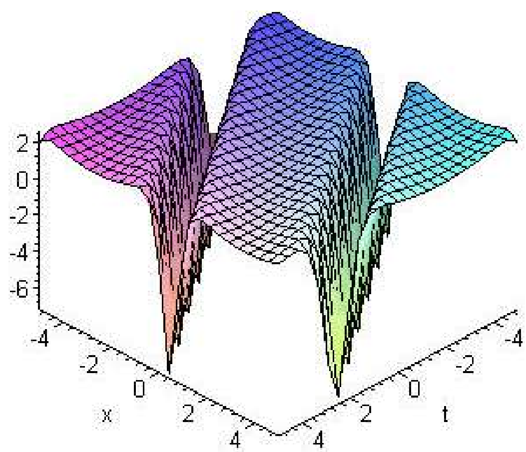

Fig. 3.8: Periodic solutions of equation (26a), when $a_1 = b_0 = K = 1$ and $\beta = 2, \gamma = 1$ $(\beta \neq 1)$

where

$$\theta=\frac{K\left(2K^2a_1^2-3\gamma a_1^2-3\beta\right)}{8a_1^2}$$

hence we get periodic solutions of equation (26a) as follows

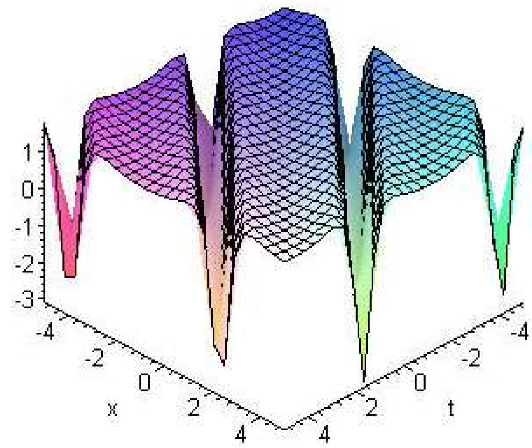

Fig. 3.9: Periodic solutions of equation (26a), when $a_1 = b_0 = K = 1$ and $\beta = 3, \gamma = 1$ $(\beta \neq 1)$

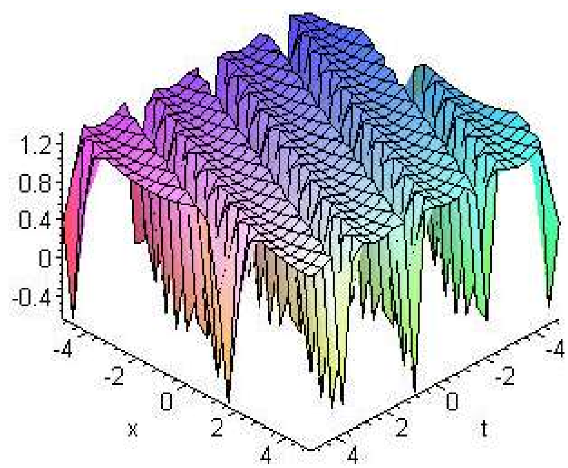

Fig. 3.10: Periodic solutions of equation (26a), when $a_1 = b_0 = K = 1$ and $\beta = 6, \gamma = 1$ $(\beta \neq 1)$. It is noticed that periodic solution is stable for long range of values of γ while wave form of the periodic solution changes with the values of β

Case 3.2.4: If $p = c = 2$ and $q = d = 1$, then equation (6a) reduces to equation (23a)

$$u(\eta)=\frac{a_2\exp[2\eta]+a_1\exp[\eta]+a_0+a_{-1}\exp[-\eta]}{b_2\exp[2\eta]+b_1\exp[\eta]+b_0+b_{-1}\exp[-\eta]}$$

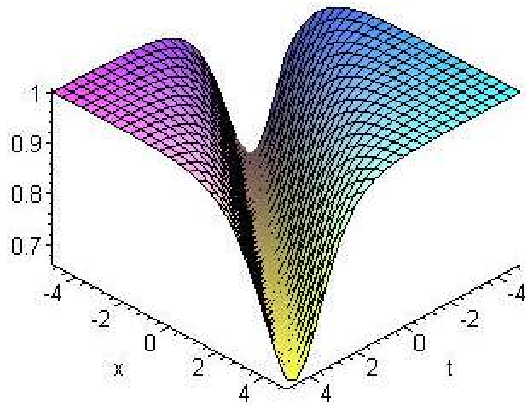

Fig. 3.11: Soliton solutions of equation (26a), when $a_2 = b_1 = K = 1$ and $\beta = \gamma = 21$

there are some free parameters in equation (24a), we set $b_2 = 1$, for simplicity, the trial-function (24a) is simplified as follows:

$$u(\eta)=\frac{a_2\exp[2\eta]+a_1\exp[\eta]+a_0+a_{-1}\exp[-\eta]}{\exp[2\eta]+b_1\exp[\eta]+b_0+b_{-1}\exp[-\eta]} \quad (36a)$$

Proceeding as before, we obtain

$$\begin{aligned}\{a_{-1}&=0,\ a_0=\frac{1}{4}a_2b_1^2,\\ a_1&=\frac{b_1a_2\left(\beta-k^2a_2^{\,2}\right)}{\beta+k^2a_2^{\,2}},\\ a_2&=a_2,\ b_1=0,\ b_{-1}=0,\ b_0=\frac{1}{4}b_1^{\,2},\\ \omega&=-\frac{1}{8}\frac{k\left(3\beta+2k^2a_2^{\,2}+3\gamma a_2^{\,2}\right)}{a_2^{\,2}}\}\end{aligned} \quad (37a)$$

hence we get the generalized solitary wave solution of equation (26a) as follows

$$u(x,t)=a_2\left(1-\frac{-2k^2a_2^{\,2}b_1}{\left(k^2a_2^{\,2}+\beta\right)\left(e^{2(kx+\omega t)}+b_1e^{(kx+\omega t)}+\frac{1}{4}b_1^{\,2}\right)}\right)$$

where

$$\omega=-\frac{1}{8}\frac{k\left(3\beta+2k^2a_2^{\,2}+3\gamma a_2^{\,2}\right)}{a_2^{\,2}}$$

and $a_2 = b_1$, β, γ and k are real numbers.

Example 3.3: [16] Consider the ZK-MEW equation (6b)

$$u_t+a(u)^3_{\ x}+\left(bu_{xt}+ru_{yy}\right)_x=0 \quad (6b)$$

Introducing a transformation as $\eta = kx+\omega y+\rho t$ we can covert equation (6b) into ordinary differential equations

$$\rho u'+3aku^2u'+\left(bk^2\rho+rk\omega^2\right)u'''=0 \quad (7b)$$

where the prime denotes the derivative with respect to η. The trial solution of the equation (7b) can be expressed as follows, as shown in equation (6b):

$$u(\eta)=\frac{a_c\exp[c\eta]+\ldots+a_{-d}\exp[-dn]}{b_p\exp[p\eta]+\ldots+b_{-q}\exp[-qn]}$$

To determine the value of c and p, we balance the linear term of highest order of equation (7b) with the highest order nonlinear term. Proceeding as before, we obtain

$$p=c \quad \text{and} \quad d=q.$$

Case 3.3.1: We can freely choose the values of p,c,d and but we will illustrate that the final solution does not strongly depend upon the choice of values of c and d. For simplicity, we set p = c = 1 and q = d = 1, then the trial solution yields:

$$u(\eta)=\frac{a_1\exp[\eta]+a_0+a_{-1}\exp[-\eta]}{b_1\exp[\eta]+a_0+b_{-1}\exp[-\eta]} \quad (8b)$$

Substituting equation (8b) into equation (7b), we have

$$\frac{1}{A}\begin{bmatrix}c_3\exp(3\eta)+c_2\exp(2\eta)+c_1\exp(\eta)+c_0\\ +c_{-1}\exp(-\eta)+c_{-2}\exp(-2\eta)+c_{-3}\exp(-3\eta)\end{bmatrix}=0 \quad (9b)$$

where

$$A=\left(b_1\exp(\eta)+b_0+b_{-1}\exp(-\eta)\right)^4$$

(38a)

c_i (i =-3,…,0,…,3) are constants obtained by Maple 11.

Equating the coefficients of exo ($n\eta$) to be zero, we obtain

$$\left\{\begin{matrix}c_{-3}=0,\ c_{-2}=0,\ c_{-1}=0,\ c_0=0,\\ c_1=0,\ c_2=0,\ c_3=0\end{matrix}\right\} \quad (10b)$$

Solution of (10b) will yield

$$b_1 = \frac{1}{8}\frac{a_0^2 a\left(bk^2+1\right)}{b_{-1}r\omega},\quad b_{-1}=b_{-1},\quad a_0=a_0$$
$$\rho = -\frac{kr\omega^2}{\left(bk^2+1\right)},\ a_{-1}=0,\quad a_1=0,\quad b_0=0 \tag{11b}$$

We, therefore, obtained the following generalized solitary solution u (x, y, t) of equation (6b) as follows

$$u(x,y,t) = \frac{a_0}{\frac{1}{8}\frac{a_0^2 a\left(1+bk^2\right)}{b_{-1}r\omega^2}e^{(kx+\omega y+\rho t)} + b_{-1}e^{-(kx+\omega y+\rho t)}} \tag{12b}$$

where $\rho = -\frac{kr\omega^2}{\left(bk^2+1\right)}$ a_0, b_{-1}, a, b, k, r and ω are real numbers.

$$u(x,y,t) = \frac{a_0}{-\frac{1}{8}\frac{a_0^2 a\left(1-bK^2\right)}{b_{-1}rW^2}e^{(iKx+iWy+\rho t)} + b_{-1}e^{-(iKx+iWy+\rho t)}}$$

The above expression can be rewritten as in expanded form

$$u(x,y,t) = \frac{-8a_0b_{-1}rW^2\left[\begin{array}{l}\cos\left(\frac{-K^3bx+Kx-WbK^2y+Wy+KrW^2t}{bK^2-1}\right)\left[-aba_0^2K^2+aa_0^2-8rW^2b_{-1}^2\right]\\ +i\sin\left(\frac{-K^3bx+Kx-WbK^2y+Wy+KrW^2t}{bK^2-1}\right)\left[-aba_0^2K^2+aa_0^2+8b_{-1}^2rW^2\right]\end{array}\right]}{\left[\begin{array}{l}32aba_0^2rb_{-1}W^2\cos\left(-K^3bx+Kx-WbK^2y+Wy+KrW^2t\right)^2\left[K^2-1\right]+a^2b^2a_0^4K^4\\ -2a^2ba_0^4K^2-16aba_0^2b_{-1}^2rK^2W^2+a^2a_0^4+16aa_0^2b_{-1}^2rW^2+64b_{-1}^4r^2W^4\end{array}\right]} \tag{14b}$$

If we search for periodic solution or compact-like solution, the imaginary part in equation (14b) must be zero, hence

$$u(x,y,t) = \frac{-8a_0b_{-1}rW^2\left[\cos\left(-K^3bx+Kx-WbK^2y+Wy+KrW^2t\right)\left[-aba_0^2K^2+aa_0^2-8rW^2b_{-1}^2\right]\right]}{\left[\begin{array}{l}32aba_0^2rb_{-1}W^2\cos\left(-K^3bx+Kx-WbK^2y+Wy+KrW^2t\right)^2\left[K^2-1\right]+a^2b^2a_0^4K^4\\ -2a^2ba_0^4K^2-16aba_0^2b_{-1}^2rK^2W^2+a^2a_0^4+16aa_0^2b_{-1}^2rW^2+64b_{-1}^4r^2W^4\end{array}\right]} \tag{15b}$$

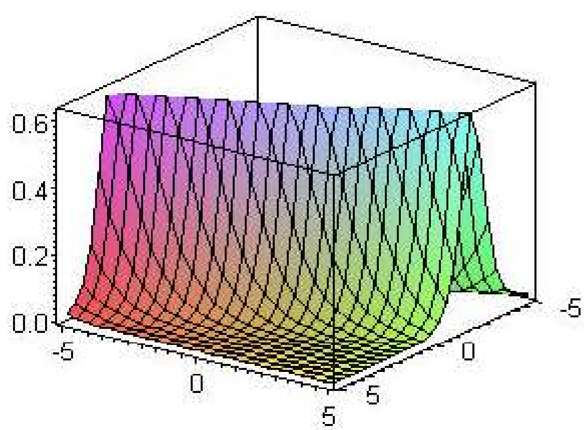

Fig. 3.12: Depicts the soliton solutions of equation (6b), when $a_0 = b_{-1} = a = b = r = k = 1$. In case k and ω are imaginary numbers, the obtained solitons solutions can be converted into periodic solutions or compact-like solutions. Therefore, we write k = iK and ω = iW consequently, equation (12b) becomes

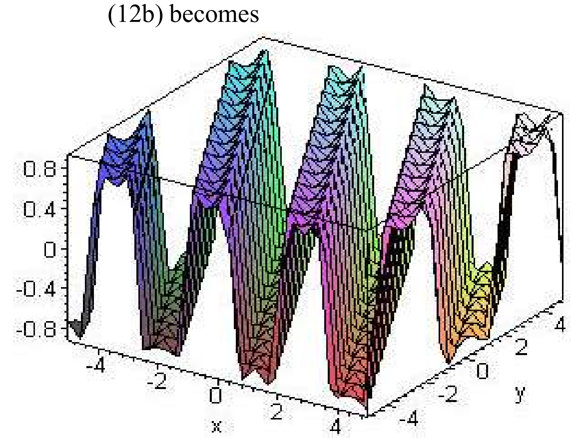

Fig. 3.13: Depicts the periodic solutions of equation (6b) when $a = b = a_0 = b_{-1}$ $k = W = r = t = 1$

Case 3.3.2: If $p = c = 2$ and $q = d = 1$, then the trial solution, equation (6b) reduces to

$$u(\eta)=\frac{a_2\exp[2\eta]+a_1\exp[\eta]+a_0+a_{-1}\exp[-\eta]}{b_2\exp[2\eta]+b_1\exp[\eta]+b_0+b_{-1}\exp[-\eta]} \qquad (16b)$$

Proceeding as before, we obtain

$$\begin{aligned}&b_2=b_2,\ \ b_1=0,\ \ b_0=0,\ \ a_1=0,\ \ \omega=\omega,\\ &b_{-1}=b_{-1},\ \ \rho=\frac{9kr\omega^2}{(9bk^2-2)},\ a_0=0,\ \ a_2=a_2,\\ &b_0=0,\ \ a_{-1}=-\frac{b_{-1}a_2}{b_2},\qquad \alpha=\frac{9{b_2}^2r\omega^2}{(9bk^2-2){a_2}^2}\end{aligned} \qquad (17b)$$

Hence, we get the generalized solitary solutions u (x,,y, t) of equation (6b) as follows

$$u(x,y,t)=-1+\frac{2b_{-1}}{b_1e^{2\left(k\,x+\,\omega y\,-\frac{9kr\omega^2t}{9bk^2-2}\right)}+b_{-1}} \qquad (18b)$$

where b_{-1}, b_1, ω and k are real numbers.

Example 3.4: Consider the following "good" Boussinesq equation

$$u_{tt}=-u_{xxxx}+u_{xx}+\left(u^2\right)_{xx} \qquad (6c)$$

Introducing a transformation as $\eta = kx+\omega t$ we can covert equation (6c) into ordinary differential equations

$$\omega^2u''+k^4u^{(iv)}-k^2u''-k^2\left(u^2\right)''=0 \qquad (7c)$$

The solution of the equation (7c) can be expressed in the form, equation (6c) as

$$u(\eta)=\frac{a_c\exp[c\eta]+...+a_{-d}\exp[-dn]}{b_p\exp[p\eta]+...+b_{-q}\exp[-qn]}$$

To determine the value of c and p, we balance the linear term of highest order of equation (7c) with the highest order nonlinear term. Proceeding as before, we obtain

$$p=c \quad \text{and} \quad d=q$$

Case 3.4.1: We can freely choose the values of p,c,d but we will illustrate that the final solution does not strongly depends upon the choice of values of c and d. For simplicity, we set $p = c = 1$ and $q = d = 1$, hence equation (6c) reduces to the following form:

$$u(\eta)=\frac{a_1\exp[\eta]+a_0+a_{-1}\exp[-\eta]}{b_1\exp[\eta]+a_0+b_{-1}\exp[-\eta]}$$

Substituting equation (7c), we have

$$\begin{aligned}&\frac{1}{A}[c_4\exp(4\eta)+c_3\exp(3\eta)+c_2\exp(2\eta)\\ &+c_1\exp(\eta)+c_0+c_{-1}\exp(-\eta)+c_{-2}\exp(-2\eta)\\ &+c_{-3}\exp(-3\eta)+c_{-4}\exp(-4\eta)]=0\end{aligned} \qquad (8c)$$

where

$$A=\left(b_1\exp(\eta)+b_0+b_{-1}\exp(-\eta)\right)^5$$

c_i (i =-4,-3,…,3,4) are constants obtained by Maple 11.

Equating the coefficients of exp $(n\eta)$ to be zero, we obtain:

$$\begin{aligned}&\{c_{-4}=0,\ c_{-3}=0,\ c_{-2}=0,\ c_{-1}=0,\\ &c_0=0,c_1=0,c_2=0,\ c_3=0,c_4=0\}\end{aligned} \qquad (9c)$$

Solution of (9c) will yield

$$\begin{aligned}&b_1=\frac{1}{4}\frac{{b_0}^2}{b_{-1}},\quad b_{-1}=b_{-1},\quad a_0=-\frac{1}{2}\frac{b_0\left(5k^4+k^2-\omega^2\right)}{k^2},\ \omega=\omega\\ &a_{-1}=\frac{1}{2}\frac{b_{-1}\left(\omega^2+k^4-k^2\right)}{k^2},\quad a_1=\frac{1}{8}\frac{{b_0}^2\left(\omega^2+k^4-k^2\right)}{k^2b_{-1}},\quad b_0=b_0\end{aligned} \qquad (10c)$$

We, therefore, obtained the following generalized solitary solution u (x, t) of equation (6c):

$$u(x,t)=\frac{\frac{1}{8}\frac{b_0^2\left(\omega^2-k^2+k^4\right)}{k^2b_{-1}}e^{(2kx+2\omega t)}-\frac{1}{2}\frac{b_0\left(5k^4+k^2-\omega^2\right)}{k^2}e^{(kx+\omega t)}+\frac{1}{2}\frac{b_{-1}\left(\omega^2-k^2+k^4\right)}{k^2}}{\frac{1}{4}\frac{b_0^2e^{(2kx+2\omega t)}}{b_{-1}}+b_0e^{(kx+\omega t)}+b_{-1}}$$

or simply, we have

$$u(x,t)=\frac{\left(\omega^2+k^4-k^2\right)}{2k^2}-\frac{3b_0k^2}{\frac{1}{4}\frac{b_0^2}{b_{-1}}e^{(2kx+2\omega t)}+b_0e^{(kx+\omega t)}+b_{-1}} \tag{11c}$$

where b_0, b_{-1}, ω and k are real numbers.

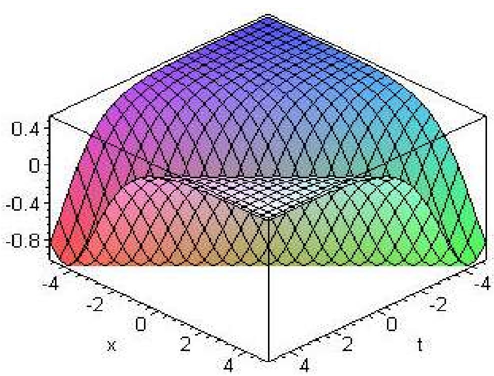

Fig. 3.14: Depicts the soliton solutions of equation (6c), when $a_0 = a_{-1} = \omega = k = 1$. In case k is an imaginary number, the obtained soliton solutions can be converted into periodic or compact-like solutions. Therefore, we write $k = iK$, consequently, equation (11c) becomes

$$u(x,t)=\frac{\left(\omega^2+K^4+K^2\right)}{-2K^2}+\frac{3b_0K^2}{\frac{1}{4}\frac{b_0^2}{b_{-1}}e^{(IKx+\omega t)}+b_0+b_{-1}e^{(-IKx-\omega t)}} \tag{12c}$$

$$u(x,t)=\frac{\left(\omega^2+K^4+K^2\right)}{-2K^2}+\frac{3b_0K^2}{\frac{1}{4}\frac{b_0^2}{b_{-1}}e^{(\omega t)}\left(\cos(Kx)+I\sin(Kx)\right)+b_0+b_{-1}e^{(-\omega t)}\left(\cos(Kx)+I\sin(Kx)\right)} \tag{13c}$$

Now, to obtain periodic solutions from soliton solutions, we put imaginary part of equation (13c) which is sin kx equal to zero, hence equation (13c) becomes

$$u(x,t)=\frac{\left(\omega^2+K^4+K^2\right)}{-2K^2}+\frac{3b_0K^2}{\frac{1}{4}\frac{b_0^2}{b_{-1}}e^{(\omega t)}\cos(Kx)+b_0+b_{-1}e^{(-\omega t)}\cos(Kx)} \tag{14c}$$

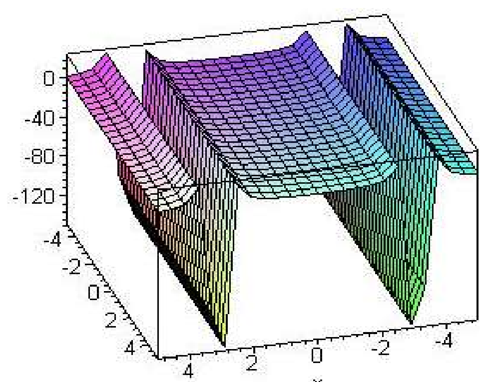

Fig. 3.15: Depicts the periodic solutions of equation (6c) drawn on the whole domain when $b_{-1} = b_0 = k = \omega = 1$

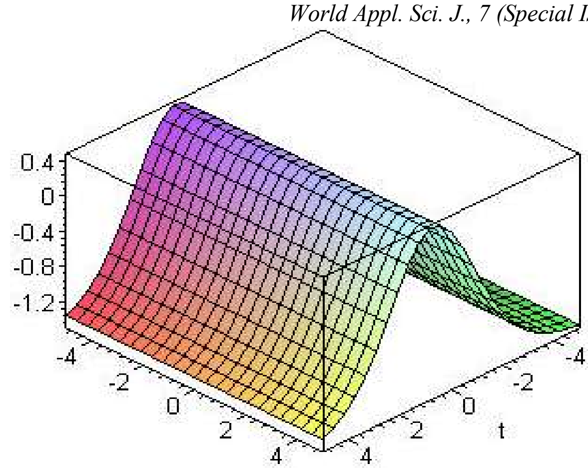

Fig. 3.16: Depicts the periodic solutions of equation (6c) drawn on the whole domain when $b_{-1} = b_0 = k = \omega = 1$

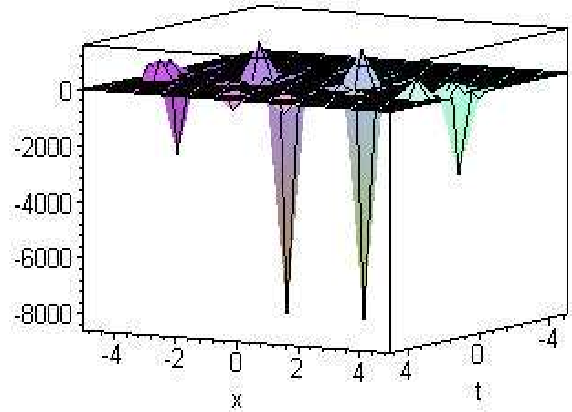

Fig. 3.17: Depicts the periodic solutions of equation (6c) when $b_{-1} = b_0 = \omega = 1$ and $K = 2$

Case 3.4.2: If p = c = 2 and q = d = 1, than equation (6c) reduces to

$$u(\eta)=\frac{a_2\exp[2\eta]+a_1\exp[\eta]+a_0+a_{-1}\exp[-\eta]}{b_2\exp[2\eta]+b_1\exp[\eta]+b_0+b_{-1}\exp[-\eta]} \tag{15c}$$

Proceeding as before, we obtain

$$\begin{aligned} &b_2=\frac{1}{4}\frac{b_1^2}{b_0},\quad b_1=b_1,\quad b_0=b_0,\; a_1=-\frac{1}{2}\frac{b_1\left(5k^4+k^2-\omega^2\right)}{k^2},\quad \omega=\omega,\, b_{-1}=0\\ &a_0=\frac{1}{2}\frac{b_0\left(\omega^2+k^4-k^2\right)}{k^2},\; a_2=\frac{1}{8}\frac{b_1^2\left(\omega^2+k^4-k^2\right)}{k^2 b_0},\quad b_0=b_0,\quad a_{-1}=0 \end{aligned} \tag{16c}$$

Hence, we get the generalized solitary solution u (x, t) of equation (6c) as follows

$$u(x,t)=\frac{\frac{1}{8}\frac{b_1^2\left(\omega^2-k^2+k^4\right)}{k^2 b_0}e^{(2kx+2\omega t)}-\frac{1}{2}\frac{b_1\left(5k^4+k^2-\omega^2\right)}{k^2}e^{(kx+\omega t)}+\frac{1}{2}\frac{b_0\left(\omega^2-k^2+k^4\right)}{k^2}}{\frac{1}{4}\frac{b_1^2 e^{(2kx+2\omega t)}}{b_0}+b_1e^{(kx+\omega t)}+b_0}$$

or simply, we have

$$u(x,t)=\frac{\left(\omega^2+k^4-k^2\right)}{2k^2}-\frac{3b_1k^2}{\frac{1}{4}\frac{b_1^2}{b_0}e^{(2kx+2\omega t)}+b_1e^{(kx+\omega t)}+b_0} \tag{17c}$$

where b_0, b_1, ω and k are real numbers.

Example 3.5: Consider the general form of the Lax equation (6d)

$$u_t+\frac{3}{10}\alpha^2u^2u_x+2\alpha u_xu_{2x}+\alpha uu_{3x}+u_{5x}=0 \tag{6d}$$

where α is an arbitrary constant. Introducing a transformation as $\eta = kx+\omega t$ we can covert equation (6d) into ordinary differential equations

$$\begin{aligned} &10\omega u'+3k\alpha^2u^2u'+20\alpha k^3u'u''\\ &\quad+10\alpha k^3u\,u'''+10k^5u^{(5)}=0 \end{aligned} \tag{7d}$$

The solution of the equation (7d) can be expressed in the following form

$$u(\eta) = \frac{a_c \exp[c\eta] + \ldots + a_{-d} \exp[-d\eta]}{b_p \exp[p\eta] + \ldots + b_{-q} \exp[-q\eta]}$$

To determine the value of c and p, we balance the linear term of highest order of equation (7d) with the highest order nonlinear term

$$u^{(5)} = \frac{c_1 \exp[(31p + c)\eta] + \cdots}{c_2 \exp[32p\eta] + \cdots} \quad (8d)$$

and

$$uu''' = \frac{c_3 \exp[(7p + 2c)\eta] + \cdots}{c_4 \exp[9p\eta] + \cdots} = \frac{c_3 \exp[(30p + 2c)\eta] + \cdots}{c_4 \exp + [32p\eta] \cdots} \quad (9d)$$

where c_i are determined coefficients only for simplicity; balancing the highest order of exp-function in (8d) and (9d), we have

$$31p + c = 30p + 2c \quad (10d)$$

which in turn gives

$$p = c \quad (11d)$$

To determine the value of d and q, we balance the linear term of lowest order of equation (6d) with the lowest order non-linear term

$$u^{(5)} = \frac{\cdots + d_1 \exp[(-d - 31q)\eta]}{\cdots + d_2 \exp[-32q\eta]} \quad (12d)$$

and

$$uu''' = \frac{\cdots + d_3 \exp[(-7q - 2d)\eta]}{\cdots + d_4 \exp[-9q\eta]} = \frac{\cdots + d_3 \exp[(-2d - 30q)\eta]}{\cdots + d_4 \exp[-32q\eta]} \quad (13d)$$

where d_i are determined coefficients only for simplicity. Now, balancing the lowest order of exp-function in (12d) and (13d), we have

$$-31q - d = -30q - 2d \quad (14d)$$

which in turn gives

$$q = d \quad (15d)$$

Case 3.5.1: We can freely choose the values of c and d, but we will illustrate that the final solution does not strongly depend upon the choice of values of c and d. For simplicity, we set p = c = 1 and q = d = 1, then the trial solution, equation (6d) reduces to

$$u(\eta) = \frac{a_1 \exp[\eta] + a_0 + a_{-1} \exp[-\eta]}{b_1 \exp[\eta] + a_0 + b_{-1} \exp[-\eta]} \quad (16d)$$

Substituting equation (16d) into equation (7d), we have

$$\begin{aligned} &\frac{1}{A}[c_5 \exp(5\eta) + c_4 \exp(4\eta) + c_3 \exp(3\eta) \\ &+ c_2 \exp(2\eta) + c_1 \exp(\eta) + c_0 + c_{-1} \exp(-\eta) \\ &+ c_{-2} \exp(-2\eta) + c_{-3} \exp(-3\eta) \\ &+ c_{-4} \exp(-4\eta) + c_{-5} \exp(-5\eta)] = 0 \end{aligned} \quad (17d)$$

where

$$A = (b_1 \exp(\eta) + b_0 + b_{-1} \exp(-\eta))^6$$

c_i (i =-5,-4,…,4,5) are constants obtained by using Maple 11. Equating the coefficients of exp (nη) to be zero, we obtain

$$\begin{aligned} &\{c_{-5} = 0,\ c_{-4} = 0,\ c_{-3} = 0,\ c_{-2} = 0, \\ &c_{-1} = 0, c_0 = 0, c_1 = 0, c_2 = 0, \\ &c_3 = 0, c_4 = 0,\ c_5 = 0\} \end{aligned} \quad (18d)$$

Solution of (18d) will yield

$$\begin{aligned} &a_{-1} = \frac{1}{4}\frac{a_1 b_0^2}{b_1^2},\ b_0 = b_0,\ b_1 = b_1,\ b_{-1} = \frac{1}{4}\frac{b_0^2}{b_1},\ a_1 = a_1 \\ &\omega = -\frac{1}{10}\frac{k(10k^4 b_1^2 + 10k^2\alpha a_1 b_1 + 3\alpha^2 a_1^2)}{b_1^2} \\ &a_0 = \frac{b_0(\alpha a_1 + 10k^2 b_1)}{\alpha b_1} \end{aligned} \quad (19d)$$

We, therefore, obtained the following generalized solitary solution u (x, t) of equation (6d)

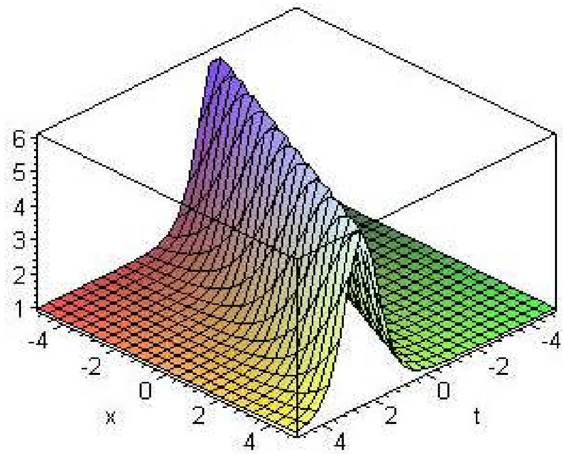

Fig. 3.18: Soliton solutions of equation (6d), when $a_1 = b_0 = b_1 = k = \alpha = 1$. In case k is an imaginary number, the obtained soliton solutions can be converted into periodic solutions or compact-like solutions. Therefore, we write $k = iK$, consequently, equation (20d) becomes

$$u(x,t) = \frac{a_1}{b_1} + \frac{1}{\alpha}\left(\frac{10k^2 b_0}{b_1 e^{(kx+\omega t)} + b_0 + \frac{b_0^2}{4b_1} e^{(-kx-\omega t)}}\right) \tag{20d}$$

where

$$\omega = -\frac{1}{10}\frac{k\left(10k^4 b_1^2 + 10k^2\alpha a_1 b_1 + 3\alpha^2 a_1^2\right)}{b_1^2}$$

and a_1, b_1, α and k are real numbers.

$$u(x,t) = \frac{a_1}{b_1} - \frac{1}{\alpha}\left(\frac{10K^2 b_0}{b_1 e^{(iKx+\omega t)} + b_0 + \frac{b_0^2}{4b_1} e^{(-iKx-\omega t)}}\right) \tag{21d}$$

where

$$\omega = -\frac{1}{10}\frac{iK\left(10K^4 b_1^2 - 10K^2\alpha a_1 b_1 + 3\alpha^2 a_1^2\right)}{b_1^2}$$

and a_1, b_1, α and K are real numbers. If we search for periodic solutions or compact-like solutions than the imaginary part of equation (21d) must be zero, consequently

$$u(x,t) = \frac{\left(\begin{array}{l} 8\alpha a_1 b_0^2 b_1^2 + 16\alpha a_1 b_1^4 + \alpha a_1 b_0^4 - 160K^2 b_0^2 b_1^3 \\ +\cos(-Kx+\lambda t)\left[32\alpha a_1 b_1^3 b_0 + 16\alpha a_1 b_1^2 b_0^2 \cos(-Kx+\lambda t) + 8\alpha a_1 b_1 b_0^3 - 160K^2 b_0 b_1^4 - 40K^2 b_0^3 b_1^2\right] \end{array}\right)}{\alpha b_1\left[\cos(-Kx+\lambda t)\left[32 b_1^3 b_0 + 16 b_1^2 b_0^2 \cos(-Kx+\lambda t) + 8 b_1 b_0^3\right] + 8 b_1^2 b_0^2 + 16 b_1^4 + b_0^4\right]} \tag{22d}$$

where $\lambda = \frac{1}{10}\frac{K\left(10K^4 b_1^2 - 10K^2\alpha a_1 b_1 + 3\alpha^2 a_1^2\right)}{b_1^2}$ which is the periodic solution of equation (6d)

Case 3.5.2: If $p = c = 1$ and $q = d = 2$, then equation (66) reduces to

$$u(\eta) = \frac{a_2 \exp[2\eta] + a_1 \exp[\eta] + a_0 + a_{-1}\exp[-\eta] + a_{-2}\exp[-2\eta]}{b_2 \exp[2\eta] + b_1 \exp[\eta] + b_0 + b_{-1}\exp[-\eta] + b_{-2}\exp[-2\eta]} \tag{23d}$$

and contains some free parameters, we set $b_1 = b_{-1} = 0$, for simplicity, the trial-function (23d) is simplified as follows:

$$u(\eta) = \frac{a_2 \exp[2\eta] + a_1 \exp[\eta] + a_0 + a_{-1}\exp[-\eta] + a_{-2}\exp[-2\eta]}{b_2 \exp[2\eta] + b_0 + b_{-2}\exp[-2\eta]}$$

Proceeding as before, we obtain

$$\begin{aligned} &a_{-1} = 0,\quad a_2 = a_2,\quad b_2 = b_2,\quad b_0 = b_0,\quad a_{-2} = \frac{1}{4}\frac{a_2 b_0^2}{b_2^2},\quad a_1 = 0 \\ &b_{-2} = \frac{1}{4}\frac{b_0^2}{b_2},\quad \omega = -\frac{1}{10}\frac{k\left(40k^2\alpha a_2 b_2 + 3\alpha^2 a_2^2 + 160k^4 b_2^2\right)}{b_2^2},\quad a_0 = \frac{b_0\left(\alpha a_2 + 40k^2 b_2\right)}{\alpha b_2} \end{aligned} \tag{24d}$$

World Applied Sciences Journal 7 (Special Issue for Applied Math): 116-147, 2009
ISSN 1818-4952

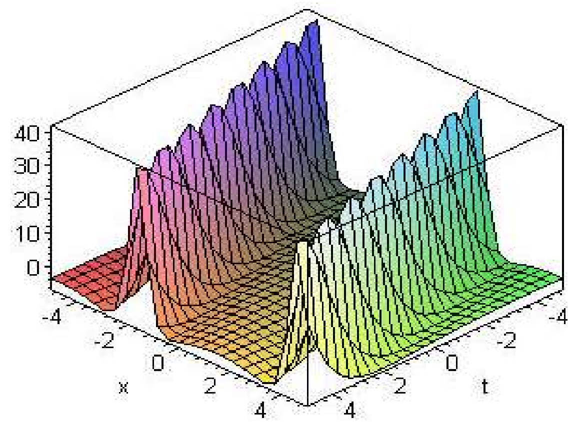

Fig. 3.19: Periodic solutions of equation (1), when $a_1 = b_0 = b_1 = k = \alpha = 1$

Hence we get the generalized solitary wave solution of equation (6d) as follows

$$u(x,t) = \frac{a_2}{b_2} + \frac{1}{\alpha}\left(\frac{60K^2 b_0}{b_2 e^{2(kx+\omega t)} + b_0 + \frac{b_0^{\,2}}{4b_2} e^{2(-kx-\omega t)}} \right) \qquad (25d)$$

where

$$\omega = -\frac{1}{10}\frac{k\left(40k^2\alpha a_2 b_2 + 3\alpha^2 a_2^{\,2} + 160k^4 b_2^{\,2}\right)}{b_2^{\,2}}$$

and a_2, b_2, α and k are real numbers.

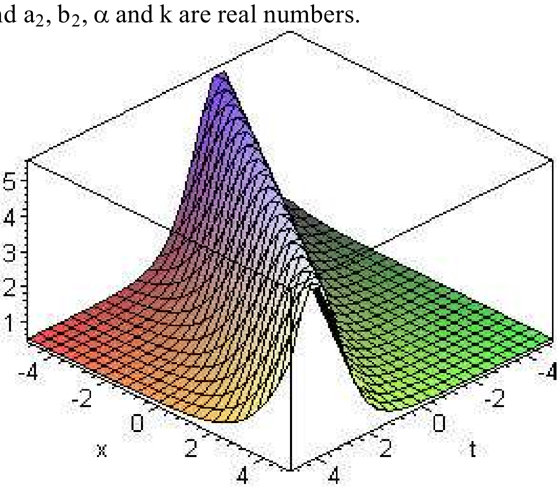

Fig. 3.20: Soliton solutions of equation (16d) when $a_2 = b_0 = b_2 = \alpha = 1$ and $k = 1/2$

Example 3.6: [1] Consider the following Master partial differential equation (6e)

$$u_{xxxx} + u\left(u_t + cu_x\right) + u_x u_{xx} = 0 \qquad (6e)$$

Introducing a transformation as $\eta = kx + \omega t$, we can covert equation (6e) into ordinary differential equations

$$k^3 u'''' + u\,u'(\omega + ck) + k^3 u' u'' = 0 \qquad (7e)$$

The solutions of the equation (7e) can be expressed in the form, equation (6e)

$$u(\eta) = \frac{a_c \exp[c\eta] + \ldots + a_{-d}\exp[-dn]}{b_p \exp[p\eta] + \ldots + b_{-q}\exp[-qn]}$$

To determine the value of c and p, we balance the linear term of highest order of equation (7e) with the highest order nonlinear term

$$u''' = \frac{c_1 \exp\left[(7p + c)\eta\right] + \cdots}{c_2 \exp\left[8p\eta\right] + \cdots} \qquad (8e)$$

and

$$uu' = \frac{c_3 \exp\left[(4p + 2c)\eta\right] + \cdots}{c_4 \exp\left[6p\eta\right] + \cdots} = \frac{c_3 \exp\left[(6p + 2c)\eta\right] + \cdots}{c_4 \exp + \left[8p\eta\right] \cdots} \qquad (9e)$$

where c_i are determined coefficients only for simplicity; balancing the highest order of Exp-function in (8e) and (9e), we have

$$6p + 2c = 7p + c \qquad (10e)$$

which in turn gives

$$p = c \qquad (11e)$$

To determine the value of d and q, we balance the linear term of lowest order of equation (6e) with the lowest order nonlinear term

$$u''' = \frac{\cdots + d_1 \exp\left[(-d - 7q)\eta\right]}{\cdots + d_2 \exp\left[-8q\eta\right]} \qquad (12e)$$

and

$$u'u'' = \frac{\cdots + d_3 \exp\left[(-4q - 2d)\eta\right]}{\cdots + d_4 \exp\left[-6q\eta\right]} = \frac{\cdots + d_3 \exp\left[(-2d - 6q)\eta\right]}{\cdots + d_4 \exp\left[-8q\eta\right]} \qquad (13e)$$

Corresponding Author: Dr. Syed Tauseef Mohyud-Din, HITEC University, Taxila Cantt., Pakistan

where d_i are determined coefficients only for simplicity. Now, balancing the lowest order of exp-function in (12e) and (13e), we have

$$-7q - d = -6q - 2d \tag{14e}$$

which in turn gives

$$q = d \tag{15e}$$

Case 3.6.1: We can freely choose the values of c and d, but we will illustrate that the final solutions do not strongly depend upon the choice of values of c and d. For simplicity, we set p = c = 1 and q = d = 1, then the trial solutions, equation (6e) reduces to

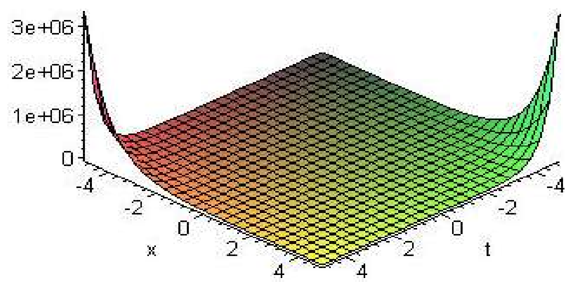

Fig. 3.21: Soliton solutions of equation (6e), when $a_1 = b_0 = a_{-1} = k = c = 1$

$$u(\eta) = \frac{a_1 \exp[\eta] + a_0 + a_{-1}\exp[-\eta]}{b_1 \exp[\eta] + a_0 + b_{-1}\exp[-\eta]} \tag{16e}$$

Substituting equation (16e) into equation (7e), we have

$$\begin{aligned}&\frac{1}{A}[c_4 \exp(4\eta) + c_3 \exp(3\eta) + c_2 \exp(2\eta)\\ &+ c_1 \exp(\eta) + c_0 + c_{-1}\exp(-\eta) + c_{-2}\exp(-2\eta)\\ &+ c_{-3}\exp(-3\eta) + c_{-4}\exp(-4\eta)] = 0\end{aligned} \tag{17e}$$

where

$$A = \left(b_1 \exp(\eta) + b_0 + b_{-1}\exp(-\eta)\right)^5$$

c_i (i =-4,-3,…,3,4) are constants obtained by Maple 11.

Equating the coefficients of exp (nη) to be zero, we obtain

$$\begin{aligned}\{&c_{-4} = 0,\ c_{-3} = 0,\ c_{-2} = 0,\ c_{-1} = 0,\\ &c_0 = 0, c_1 = 0, c_2 = 0,\ c_3 = 0, c_4 = 0\}\end{aligned} \tag{18e}$$

Solutions of (18e) will yield

$$\begin{aligned}\{&b_1 = 0,\ b_{-1} = 0,\ a_0 = b_0,\ a_{-1} = a_{-1},\\ &a_1 = a_1,\ \ \omega = -k^3 - ck,\ \ b_0 = b_0\}\end{aligned} \tag{19e}$$

We, therefore, obtained the following generalized solitary solutions u (x, t) of equation (6e)

$$u(x,t) = \frac{a_1 e^{\left(kx + \left(-k^3 - ck\right)t\right)} + b_0 + a_{-1} e^{\left(-kx - \left(-k^3 - ck\right)t\right)}}{b_0} \tag{20e}$$

where a_1, b_0, a_{-1}, c and k are real numbers.

In case k is an imaginary number, the obtained soliton solutions can be converted into periodic solutions or compact-like solutions. Therefore, we write k = iK, consequently, equation (20e) becomes

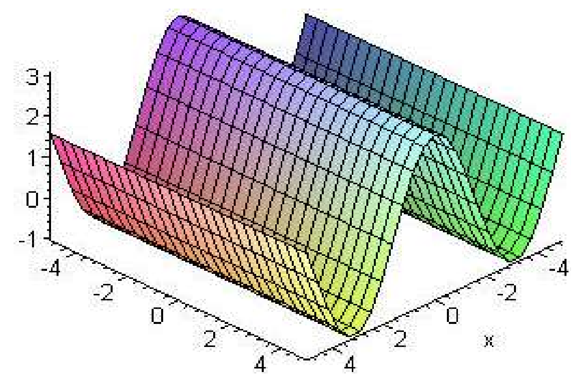

Fig. 3.22: Periodic solutions of equation (1), when $a_0 = b_{-1} = b_0 = K = c = 1$

$$u(x,t) = \frac{a_1 e^{\left(iKx + i\left(K^3 - cK\right)t\right)} + b_0 + a_{-1} e^{\left(-iKx + \left(K^3 - cK\right)t\right)}}{b_0} \tag{21e}$$

If we search for periodic solutions or compact-like solutions, the imaginary part in equation (21e) must be zero that requires, therefore equation (21e) becomes

$$u(x,t) = 1 + \frac{(a_1 + a_{-1})}{b_0}\cos\left(-Kx - tK^3 + tcK\right) \tag{22e}$$

which is periodic solutions of equation (6e)

Case 3.6.2: If p = c = 2 and q = d = 1, then equation (6e) reduces to

$$u(\eta) = \frac{a_2 \exp[2\eta] + a_1 \exp[\eta] + a_0 + a_{-1}\exp[-\eta]}{b_2 \exp[2\eta] + b_1 \exp[\eta] + b_0 + b_{-1}\exp[-\eta]} \tag{23e}$$

Proceeding as before, we obtain

$$\{a_{-1} = a_{-1},\ a_0 = a_0,\ a_1 = a_1,\ a_2 = 0, b_1 = 0,\ b_{-1} = 0\\ b_0 = a_0,\quad b_2 = 0,\quad \omega = -k^3 - ck\}$$

Hence we get the generalized solitary wave solutions of equation (6e) as follows

$$u(x,t) = \frac{a_1 e^{\left(kx+\left(k^3-ck\right)t\right)} + b_0 + a_{-1}e^{\left(-kx-\left(-k^3-ck\right)t\right)}}{b_0}$$

Case 3.6.3: If p = c = 2 and q = d = 2, then equation (6e) reduces to

$$u(\eta) = \frac{a_2\exp[2\eta] + a_1\exp[\eta] + a_0 + a_{-1}\exp[-\eta] + a_{-2}\exp[-2\eta]}{b_2\exp[2\eta] + b_1\exp[\eta] + b_0 + b_{-1}\exp[-\eta] + b_{-2}\exp[-2\eta]} \quad (25e)$$

In equation (25e), there are some parameters, we set $[b_{-1} = b_1 = 0]$ for simplicity and the trial function is simplified as follows

$$u(\eta) = \frac{a_2\exp[2\eta] + a_1\exp[\eta] + a_0 + a_{-1}\exp[-\eta] + a_{-2}\exp[-2\eta]}{b_2\exp[2\eta] + b_0 + b_{-2}\exp[-2\eta]} \quad (26e)$$

Proceeding as before, we obtain

$$\{a_{-2} = 0,\ a_{-1} = a_{-1},\ a_0 = b_0,\ a_1 = a_1,\ a_2 = 0,\\ b_{-2} = 0,\ b_0 = b_0,\ b_2 = 0,\quad \omega = -k^3 - ck\} \quad (27e)$$

Hence we get the generalized solitary solutions of equation (6e) as follows

$$u(x,t) = \frac{a_1 e^{\left(kx+\left(-k^3-ck\right)t\right)} + b_0 + a_{-1}e^{\left(-kx-\left(-k^3-ck\right)t\right)}}{b_0}$$

In all three cases, for different choices of c, p, d and q we get the same soliton solutions which clearly illustrate that final solutions do not strongly depends upon these parameters.

Example 3.7: Consider the following Burgers equation:

$$u_t + \alpha u u_x - u_{xx} = 0 \quad (6f)$$

Introducing a transformation as $\eta = kx+\omega t$, we can covert equation (6f) into ordinary differential equations

$$\omega u' + \alpha k u u' - k^2 u'' = 0 \quad (7f)$$

where the prime denotes the derivative with respect to η.. The solution of the equation (7f) can be expressed as follows

$$u(\eta) = \frac{a_c\exp[c\eta] + \ldots + a_{-d}\exp[-dn]}{b_p\exp[p\eta] + \ldots + b_{-q}\exp[-qn]} \quad (24e)$$

To determine the value of c and p, we balance the linear term of highest order of equation (7f) with the highest order nonlinear term

$$u'' = \frac{c_1\exp[(3p+c)\eta] + \cdots}{c_2\exp[4p\eta] + \cdots} \quad (8f)$$

and

$$uu' = \frac{c_3\exp[(p+2c)\eta] + \cdots}{c_4\exp[3p\eta] + \cdots} = \frac{c_3\exp[(2p+2c)\eta] + \cdots}{c_4\exp + [4p\eta]\cdots} \quad (9f)$$

where c_i are determined coefficients only for simplicity; balancing the highest order of exp-function in (8f) and (9f), we have

$$3p + c = 2p + 2c \quad (10f)$$

which in turn gives

$$p = c \quad (11f)$$

To determine the value of d and q, we balance the linear term of lowest order of equation (7f) with the lowest order nonlinear term

$$u'' = \frac{\cdots + d_1\exp[(-d-3q)\eta]}{\cdots + d_2\exp[-4q\eta]} \quad (12f)$$

and

$$u'u = \frac{\cdots + d_3\exp[(-q-2d)\eta]}{\cdots + d_4\exp[-3q\eta]} = \frac{\cdots + d_3\exp[(-2d-2q)\eta]}{\cdots + d_4\exp[-4q\eta]} \quad (13f)$$

where d_i are determined coefficients only for simplicity. Now, balancing the lowest order of exp-function in (12f) and (13f), we have

$$-3q - d = -2q - 2d \quad (14f)$$

which in turn gives

$$q = d \quad (15f)$$

Case 3.7.1: We can freely choose the values of c and d, but we will illustrate that the final solution does not strongly depend upon the choice of values of c and d. For simplicity, we set p = c = 1 and q = d = 1, then the trial solution, equation (6f) reduces to

$$u(\eta)=\frac{a_1\exp[\eta]+a_0+a_{-1}\exp[-\eta]}{b_1\exp[\eta]+a_0+b_{-1}\exp[-\eta]} \tag{16f}$$

Substituting equation (16f) into equation (7f), we have

$$\frac{1}{A}\left[\begin{array}{c} C_2\exp(2\eta)+C_1\exp(\eta)+C_0 \\ +C_{-1}\exp(-\eta)+C_{-2}\exp(-2\eta)\end{array}\right]=0 \tag{17f}$$

where

$$A=\left(b_1\exp(\eta)+b_0+b_{-1}\exp(-\eta)\right)^3$$

C_i (i =-2,-1, 0, 1, 2) are constants obtained by Maple 11. Equating the coefficients of exp ($n\eta$) to be zero, we obtain

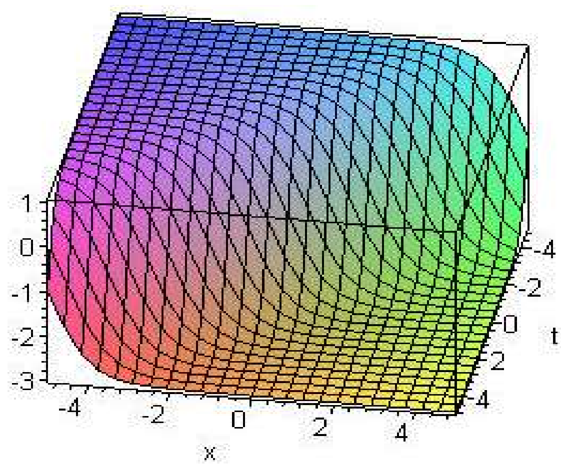

Fig. 3.23: Depicts soliton solutions of equation (6f), when $a_{-1} = b_{-1} = b_1 = k = \alpha = 1$. In case k is imaginary numbers, the obtained soliton solution can be converted into periodic or compact-like solutions. Therefore, we write k = iK. Consequently, equation (20f) becomes

$$\{C_{-2}=0,\ C_{-1}=0,\ C_0=0,\ C_1=0,\ C_2=0\} \tag{18f}$$

Solution of (18f) will yield

$$a_{-1}=a_{-1},\ b_0=0,\ b_1=b_1,\ b_{-1}=b_{-1},$$
$$a_1=\frac{b_1(\alpha a_{-1}-4kb_{-1})}{\alpha b_{-1}},\ \omega=\frac{k(\alpha a_{-1}-2kb_{-1})}{b_{-1}},\ a_0=0 \tag{19f}$$

We, therefore, obtained the following generalized solitary solution u (x, t) of equation (6f)

$$u(x,t)=\frac{\dfrac{b_1(\alpha a_{-1}-4kb_{-1})e^{(kx+\omega t)}}{\beta b_{-1}}+a_{-1}e^{(-kx-\omega t)}}{b_1e^{(kx+\omega t)}+b_{-1}e^{(-kx-\omega t)}} \tag{20f}$$

where

$$\omega=\frac{k(\alpha a_{-1}-2kb_{-1})}{b_{-1}}$$

and a_{-1}, b_{-1}, b_1, k and α are real numbers.

$$u(x,t)=\frac{\dfrac{b_1(\alpha a_{-1}-4iKb_{-1})e^{(iKx+\omega t)}}{\beta b_{-1}}+a_{-1}e^{(-iKx-\omega t)}}{b_1e^{(iKx+\omega t)}+b_{-1}e^{(-iKx-\omega t)}} \tag{22f}$$

where

$$\omega=\frac{iK(\alpha a_{-1}-2iKb_{-1})}{b_{-1}}$$

and a_{-1}, b_{-1}, b_1, K and α are real numbers. Equivalently above expression can be rewritten as in expanded form

$$u(x,t)=\frac{\left[\begin{array}{l} 4b_1a_{-1}\cos\left(\frac{K(-xb_{-1}+t\alpha a_{-1})}{b_{-1}}\right)^2\alpha b_{-1}+b_1^2a_{-1}\alpha e^{(-4K^2t)}-2\alpha a_{-1}b_1b_{-1}+a_{-1}\alpha b_{-1}^2e^{(4K^2t)}-8Kb_{-1}^2\cos\left(\frac{K(-xb_{-1}+t\alpha a_{-1})}{b_{-1}}\right) \\ \sin\left(\frac{K(-xb_{-1}+t\alpha a_{-1})}{b_{-1}}\right)-i\left(8b_1Kb_{-1}^2\cos\left(\frac{K(-xb_{-1}+t\alpha a_{-1})}{b_{-1}}\right)^2-4b_1^2b_{-1}Ke^{(-4K^2t)}+4Kb_1b_{-1}^2\right)\end{array}\right]}{\left[b_{-1}\alpha\left(4b_1b_{-1}\cos\left(\frac{K(-xb_{-1}+t\alpha a_{-1})}{b_{-1}}\right)^2+\right)+b_1^2e^{(-4K^2t)}-2b_1b_{-1}+b_{-1}^2e^{(4K^2t)}\right]} \tag{22f}$$

If we search for periodic or compact-like solutions, the imaginary part in equation (22f) must be zero, therefore equation (22f) becomes

$$u(x,t)=\frac{\left[\begin{array}{l}4b_1a_{-1}\cos\left(\frac{K(-xb_{-1}+t\alpha a_{-1})}{b_{-1}}\right)^2\alpha b_{-1}-8Kb_{-1}^2\cos\left(\frac{K(-xb_{-1}+t\alpha a_{-1})}{b_{-1}}\right)\sin\left(\frac{K(-xb_{-1}+t\alpha a_{-1})}{b_{-1}}\right)\\+b_1^2a_{-1}\alpha e^{(-4K^2t)}-2\alpha a_{-1}b_1b_{-1}+a_{-1}\alpha b_{-1}^2e^{(4K^2t)}\end{array}\right]}{\left[b_{-1}\alpha\left(4b_1b_{-1}\cos\left(\frac{K(-xb_{-1}+t\alpha a_{-1})}{b_{-1}}\right)^2+\right)+b_1^2e^{(-4K^2t)}-2b_1b_{-1}+b_{-1}^2e^{(4K^2t)}\right]} \quad (23f)$$

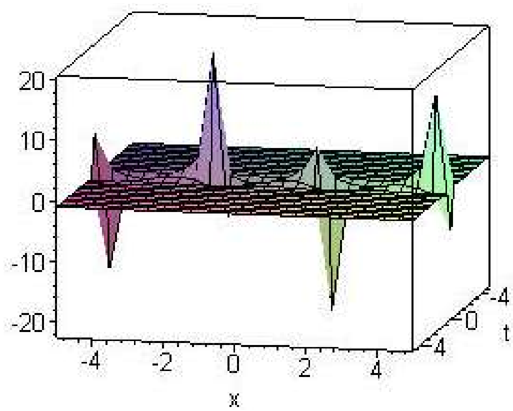

Fig. 3.24: Depicts periodic solutions of equation (6f), when $a_{-1} = b_{-1} = b_1 = \alpha = 1$

which is the periodic solutions of equation (6f).

Case 3.7.2: If p = c = 2 and q = d = 1, then equation (6f) reduces to

$$u(\eta)=\frac{a_2\exp[2\eta]+a_1\exp[\eta]+a_0+a_{-1}\exp[-\eta]}{b_2\exp[2\eta]++b_1\exp[\eta]+b_0+b_{-1}\exp[-\eta]} \quad (24f)$$

Proceeding as before, we obtain

$$\begin{array}{lll} a_{-1}=-\dfrac{b_{-1}(\alpha a_1+4kb_1)}{\alpha b_1}, & a_2=0, \quad b_2=0, & \\ b_0=0, \quad a_{-2}=0, & a_1=a_1\, b_{-2}=b_{-2}, & (25f) \\ \omega=-\dfrac{k(\alpha a_1+2kb_1)}{b_1}, & a_0=0, b_{-1}=b_{-1}, \; b_1=b_1 & \end{array}$$

Hence we get the generalized solitary wave solution u (x, t) of equation (6f) as follows

$$u(x,t)=\frac{a_1e^{(kx+\omega t)}+\dfrac{b_{-1}(\alpha a_1+4kb_1)e^{-(kx+\omega t)}}{\alpha b_1}}{b_1e^{(kx+\omega t)}+b_{-1}e^{-(kx+\omega t)}} \quad (26f)$$

where

$$\omega=-\frac{k(\alpha a_1+2kb_1)}{b_1}$$

and a_1, b_1, b_{-1}, α and k are real numbers.

Case 3.7.3: Consider the Newell-Whitehead equation as follows

$$u_t-u_{xx}=\beta u(1-u)(u+1) \quad (27f)$$

Introducing a transformation as $\eta = kx+\omega t$, we can covert equation (27f) into ordinary differential equations

$$\omega u'-k^2u''=\beta u(1-u^2) \quad (28f)$$

where the prime denotes the derivative with respect to η. The trial solution of the equation (28f) can be expressed as follows, as shown in equation (6f):

$$u(\eta)=\frac{a_c\exp[c\eta]+\ldots+a_{-d}\exp[-dn]}{b_p\exp[p\eta]+\ldots+b_{-q}\exp[-qn]}$$

To determine the value of c and p, we balance the linear term of highest order of equation (28f) with the highest order non-linear term. Proceeding as before, we obtain

$$p=c \text{ and } d=q$$

Case 3.7.4: We can freely choose the values of p,c,d and but we will illustrate that the final solution does not strongly depend upon the choice of values of c and d. For simplicity, we set p = c = 1 and q = d = 1, then the trial solution, equation (7f) reduces to equation (17f) as follows:

$$u(\eta)=\frac{a_1\exp[\eta]+a_0+a_{-1}\exp[-\eta]}{b_1\exp[\eta]+a_0+b_{-1}\exp[-\eta]}$$

Substituting equation (17f) into equation (28f), we have

$$\frac{1}{A_1}\left[\begin{array}{l}C_3\exp(3\eta)+C_2\exp(2\eta)\\+C_1\exp(\eta)+C_0+C_{-1}\exp(-\eta)\\+C_{-2}\exp(-2\eta)+C_{-3}\exp(-3\eta)\end{array}\right]=0 \quad (29f)$$

Where C_i (i =-3,…,0,…,3) are constants obtained by Maple 11. Equating the coefficients of exp ($n\eta$) to be zero, we obtain

$$\{C_{-3}=0,\ C_{-2}=0,\ C_{-1}=0,\ C_0=0,\ C_1=0,\ C_2=0,\ C_3=0\} \qquad (30f)$$

Solution of (30) will yield

$$b_1=b_1,\ b_{-1}=0,\ a_0=a_0,\ \omega=-k^2, a_{-1}=-\frac{b_0\left(a_1b_0-b_1a_0\right)}{b_1^2}b_{-1}\gamma,\ a_1=a_1,\ b_0=b_0 \qquad (31f)$$

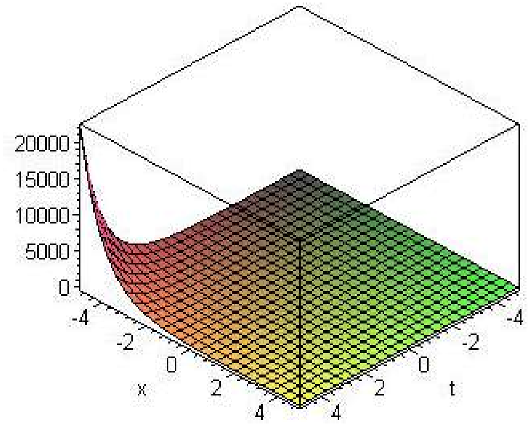

Fig. 3.25: Depicts the soliton solutions of equation (27f), when $a_0 = a_1 = b_1 = k = 1$ and $b_0 = 2$. In case k is imaginary numbers, the obtained solitons solution can be converted into periodic solution or compact-like solution. Therefore, we write k = iK, equation (31f) becomes

We, therefore, obtained the following generalized solitary solution u (x, t) of equation (27f) as follows

$$u(x,t)=\frac{a_1e^{(kx-k^2t)}+a_0-\frac{b_0\left(a_1b_0-b_1a_0\right)}{b_1^2}e^{\left(-kx+k^2t\right)}}{b_1e^{\left(kx-k^2t\right)}+b_0} \qquad (32f)$$

where a_1, b_0, a_0, b_1 and k are real numbers but at the same time at least one of them should have different value.

$$u(x,t)=\frac{a_1e^{(iKx+K^2t)}+a_0-\frac{b_0\left(a_1b_0-b_1a_0\right)}{b_1^2}e^{\left(-iKx-K^2t\right)}}{b_1e^{\left(iKx+K^2t\right)}+b_0} \qquad (32f)$$

Equivalently above expression can be rewritten as in expanded form

$$u(x,t)=\frac{\left[\begin{array}{l}\cos(Kx)e^{K^2t}\left[a_1b_0b_1^2+a_0b_1^3\right]+\cos(Kx)e^{-K^2t}\left[a_0b_1b_0^2-a_1b_0^3\right]+a_1b_1^3e^{2K^2t}\\+\cos(Kx)^2\left[2a_0b_0b_1^2-2a_1b_1b_0^2\right]+a_1b_1b_0^2+i\sin(Kx)e^{K^2t}\left[a_1b_0b_1^2-a_0b_1^3\right]\\+i\sin(Kx)e^{-K^2t}\left[-a_0b_1b_0^2+a_1b_0^3\right]+i\sin(Kx)\cos(Kx)\left[2a_1b_1b_0^2-2a_0b_0b_1^2\right]\end{array}\right]}{b_1^2\left[2b_1b_0\cos(Kx)e^{K^2t}+b_0^2+b_1^2e^{2K^2t}\right]} \qquad (32f)$$

If we search for periodic or compact-like solutions, the imaginary part in equation (33f) must be zero, therefore equation (33f) becomes

$$u(x,t)=\frac{\begin{bmatrix}\cos(Kx)e^{K^2t}\left[a_1b_0b_1^2+a_0b_1^3\right]+\cos(Kx)e^{-K^2t}\left[a_0b_1b_0^2-a_1b_0^3\right]+a_1b_1^3e^{2K^2t}\\+\cos(Kx)^2\left[2a_0b_0b_1^2-2a_1b_1b_0^2\right]+a_1b_1b_0^2\end{bmatrix}}{b_1^2\left[2b_1b_0\cos(Kx)e^{K^2t}+b_0^2+b_1^2e^{2K^2t}\right]} \quad (34f)$$

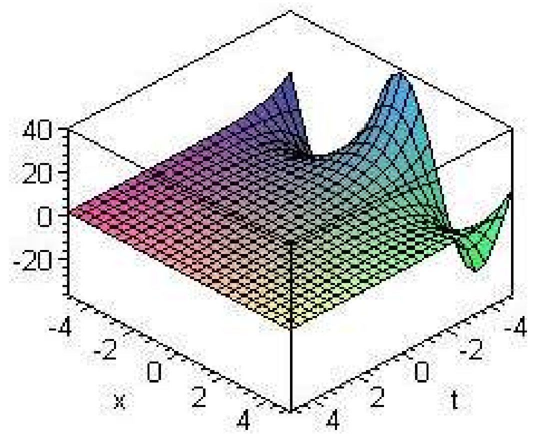

Fig. 3.26: Depicts the periodic solution of equation (27f) when $b_0 = a_1 = b_1 = K = 1$ and $a_0 = 2$.

Case 3.7.5: If p = c =2 and q = d = 1, then the trial solution, equation (6f) reduces to

$$u(\eta)=\frac{a_2\exp[2\eta]+a_1\exp[\eta]+a_0+a_{-1}\exp[-\eta]}{b_2\exp[2\eta]+b_1\exp[\eta]+b_0+b_{-1}\exp[-\eta]} \quad (35f)$$

Proceeding as before, we obtain

$$\begin{aligned}&b_2=0,\, b_1=b_1,\quad b_0=b_0,\; a_1=-\frac{b_1(a_{-1}b_1-a_0b_0)}{b_0^2}\\&b_{-1}=0,\;\omega=-k^2,\; a_0=a_0,\; a_2=0,\quad a_{-1}=a_{-1}\end{aligned} \quad (36f)$$

Hence, we get the generalized solitary solution u (x, t) of equation (27f) as follows

$$u(x,t)=\frac{-\dfrac{b_1(a_{-1}b_1-a_0b_0)e^{(kx-k^2t)}}{b_0^2}+a_0+a_{-1}e^{(-kx+k^2t)}}{b_1e^{(kx-k^2t)}+b_0} \quad (37f)$$

where b_0, b_1, a_{-1}, a_0 and k are real numbers and for some particulars values of these numbers we have the same form as equation (31f).

Example 3.8: Consider the following reaction diffusion equation

$$y''(x)+y^n(x)=0,\qquad 0<x<L \quad (6g)$$

with boundary conditions

$$y(0)=y(L)=0$$

Equation (6g) can be re-written in the following equivalent form:

$$y'''(x)+n\,y^{n-1}(x)y'(x)=0 \quad (7g)$$

with boundary conditions

$$y(0)=0,\quad y'(0)=\alpha,\quad y(L)=0$$

where α is an arbitrary constant.

Case 3.8.1: p = c = 1 and q = d = 1. Equation (7g) reduces to

$$y(x)=\frac{a_1\exp[x]+a_0+a_{-1}\exp[-x]}{b_1\exp[x]+b_0+b_{-1}\exp[-x]} \quad (8g)$$

Substituting (8g) into (7g) with n = 2, we have

$$\begin{aligned}&\frac{1}{A}[C_3\exp(3x)+C_2\exp(2x)+C_1\exp(x)\\&+C_0+C_{-1}\exp(-x)+C_{-2}\exp(-2x)\\&+C_{-3}\exp(-3x)]=0\end{aligned} \quad (9g)$$

where

$$A=\left(b_1\exp(x)+b_0+b_{-1}\exp(-x)\right)^4$$

C_i (i =-3,-2,…, 2, 3) are constants obtained by Maple 11.
Equating the coefficients of exp (nx) to zero, we obtain

$$\begin{aligned}&\{C_{-3}=0,\; C_{-2}=0,\, C_{-1}=0,\\&C_0=0,\; C_1=0,\; C_2=0,\; C_3=0\}\end{aligned} \quad (10g)$$

Solving the system, we obtain the following solutions

$$\begin{aligned}&\{a_{-1}=-\frac{1}{2}b_{-1},\; b_1=\frac{1}{4}\frac{b_0^2}{b_{-1}},\; a_0=\frac{5}{2}b_0,\\&b_0=b_0,\qquad b_{-1}=b_{-1},\qquad a_1=-\frac{1}{8}\frac{b_0^2}{b_{-1}}\}\end{aligned} \quad (11g)$$

The soliton solutions of equation (7g) are given as:

$$y(x) = \frac{-\frac{1}{8}\frac{b_0^2}{b_{-1}}e^{x} + \frac{5}{2}b_0 - \frac{1}{2}b_{-1}e^{-x}}{\frac{1}{4}\frac{b_0^2}{b_{-1}}e^{x} + b_0 + b_{-1}e^{-x}} \quad (12g)$$

Figure 3.27 and 3.28 depict the solitons s of equation (7g9)

Case 3.8.2: $p = c = 2$ and $q = d = 2$.

The trial function of equation (7g) becomes

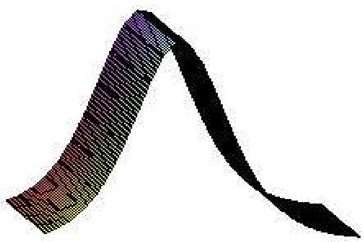

Fig. 3.27

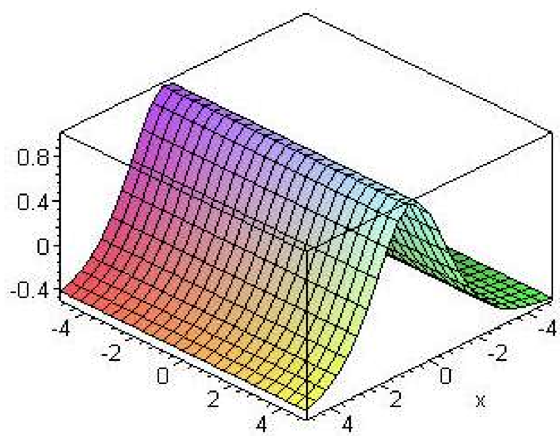

Fig. 3.28

$$y(x) = \frac{a_2 \exp[2x] + a_1 \exp[x] + a_0 + a_{-1}\exp[-x]}{b_2 \exp[2x] + b_1 \exp[x] + b_0 + b_{-1}\exp[-x]} \quad (13g)$$

Proceeding as before, we obtain

$$\{a_{-1} = 0,\ b_1 = b_1,\quad a_1 = \frac{5}{2}b_1,\quad b_0 = \frac{1}{4}\frac{b_1^2}{b_2},$$
$$b_{-1} = 0,\ a_0 = -\frac{1}{8}\frac{b_1^2}{b_2}, b_2 = b_2,\quad a_2 = -\frac{1}{2}b_2\} \quad (14g)$$

Consequently

$$y(x) = \frac{-\frac{1}{2}b_2e^{2x} + \frac{5}{2}b_1e^{x} - \frac{1}{8}\frac{b_1^2}{b_2}e^{x}}{b_2e^{2x} + b_1e^{x} + \frac{1}{4}\frac{b_1^2}{b_2}}$$

Figure 3.29 and 3.30 depict the soliton solutions of equation (7g)

Example 3.9: Consider the following Kuramoto-Sivashinsky equation (6h)

$$u_t = -uu_x - u_{xx} - u_{xxxx} \quad (6h)$$

with periodic conditions

Fig. 3.29

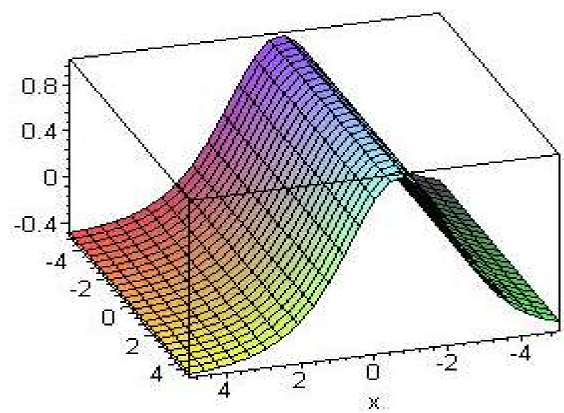

Fig. 3.30

$$u(x, t = 0) = \cos(\frac{x}{16})(1 + \sin(\frac{x}{16}))$$

with the simulation running to $t = 30$. Introducing a transformation as

$\eta = kx+\omega t$, we can covert equation (6h) into ordinary differential equations

$$\omega u' + kuu' + k^2u'' + k^4u^{(iv)} = 0 \quad (7h)$$

The solution of the equation (7h) can be expressed in the form

$$u(\eta)=\frac{a_c\exp[c\eta]+\ldots+a_{-d}\exp[-d\eta]}{b_p\exp[p\eta]+\ldots+b_{-q}\exp[-q\eta]}$$

To determine the value of c and p, we balance the linear term of highest order of equation (22h) with the highest order non-linear term

$$u^{(vi)}=\frac{c_1\exp[(15p+c)\eta]+\cdots}{c_2\exp[16p\eta]+\cdots} \tag{8h}$$

and

$$uu'=\frac{c_3\exp[(p+2c)\eta]+\cdots}{c_4\exp[3p\eta]+\cdots}=\frac{c_3\exp[(14p+2c)\eta]+\cdots}{c_4\exp+[16p\eta]\cdots} \tag{9h}$$

where c_i are determined coefficients only for simplicity; balancing the highest order of exp-function in (8h) and (9h), we have

$$14p+2c = 15p+c \tag{10h}$$

which in turn gives

$$p = c \tag{11h}$$

To determine the value of d and q, we balance the linear term of lowest order of equation (7h) with the lowest order non-linear term

$$u^{(vi)}=\frac{\cdots+d_1\exp[(-d-15q)\eta]}{\cdots+d_2\exp[-16q\eta]} \tag{12h}$$

and

$$uu'=\frac{\cdots+d_3\exp[(-2q-d)\eta]}{\cdots+d_4\exp[-3q\eta]}=\frac{\cdots+d_3\exp[(-2d-14q)\eta]}{\cdots+d_4\exp[-16q\eta]} \tag{13h}$$

where d_i are determined coefficients only for simplicity. Now, balancing the lowest order of exp-function in (12h) and (13h), we have

$$-15q-d=-14q-2d \tag{14h}$$

which in turn gives

$$q = d \tag{15h}$$

Case 3.9.1: We can freely choose the values of c and d, but we will illustrate that the final solution does not strongly depend upon the choice of values of c and d. For simplicity, we set p = c = 1 and q = d = 1, we get

$$u(\eta)=\frac{a_1\exp[\eta]+a_0+a_{-1}\exp[-\eta]}{b_1\exp[\eta]+a_0+b_{-1}\exp[-\eta]} \tag{16h}$$

Substituting equation (16h) into equation (7h), we have

$$\begin{aligned}&\frac{1}{A}[c_4\exp(4\eta)+c_3\exp(3\eta)+c_2\exp(2\eta)\\&+c_1\exp(\eta)+c_0+c_{-1}\exp(-\eta)+c_{-2}\exp(-2\eta)\\&+c_{-3}\exp(-3\eta)+c_{-4}\exp(-4\eta)]=0\end{aligned} \tag{17h}$$

where

$$A=(b_1\exp(\eta)+b_0+b_{-1}\exp(-\eta))^5$$

c_i (i =-4,-3,..., 3, 4) are constants obtained by Maple 11. Equating the coefficients of exp (nη) to be zero, we obtain

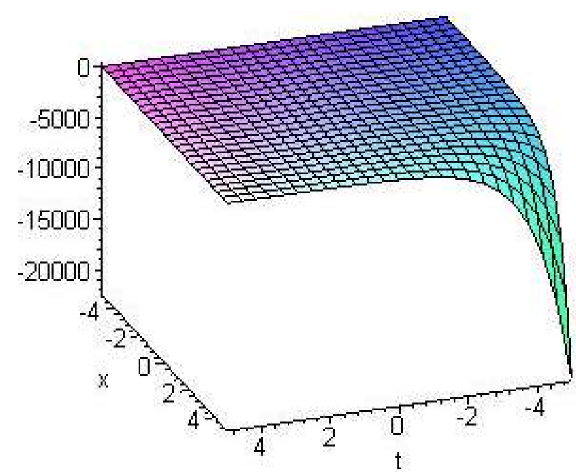

Fig. 3.31: Soliton solution equation (6h), when a_0, $a_{-1} = k = 1$

$$\begin{aligned}\{&c_{-4}=0,\ c_{-3}=0,\ c_{-2}=0,\\&c_{-1}=0,\ c_0=0,\ c_1=0,\\&c_2=0,\ c_3=0,\ c_4=0\}\end{aligned} \tag{18h}$$

Solution of (18h) will yield

$$\begin{aligned}&b_1=0,\quad b_{-1}=-\frac{a_{-1}^2}{a_0^2-ka_{-1}+a_{-1}k^3},\quad a_0=a_0\\&a_{-1}=a_{-1},\ a_1=0,\quad \omega=-\frac{(-a_0^2+2ka_{-1})k}{a_{-1}},\ b_0=0\end{aligned} \tag{19h}$$

We, therefore, obtained the following generalized solitary solutions u (x, t) of equation (6h)

$$u(\ ) = -\frac{\left(a_0 + a_{-1} e^{\left(-kx + \frac{(-a_0^2 + 2ka_{-1})kt}{a_{-1}}\right)}\right)\left(a_0^2 - ka_{-1} + a_{-1}k^3\right)}{a_{-1}^2 e^{\left(-kx + \frac{(-a_0^2 + 2ka_{-1})kt}{a_{-1}}\right)}} \quad (20h)$$

$$u(x,t) = -\frac{\left(a_0 + a_{-1} e^{\left(-iKx + \frac{(-a_0^2 + 2iKa_{-1})iKt}{a_{-1}}\right)}\right)\left(a_0^2 - iKa_{-1} - a_{-1}iK^3\right)}{a_{-1}^2 e^{\left(-iKx + \frac{(-a_0^2 + 2iKa_{-1})iKt}{a_{-1}}\right)}} \quad (21h)$$

where a_0, a_{-1} and k are real numbers.

In case k is an imaginary number, the obtained soliton solution can be converted into periodic solution or compact-like solution. Therefore, we write $k = iK$, consequently, equation (20h) becomes

$$u(x,t) = \frac{\left(\begin{array}{l}\left[\cos\left(\frac{K(xa_{-1} + ta_0^2)}{a_{-1}}\right)a_0^3 - a_0^2 a_{-1} e^{(-2K^2 t)} - a_{-1}a_0 \sin\left(\frac{K(xa_{-1} + ta_0^2)}{a_{-1}}\right)K(1+K^2)\right] \\ +i\left[a_{-1}a_0\cos\left(\frac{K(xa_{-1} + ta_0^2)}{a_{-1}}\right)K(1+K^2) - \sin\left(\frac{K(xa_{-1} + ta_0^2)}{a_{-1}}\right)a_0^3 + 2a_{-1}^2 K e^{(-2K^2 t)}(1+K^2)\right]\end{array}\right) e^{(2K^2 t)}}{a_{-1}^2} \quad (22h)$$

If we search for periodic solution or compact-like solution, the imaginary part in equation (22h) must be zero that requires, therefore equation (22h) becomes

$$u(x,t) = \frac{\left[\cos\left(\frac{K(xa_{-1} + ta_0^2)}{a_{-1}}\right)a_0^3 - a_0^2 a_{-1} e^{(-2K^2 t)} - a_{-1}a_0 \sin\left(\frac{K(xa_{-1} + ta_0^2)}{a_{-1}}\right)K(1+K^2)\right] e^{(2K^2 t)}}{a_{-1}^2} \quad (23h)$$

If we take $a_{-1} = a_0^2$ then equation (24h) turns to be the following periodic solution of equation (6h)

$$u(x,t) = -\frac{\left[\cos(K(x+t)) + a_0 e^{(-2K^2 t)} + K(1+K^2)\sin(K(x+t))\right]}{a_0} e^{(2K^2 t)} \quad (24h)$$

Case 3.9.2: If p = c = 2 and q = d = 2 then

$$u(\eta) = \frac{a_2 \exp[2\eta] + a_1 \exp[\eta] + a_0 + a_{-1}\exp[-\eta] + a_{-2}\exp[-2\eta]}{b_2 \exp[2\eta] + b_1 \exp[\eta] + b_0 + b_{-1}\exp[-\eta] + b_{-2}\exp[-2\eta]} \quad (25h)$$

In equation (25h), there are some parameters, we set $[b_{-1} = b_1 = 0]$ for simplicity and the trial function is simplified as follows

$$u(\eta) = \frac{a_2 \exp[2\eta] + a_1 \exp[\eta] + a_0 + a_{-1}\exp[-\eta] + a_{-2}\exp[-2\eta]}{b_2 \exp[2\eta] + b_0 + b_{-2}\exp[-2\eta]} \quad (26h)$$

Proceeding as before, we obtain

$$\{a_{-2} = a_{-2},\quad a_{-1} = 0,\quad a_0 = a_0,\quad a_1 = 0,\quad a_2 = a_2,$$
$$b_{-2} = b_{-2},\quad b_0 = \frac{b_{-2}a_0}{a_{-2}},\quad b_2 = 0,\quad \omega = -\frac{k\left(a_{-2} + 8k^3 b_{-2} + 2kb_{-2}\right)}{b_{-2}}\} \quad (27h)$$

World Applied Sciences Journal 7 (Special Issue for Applied Math): 116-147, 2009
ISSN 1818-4952

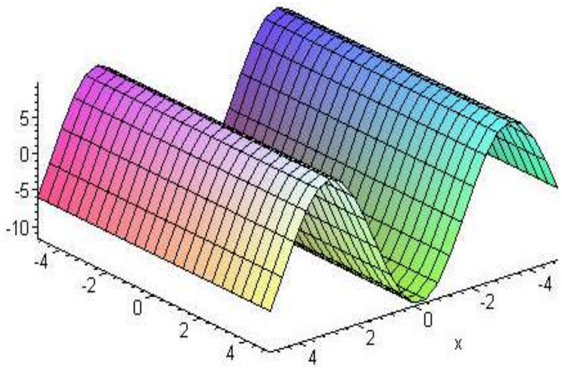

Fig. 3.32: Periodic solution equation (6h), when $a_0 = K = 1$

Hence we get the generalized solitary wave solution of equation (6h) as follows

$$u(\eta) = \frac{\left(a_2 e^{(2\eta)} + a_0 + a_{-2} e^{(-2\eta)}\right) a_{-2}}{b_2 a_{-2} e^{(2\eta)} + b_{-2} a_0 + b_{-2} e^{(-2\eta)}}$$

where

$$\eta = kx - \frac{k\left(a_{-2} + 8k^3 b_{-2} + 2kb_{-2}\right)}{b_{-2}} t \qquad (28h)$$

Example 3.10: Consider the singularly perturbed sixth-order Boussinesq equation

$$u_{tt} = u_{xx} + (p(u))_{xx} + \alpha u_{xxxx} + \beta u_{xxxxxx} \qquad (6j)$$

taking $\alpha = 1$, $\beta = 0$ and $p(u) = 3u^2$ the model equation is given as

$$u_{tt} = u_{xx} + 3(u^2)_{xx} + u_{xxxx} \qquad (7j)$$

with initial conditions

$$u(x,0) = \frac{2ak^2 e^{kx}}{(1+ae^{kx})},\ u_t(x,0) = \frac{2ak^3\sqrt{1+k^2}(1-ae^{kx})e^{kx}}{(1+ae^{kx})^3}$$

where *a* and *k* are arbitrary constants. The exact solution u (x, t) of the problem is given as

$$u(x,t) = 2\frac{ak^2 \exp(kx + k\sqrt{1+k^2}t)}{(1+a\exp(kx+k\sqrt{1+k^2}t))^2} \qquad (8j)$$

Introducing a transformation $\eta = kx + \omega t$ we get

$$[k^2 - \omega^2]u'' + 6k^2[u'^2 + uu''] + k^4 u^{(iv)} = 0 \qquad (9j)$$

The trial solution of the above problem can be expressed in the following form

$$u(x) = \frac{a_c \exp[c\eta] + \ldots + a_{-d}\exp[-d\eta]}{b_p \exp[p\eta] + \ldots + b_{-q}\exp[-q\eta]}$$

Corresponding Author: Dr. Syed Tauseef Mohyud-Din, HITEC University, Taxila Cantt., Pakistan

Proceeding as before, we obtain

$$p = c, q = d$$

Case 3.10.1: If p = c = 1 and q = d = 1 then

$$u(\eta) = \frac{a_1 \exp[\eta] + a_0 + a_{-1} \exp[-\eta]}{b_1 \exp[\eta] + a_0 + b_{-1} \exp[-\eta]}$$

Proceeding as before, we have

$$\begin{aligned} &\frac{1}{A}[c_4 \exp(4\eta) + c_3 \exp(3\eta) + c_2 \exp(2\eta) \\ &+ c_1 \exp(\eta) + c_0 + c_{-1} \exp(-\eta) + c_{-2} \exp(-2\eta) \\ &+ c_{-3} \exp(-3\eta) + c_{-4} \exp(-4\eta)] = 0 \end{aligned} \quad (10j)$$

where

$$A = (b_1 \exp(\eta) + b_0 + b_{-1} \exp(-\eta))^5$$

c_i (i =-4,-3,…,3,4) are constants obtained by Maple 11.

Equating the coefficients of exp (nη) to be zero, we obtain

$$\begin{aligned} &\{c_{-4} = 0,\ c_{-3} = 0,\ c_{-2} = 0,\ c_{-1} = 0, \\ &c_0 = 0,\ c_1 = 0,\ c_2 = 0,\ c_3 = 0, c_4 = 0\} \end{aligned} \quad (11j)$$

Solution of (11j) will yield

$$\begin{aligned} &\{a_{-1} = \frac{b_{-1}(-k^4 + \omega^2 - k^2)}{6k^2},\quad b_1 = \frac{b_0^2}{4b_{-1}},\quad a_0 = \frac{b_0(5k^4 + \omega^2 - k^2)}{6k^2} \\ &b_0 = b_0,\quad b_{-1} = b_{-1},\quad a_1 = \frac{b_0^2(-k^4 + \omega^2 - k^2)}{24k^2 b_{-1}}\} \end{aligned} \quad (12j)$$

Therefore, we obtained the following generalized solitary solution u (x, t)

$$\begin{aligned} u(x,t) &= \frac{\left(\frac{1}{24}\frac{b_0^2(-k^4 + \omega^2 - k^2)\exp(kx + \omega t)}{k^2 b_{-1}} + \frac{1}{6}\frac{b_0\ (5k^4 + \omega^2 - k^2)}{k^2} + \frac{1}{6}\frac{b_{-1}\ (-k^4 + \omega^2 - k^2)\exp(-kx - \omega t)}{k^2}\right)}{\frac{1}{4}\frac{b_0^2 e\exp(kx + \omega t)}{b_{-1}} + b_0 + b_{-1}\exp(-kx - \omega t)} \\ &= \frac{(-k^4 + \omega^2 - k^2)}{6k^2} + \frac{4 b_0 b_{-1} k^2}{b_0^2 \exp(kx + \omega t) + 4b_0 b_{-1} + 4b_{-1}^2 \exp(-kx - \omega t)} \end{aligned} \quad (13j)$$

where b_0, b_{-1} and k are real numbers.

This graphical representation is similar to the exact solution at a = k = 1, in case k is an imaginary number, the obtained soliton solution can be converted into periodic solution or compact-like solution. Therefore, we write k = iK and ω = iα, hence

$$u(x,t) = \frac{(K^4 + \alpha^2 - K^2)}{6K^2} - \frac{4b_0 b_{-1} K^2}{4 b_0 b_{-1} + \cos(Kx + \alpha t)(b_0^2 + 4b_{-1}^2) + i\sin(Kx + \alpha t)(b_0^2 - 4b_{-1}^2)} \quad (14j)$$

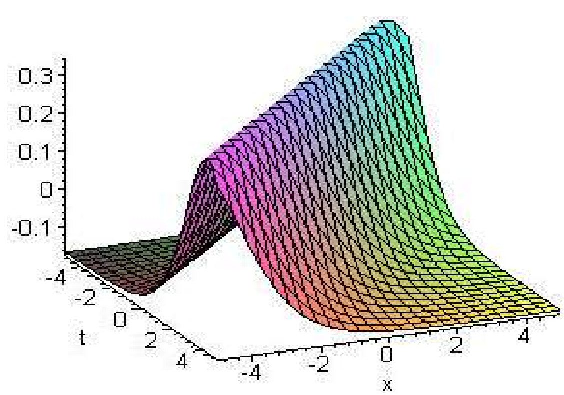

Fig. 3.33: Soliton solutions, when k = ω = 1 and b_0, b_{-1} = 1

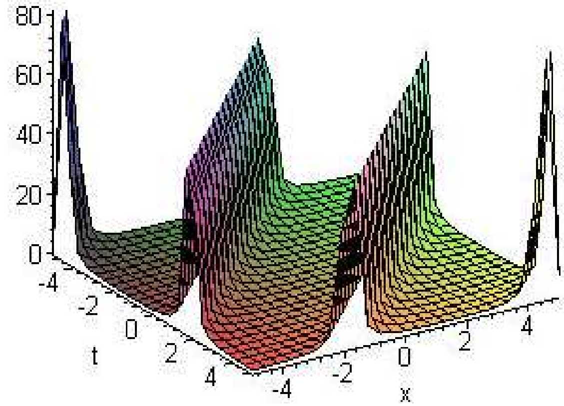

Fig. 3.34: Periodic solutions, when K = α = 1

For periodic solution or compacton-like solution, the imaginary part in equation (14j) must be zero, which requires

$$b_0^2 - 4b_{-1}^2 = 0 \quad \text{or} \quad b_0 = \pm 2b_{-1} \tag{15j}$$

substituting the value of equation(14j) in equation (15j), we obtain compacton-like solution as follows

$$u(x,t) = \frac{(K^4 + \alpha^2 - K^2)}{6K^2} - \frac{\pm 2b_0^2 K^2}{2b_0^2 + 2b_0^2 \cos(Kx + \alpha t)} = \frac{(K^4 + \alpha^2 - K^2)}{6K^2} \pm \frac{K^2}{1 + \cos(Kx + \alpha t)} \tag{16j}$$

Case 3.10.2: If p = c = 2 and q = d = 2 then

$$u(\eta) = \frac{a_2 \exp[2\eta] + a_1 \exp[\eta] + a_0 + a_{-1} \exp[-\eta] + a_{-2} \exp[-2\eta]}{b_2 \exp[2\eta] + b_1 \exp[\eta] + b_0 + b_{-1} \exp[-\eta] + b_{-2} \exp[-2\eta]}$$

Proceeding as before, we obtain

$$\begin{aligned} &\{a_{-2} = \frac{1}{96}\frac{a_1^4(-k^2 + \omega^2 - k^4)}{k^{10}},\ a_{-1} = \frac{1}{4}\frac{a_1^3}{k^4},\ a_0 = -\frac{1}{12}\frac{a_1^2(-k^2 + \omega^2 + 11k^4)}{k^6} \\ &a_1 = a_1,\ \ a_2 = \frac{1}{6}\frac{(-k^2 + \omega^2 - k^4)}{k^2},\ b_{-2} = \frac{1}{16}\frac{a_1^4}{k^8},\ \ b_0 = -\frac{1}{2}\frac{a_1^4}{k^4},\ \omega = \omega\} \end{aligned} \tag{17j}$$

Equation (17j) leads the following solution

$$u(x,t) = \frac{\left(\begin{aligned} &\frac{1}{6}\frac{(-k^2 + \omega^2 - k^4)}{k^2} e^{(2kx + 2\omega t)} + a_1 e^{(kx + \omega t)} - \frac{1}{12}\frac{a_1^2(-k^2 + \omega^2 + 11k^4)}{k^6} \\ &+ \frac{1}{4}\frac{a_1^3}{k^4} e^{(-kx - \omega t)} + \frac{1}{96}\frac{a_1^4(-k^2 + \omega^2 - k^4)}{k^{10}} e^{(-2kx - 2\omega t)} \end{aligned}\right)}{e^{(2kx + 2\omega t)} + -\frac{1}{2}\frac{a_1^4}{k^4} + \frac{1}{16}\frac{a_1^4}{k^8} e^{(-2kx - 2\omega t)}} \tag{18j}$$

Simplifying equation (18j), we get following soliton solution

$$u(x,t) = \frac{(-k^2 + \omega^2 - k^4)}{6k^2} + \frac{4k^2 a_1}{4\exp(kx + \omega t)k^4 + 4a_1 k^2 + a_1^2 \exp(-kx - \omega t)}, \tag{19j}$$

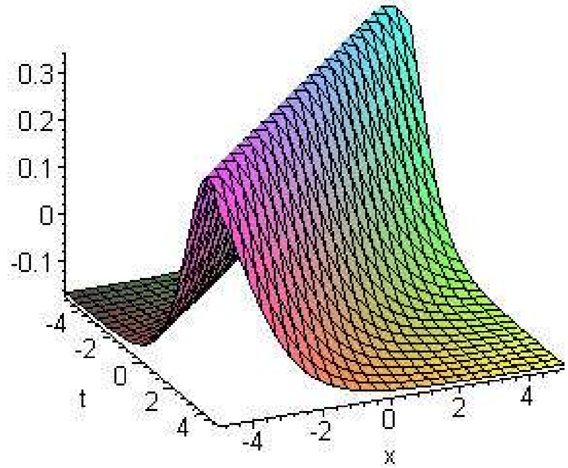

Fig. 3.35: Soliton solutions, when $k = \omega = 1$ and $a_1 = 1$

Example 3.11: Consider the following nonlinear boundary value problem of sixth-order

$$y^{(vi)}(x) = e^{-x} y^2(x), \quad 0 < x < 1 \qquad (6k)$$

with boundary conditions

$$y(0) = y'(0) = y^{(iv)}(0) = 1, \quad y(1) = y''(1) = y^{(iv)}(1) = e$$

The exact solution for this problem is

$$y(x) = e^x$$

The above boundary value problem can be expressed in the following form

$$y(x) = \frac{a_c \exp[cx] + ... + a_{-d} \exp[-dx]}{b_p \exp[px] + ... + b_{-q} \exp[-qx]} \qquad (3k)$$

The appropriate simplification would yield

$$y^{(vi)} = \frac{c_1 \exp[(63p + c)x] + \cdots}{c_2 \exp[64px] + \cdots} \qquad (7k)$$

and

$$y^2 = \frac{c_3 \exp[2cx] + \cdots}{c_4 \exp[2px] + \cdots} = \frac{c_3 \exp[(62p + 2c)x] + \cdots}{c_4 \exp[64px] + \cdots} \qquad (8k)$$

where c_i are determined coefficients only for simplicity; balancing the highest order of exp-function in (7k) and (8k), we have

$$63p+c = 62p+2c \qquad (9k)$$

which in turn gives

$$p = c \qquad (10k)$$

The values of d and q can also be determined by balancing the linear term of the lowest order

$$y^{(vi)} = \frac{\cdots + d_1 \exp[(-63q - d)x]}{\cdots + d_2 \exp[-64qx]} \qquad (11k)$$

and

$$y^2 = \frac{\cdots + d_3 \exp[-2dx]}{\cdots + d_4 \exp[-2qx]} = \frac{\cdots + d_3 \exp[(-62q - 2d)x]}{\cdots + d_4 \exp[-64qx]} \qquad (12k)$$

where d_i are determined coefficients only for simplicity. Now, balancing the lowest order of Exp-function in (11k) and (12k), we have

$$-63q-d = -62q-2d \qquad (13k)$$

which in turn gives

$$q = d \qquad (14k)$$

Case 3.11.1: $p = c = 1$ and $q = d = 1$

Equation (3k) reduces to

$$y(x) = \frac{a_1 \exp[x] + a_0 + a_{-1} \exp[-x]}{b_1 \exp[x] + b_0 + b_{-1} \exp[-x]} \qquad (15k)$$

Consequently, we have

$$\begin{aligned}&\frac{1}{A}[c_6 \exp(6x) + c_5 \exp(5x) + c_4 \exp(4x) + c_3 \exp(3x)\\ &+ c_2 \exp(2x) + c_1 \exp(x) + c_0 + c_{-1} \exp(-x) + c_{-2} \exp(-2x)\\ &+ c_{-3} \exp(-3x) + c_{-4} \exp(-4x) + c_{-5} \exp(-5x)\\ &+ c_{-6} \exp(-6x) + c_{-7} \exp(-7x) + c_{-8} \exp(-8x)] = 0\end{aligned} \qquad (16k)$$

where

$$A = \left(b_1 \exp(x) + b_0 + b_{-1} \exp(-x)\right)^7$$

C_i (i =-8,-7,-6,-4,…,4, 5, 6) are constants obtained by Maple 11. Equating the coefficients of exp (nx) to be zero, we obtain

$$\begin{aligned}\{&c_{-8} = 0, c_{-7} = 0, c_{-6} = 0, c_{-5} = 0, c_{-4} = 0,\\ &c_{-3} = 0, c_{-2} = 0, c_{-1} = 0, c_0 = 0, c_1 = 0,\\ &c_2 = 0, c_3 = 0, c_4 = 0, c_5 = 0, c_6 = 0\}\end{aligned} \qquad (17k)$$

Solution of (17k) will yield

$$\begin{aligned}\{&a_{-1} = 0, \quad b_1 = 0, \quad a_0 = b_{-1},\\ &b_0 = 0, \quad b_{-1} = b_{-1}, \quad a_1 = 0\}\end{aligned} \qquad (18k)$$

Consequently, the following exact solution is obtained.

$$y(x) = e^{x} \qquad (19k)$$

Case 3.11.2: p = c = 2 and q = d = 2

Equation (3k) reduces to

$$y(x) = \frac{a_2\exp[2x] + a_1\exp[x] + a_0 + a_{-1}\exp[-x] + a_{-2}\exp[-2x]}{b_2\exp[2x] + b_1\exp[x] + b_0 + b_{-1}\exp[-x] + b_{-2}\exp[-2x]} \qquad (20k)$$

Proceeding as before, we obtain

$$\begin{aligned}\{a_{-2} = 0,\ \ a_{-1} = b_{-2},\ \ a_0 = a_0,\ \ a_1 = 0,\ \ a_2 = 0\\ b_{-2} = b_{-2},\ \ b_{-1} = a_0,\ \ b_0 = 0,\ \ b_1 = 0,\ \ b_2 = 0\end{aligned} \qquad (21k)$$

$$y(x) = \frac{a_0 + b_{-2}e^{(-x)}}{a_0e^{(-x)} + b_{-2}e^{(-2x)}} = \frac{a_0 + b_{-2}e^{(-x)}}{e^{(-x)}(a_0 + b_{-2}e^{(-x)})}$$

where $a_0 + b_{-2}e^{(-x)} \neq 0$ Consequently, the exact solution is obtained as

$$y(x) = e^{x} \qquad (22k)$$

CONCLUSION

In this paper, we applied the exp-function method which was developed by He and Wu [25] for finding solutions of a wide class of nonlinear differential equations. The proposed method has been successfully tested on a variety of physical problems. Numerical results reveal the complete reliability and efficiency of the proposed algorithm. It is concluded that the exp-function method can be viewed as an efficient alternative for solving nonlinear differential equations.

ACKNOWLEDGEMENT

The author is highly grateful to Brig (Rtd) Qamar Zaman Vice Chancellor HITEC University Pakistan for providing the excellent research facilities and environment.

REFERENCES

1. Abbasbandy, S., 2007. Numerical solutions of nonlinear Klein-Gordon equation by variational iteration method. Internat. J. Numer. Meth. Engrg., 70: 876-881.
2. Abdou, M.A. and A.A. Soliman, 2005. Variational iteration method for solving Burger's and coupled Burger's equations. J. Comput. Appl. Math., 181: 245-251.
3. Abdou, M.A. and A.A. Soliman, 2005. New applications of variational iteration method, Phys., D211 (1-2): 1-8.
4. Abdou, M.A., A.A. Soliman and S.T. Basyony, 2007. New application of exp-function method for improved Boussinesq equation. Phys. Lett. A, 369: 469-475.
5. Caudrey, P.J., R.K. Dodd and J.D. Gibbon, 1976. A new hierarchy of Korteweg-de Vries equations. Proc. Roy. Soc. Lond. A 351: 407-422.
6. El-Wakil, S.A., M.A. Madkour and M.A. Abdou, 2007. Application of exp-function method for nonlinear evolution equations with variable co-efficient. Phys. Lett. A, 369: 62-69.
7. Ghorbani, A. and J.S. Nadjfi, 2007. He's homotopy perturbation method for calculating Adomian's polynomials. Int. J. Nonlin. Sci. Num. Simul., 8 (2): 229-332.
8. Ghorbani, A., 2007. Beyond Adomian's polynomials: He polynomials, Chaos, Solitons and Fractals, In Press.
9. He, J.H., 2003. Variational approach to the Thomas-Fermi equation. Appl. Math. Comput., 143: 533-535.
10. He, J.H., 2006. Some asymptotic methods for strongly nonlinear equation. Int. J. Mod. Phys. 20 (20) 10: 1144-1199.
11. He, J.H., 1999. Variational iteration method. A kind of non-linear analytical technique, some examples, Internat. J. Nonlin. Mech., 34 (4): 699-708.
12. He, J.H., 2000. Variational iteration method for autonomous ordinary differential systems. Comput. Math. Appl. 114 (2-3): 115-123.
13. He, J.H. and X.H. Wu, 2006. Construction of solitary solution and compaction-like solution by variational iteration method. Chaos, Solitons & Fractals, 29 (1): 108-113.
14. He, J.H., 2007. Variational iteration method-some recent results and new interpretations. J. Comput. Appl. Math., 207: 3-17.
15. He, J.H. and X.H. Wu, 2007. Variational iteration method: New development and applications. Comput. Math. Appl., 54: 881-894.
16. He, J.H., 2007. The variational iteration method for eighth-order initial boundary value problems. Phys. Scr. 76 (6): 680-682.
17. He, J.H., 2008. Recent developments of the homotopy perturbation method. Top. Meth. Nonlin. Anal., 31: 205-209.

18. He, J.H., 1999. Homotopy perturbation technique. Comput. Math. Appl. Mech, Engy., pp: 178-257.
19. He, J.H., 2006. Homotopy perturbation method for solving boundary value problems. Phys. Lett., A 350: 87-88.
20. He, J.H., 2004. Comparison of homotopy perturbation method and homotopy analysis method. Appl. Math. Comput., 156: 527-539.
21. He, J.H., 2005. Homotopy perturbation method for bifurcation of nonlinear problems. Int. J. Nonlin. Sci. Num. Simul., 6 (2): 207-208.
22. He, J.H., 2004. The homotopy perturbation method for nonlinear oscillators with discontinuities. Appl. Math. Comput., 151: 287-292.
23. He, J.H., 2000. A coupling method of homotopy technique and perturbation technique for nonlinear problems. Int. J. Nonlin. Mech., 35 (1): 115-123.
24. He, J.H., 2008. An elementary introduction of recently developed asymptotic methods and nanomechanics in textile engineering. Int. J. Mod. Phys., B 22 (21): 3487-4578.
25. He, J.H. and X.H. Wu, 2006. Exp-function method for nonlinear wave equations, Chaos, Solitons Fractals, 30 (3): 700–708.
26. He, J.H. and M.A. Abdou, 2007. New periodic solutions for nonlinear evolution equation using exp-method. Chaos, Solitons and Fractals, 34: 1421-1429.
27. Kaya, D., 2003. Explicit and numerical solutions of some fifth-order KdV equations by decomposition method. Appl. Math. Comput., 144: 353-363.
28. Kaya, D., 1999. On the solution of a Korteweg-de Vries like equation by the decomposition method. Intern. J. Comput. Math., 72: 531-539.
29. Ma, W.X. and D.T. Zhou, 1997. Explicit exact solution of a generalized KdV equation, Acta Math. Scita., 17: 168-174.
30. Mohyud-Din, S.T., M.A. Noor and K.I. Noor, 2009. Travelling wave solutions of seventh-order generalized KdV equations using He’s polynomials. Int. J. Nonlin. Sci. Num. Sim., 10 (2): 223-229.
31. Mohyud-Din, S.T. and M.A. Noor, 2006. Homotopy perturbation method for solving fourth order boundary value problems. Math. Prob. Engg. ID 98602, doi:10.1155/2007/98602, pp: 1-15.
32. Mohyud-Din, S.T., M.A. Noor and K.I. Noor, 2009. Travelling wave solutions of seventh-order generalized KdV equations by variational iteration method using Adomian’s polynomials. Int. J. Mod. Phys. B.
33. Mohyud-Din, S.T., M.A. Noor, K.I. Noor and A. Waheed, 2008. Exp-function method for generalized travelling solutions of Lax equations. Int. J. Mod. Phys. B.
34. Mohyud-Din, S.T., M.A. Noor and A. Waheed, 2008. Exp-function method for generalized travelling solutions of good Boussinesq equations. J. Appl. Math. Computg. (2008), DOI 10.1007/s12190-008-0183-8.
35. Noor, M.A. and S.T. Mohyud-Din, 2008. Homotopy perturbation method for nonlinear higher-order boundary value problems. Int. J. Nonlin. Sci. Num. Simul., 9 (4): 395-408.
36. Noor, M.A. and S.T. Mohyud-Din, 2008. Variational iteration method for solving higher-order nonlinear boundary value problems using He’s polynomials. Int. J. Nonlin. Sci. Num. Simul., 9 (2): 141-157.
37. Noor, M.A. and S.T. Mohyud-Din, 2008. Variational iteration method for solving fifth-order boundary value problems using He’s polynomials. Math. Prob. Engg. 2008, Article ID 954794, doi: 10:1155/2008/954794.
38. Noor, M.A. and S.T. Mohyud-Din, 2008. Modified variational iteration method for heat and wave-like equations, Acta Applnda. Mathmtce, DOI: 10.1007/s10440-008-9255-x.
39. Noor, M.A. and S.T. Mohyud-Din, 2008. Variational iteration method for unsteady flow of gas through a porous medium using He’s polynomials and Pade´ approximants, Comput. Math. Appl.
40. Noor, M.A. and S.T. Mohyud-Din, 2008. Variational homotopy perturbation method for solving higher dimensional initial boundary value problems, Math. Prob. Engg. 2008, Article ID 696734, doi:10.1155/2008/696734.
41. Noor, M.A. and S.T. Mohyud-Din, Solution of singular and nonsingular initial and boundary value problems by modified variational iteration method. Math. Prob. Engg. 2008, Article ID 917407, 23 pages, doi:10.1155/2008/917407.
42. Noor, M.A. and S.T. Mohyud-Din, 2007. Variational iteration technique for solving higher order boundary value problems. Appl. Math. Comput., 189: 1929-1942.
43. Noor, M.A., S.T. Mohyud-Din and A. Waheed, 2008. Exp-function method for solving Kuramoto-Sivashinsky and Boussinesq equations. J. Appl. Math. Computg.,DOI:10.1007/s12190-008-0083-y.
44. Noor, M.A., S.T. Mohyud-Din and A. Waheed, 2008. Exp-function method for generalized travelling solutions of master partial differential equations. Acta Applnda. Mathmtce, DOI: 10.1007/s10440-008-9245-z.
45. Wazwaz, A.M., 1999. The modified decomposition method and Pade approximants for solving Thomas-Fermi equation. Appl. Math. Comput., 105: 11-19.

46. Wazwaz, A.M., 2001. Analytic treatment for variable coefficient fourth-order parabolic partial differential equations. Appl. Math. Comput., 123: 219-227.
47. Wazwaz, A.M., 1995. The decomposition method for the approximate solution to the Goursat problem. Appl. Math. Comput., 69: 299-311.
48. Wu, X.-H. and J.H. He, 2007. Solitary solutions, periodic solutions and compacton like solutions using the Exp-function method. Comput. Math. Appl., 54: 966-986.
49. Wu, X.H. and J.H. He, 2007. Exp-function method and its applications to nonlinear equations, Chaos, Solitons and Fractals, In Press.
50. Wu, X.H. and J.H. He, 2007. Solitary solutions, periodic solutions and compacton like solutions using the Exp-function method, Comput. Math. Appl., 54: 966-986.
51. Xu, L., 2007. The variational iteration method for fourth-order boundary value problems, Chaos, Solitons & Fractals, In Press.
52. Yusufoglu, E., 2008. New solitonary solutions for the MBBN equations using exp-function method. Phys. Lett. A. 372: 442-446.
53. Zhou, X.W., Y.X. Wen and J.H. He, 2008. Exp-function method to solve the nonlinear dispersive k (m, n) equations. Int. J. Nonlin. Sci. Num. Sim., 9 (3): 301-306.
54. Zhang, S., 2007. Application of Exp-function method to high-dimensional nonlinear evolution equation, Chaos, Solitons and Fractals, 365: 448-455.
55. Zhu, S.D., 2007. Exp-function method for the Hybrid-Lattice system. Inter. J. Nonlin. Sci. Numer. Sim. 8: 461-464.
56. Zhu, S.D., 2007. Exp-function method for the discrete mKdv lattice. Inter. J. Nonlin. Sci. Numer. Sim., 8: 465-468.