World Applied Sciences Journal 7 (Special Issue for Applied Math): 109-115, 2009
ISSN 1818-4952

Dynamic Analysis of the Fractional-order Chen Chaotic System

[1]A. Hajipoor, [2]H.T. Shandiz and [2]H. Marvi

[1]Islamic Azad University of Sabzevar, Iran
[2]Shahrood University of Technology, P. Code: 3619995161, P.O. Box: 317, Shahrood, Iran

Abstract: In this paper, the existence and uniqueness of solution for fractional order Chen chaotic system is investigated theoretically based on the qualitative theory. The stability of the corresponding equilibria is also investigated similar to the integer order counterpart. According to the obtained results, the bifurcation conditions of these two systems are significantly different. Numerical simulations are presented to confirm the given analytical results.

Key words: Chen system . fractional order nonlinear system . stability

INTRODUCTION

Nowadays the behavior of many dynamical systems can be properly described by using the fractional order system theory. For example, Phenomena in electromagnetic [1], quantitative finance [2], electrochemistry, material science [3] have been described using fractional differ-integration operators. Due to fundamental differences between Fractional Order Differential Equations (FODE) and Ordinary Differential Equations (ODE), most of characteristics or conclusions of the ODE systems cannot be directly extended to the case of the FODE systems. In recent times, many attempts have been dedicated to the study of chaotic dynamics of fractional-order differential systems [4-7]. Many current results about fractional-order chaotic systems, however, are attained only by numerical simulations. The aim of this paper is to examine the stability and bifurcation for the fractional-order Chen system. Similar to the results presented in [8], the existence and uniqueness about solutions of the fractional-order Chen system will be established. At the same time, the stability of equilibria of the system will also be analyzed. More importantly, the fractional-order system can display a Hopf bifurcation under certain conditions which are entirely different from the corresponding integer-order Chen system. The paper is organized as follows. Section 2 briefs basic concepts in fractional calculus and fractional systems. In Section 3, the stability of equilibria and bifurcation for fractional-order Chen system are analyzed according to the qualitative theory. Numerical simulations to illustrate the validity of the results are presented in Section 4 and finally, conclusions in Section 5 close the paper.

MATHEMATICAL BACKGROUND

The operator ${}_aD_t^\alpha$, where a and t are the limits of the operation and $\alpha \in R$, is a combined differentiation and integration operator commonly used in fractional calculus. The continuous integro-differential operator is defined as:

$$ {}_aD_t^\alpha = \begin{cases} \dfrac{d^\alpha}{dt^\alpha} & \alpha > 0 \\ 1 & \alpha = 0 \\ \displaystyle\int_a^t (d\tau)^\alpha & \alpha < 0 \end{cases} \tag{1} $$

There are different definitions for fractional derivatives [9]. The Grunwald-Letnikov, Riemann-Liouville and Caputo definitions are used for the general fractional differ-integral. These definitions are briefly introduced in the following lines. The Grunwald-Letnikov fractional order derivative of $f(t)$ is defined as:

$$ {}_aD_t^\alpha f(t) = \frac{d^\alpha f(t)}{d(t-a)^\alpha} = \lim_{N\to\infty}\left[\frac{t-a}{N}\right]^{-\alpha}\sum_{j=0}^{N-1}(-1)^j\binom{\alpha}{j} f\left(t - j\left[\frac{t-a}{N}\right]\right) \tag{2} $$

The Riemann-Liouville fractional order derivative of $f(t)$ is defined as:

$$ D_{t_0}^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)}\int_{t_0}^t \frac{x^{(n)}(u)}{(t-u)^{\alpha-n+1}}du \tag{3} $$

Corresponding Author: Dr. A. Hajipoor, Islamic Azad University of Sabzevar, Iran

where $\alpha>0$ and n is the first integer which is not less than α, i.e., $n-1\le\alpha\le n$ and $\Gamma(.)$ is the Gamma function.

$$\Gamma(z)=\int_0^{\infty} t^{z-1}e^{-t}dt$$

For a wide class of functions, the Grunwald-Letnikov and the Riemann-Liouville definition are equivalent [9]. The initial value problem related to Eq.(??) is

$$\begin{cases} D^{\alpha}x(t)=f(t,x(t)) \\ x(t)|_{t=0_+}=x_0 \end{cases} \qquad (4)$$

where $0<\alpha<1$ and $D^{\alpha}=D_0^{\alpha}$. Looking at the questions of existence and uniqueness of the solution Eq. (4), we can present the following result that is very similar to the corresponding classical theorem known in the case of first-order equations.

Theorem 1: (Existence and Uniqueness Theorem [8]): Let $f(t, x)$ be a real-valued continuous function, defined in the domain G, satisfying in G the Lipschitz condition with respect to x, i.e.

$$f(t,x_1)-f(t,x_2)|\le M|x_1-x_2|$$

where M is a positive constant, such that $|f(t,x)|=M<\infty$ for all $(t,x)\in G$. Let also

$$K\ge\frac{Mh^{s_n-s_1+1}}{\Gamma(1+\sigma_n)} \qquad (5)$$

Then there exists in a region a $R(h,K)$ a unique and continuous solution $x(t)$ of the following initial-value problem,

$$D^{\sigma_n}x(t)=f(t,x) \qquad (6)$$

$$[D^{\sigma_{k-1}}x(t)]_{t=0}=b_k, \qquad k=1,2,\ldots,n$$

where

$$D^{\sigma_k}\equiv D^{\alpha_k}D^{\alpha_{k-1}}\ldots D^{\alpha_1},\quad D^{\sigma_{k-1}}\equiv D^{\alpha_{k-1}}D^{\alpha_{k-2}}\ldots D^{\alpha_1}$$

$$\sigma_k=\sum_{j=1}^{k}\alpha_j,\qquad (k=1,2,\ldots,n);\quad 0<a_j\le 1$$

Furthermore, the above definition in one dimension can naturally be generalized to the case of multiple dimensions. That is, let

$$X(t)=(x_1(t),x_2(t),\ldots,x_n(t))^T\in R$$

and

$$\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_n)^T\in R,\ 0<\alpha_i<1, i=1,2,\ldots,n$$

The n-dimension FODE is described as follows:

$$D_{t_0}^{\alpha}X(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{X'(u)}{(t-u)^{\alpha}}du \equiv F(t,X(t)) \qquad (7)$$

where

$$\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{X'(u)}{(t-u)^{\alpha}}du = (D^{\alpha_1}x_1(t),D^{\alpha_1}x_1(t),\ldots,D^{\alpha_n}x_n(t))^T$$

and

$$F(t,X(t))=\begin{pmatrix} f_1(t,x_1(t),x_2(t),\ldots,x_n(t)) \\ f_2(t,x_1(t),x_2(t),\ldots,x_n(t)) \\ \ldots \\ f_n(t,x_1(t),x_2(t),\ldots,x_n(t)) \end{pmatrix}$$

The results of Theorem 2 can be easily generalized to the initial value problem of the vector-valued functions (7).

STABILITY ANALYSIS OF FRACTIONAL-ORDER CHEN SYSTEM

Existence and uniqueness of solutions: The canonical integer-order Chen system can be described by the following autonomous ODE [10]

$$\begin{cases} \frac{dx_1}{dt}=a(x_2-x_1) \\ \frac{dx_2}{dt}=(c-a)x_1-x_1x_3+cx_2 \\ \frac{dx_3}{dt}=x_1x_2-bx_3 \end{cases} \qquad (8)$$

which has a chaotic attractor with the parameters: a = 35, b = 3 and c = 28. The corresponding fractional-order Chen system can be written in the form as below:

$$\begin{cases} \frac{d^{\alpha_1}x_1}{dt^{\alpha_1}}=a(x_2-x_1) \\ \frac{d^{\alpha_2}x_2}{dt^{\alpha_2}}=(c-a)x_1-x_1x_3+cx_2 \\ \frac{d^{\alpha_3}x_3}{dt^{\alpha_3}}=x_1x_2-bx_3 \end{cases} \qquad (9)$$

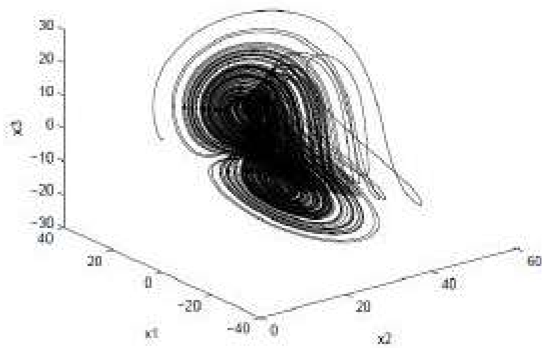

Fig. 1: The fractional-order Chen chaotic attractor

where a,b and c are real parameters and $0<\alpha_i<1$, i = 1,2,3. Especially, for a set of parameter values: a=35, b=3 and c = 28 and α = (0.985, 0.99, 0.98), the fractional-order Chen system can display chaotic attractors [11] as shown in Fig. 1.

The chaotic dynamics of fractional-order Chen system, as mentioned previously, is mainly investigated by some researchers only through numerical simulations [11-14]. Based on the qualitative theory and some existing results about FODE, the following results can be deduced for the fractional-order Chen system (9).

Theorem 2: The initial value problem of the fractional-order Chen system (9) can be represented in the following form:

$$\begin{cases} D^{\alpha}X(t)=AX(t)+x_1(t)BX(t) \\ X(0)=X_0, t\in(0,T] \end{cases} \quad (10)$$

where

$$X(t)=(x_1(t),x_2(t),x_3(t))^T\in R^3, X_0=(x_{10},x_{20},x_{30})^T$$

and

$$A=\begin{pmatrix} -a & a & 0 \\ c-a & c & 0 \\ 0 & 0 & -b \end{pmatrix}, \ B=\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$\alpha = (\alpha_1, \alpha_2, \alpha_3)^T$ and $0<\alpha_i<1$(i = 1,2,3), some constant T>0, then it has a unique solution.

Proof: Let F(X(t)) = AX(t) + x_1BX(t) which is obviously continuous and bounded on the interval $[X_0-\delta, X_0+\delta]$ for any $\delta>0$. Furthermore, one has

$$|F(X(t)-F(Y(t))|=$$
$$|A(X(t)-Y(t))+x_1(t)BX(t)-y_1(t)BY(t)|\le$$
$$(||A||+||B||(|X(t)|+|y(t)|))$$
$$|(X(t)-Y(t))|\le M|(X(t)-Y(t))|$$

where

$$M=||A||+||B||(2|X_0|+\delta)>0, X(t), Y(t)\in R^3, ||.|| \text{and} |.|$$

denote matrix norm and vector norm, respectively. The above inequality manifests that F(x(t)) satisfies a Lipschitz condition. Based on the results of Theorem 1, we can conclude that the initial value problem of fractional-order Chen system has a unique solution.

The analysis of equilibria and bifurcation: Now, we state two stability theorems on fractional order systems and their related results. The first theorem has been given for commensurate fractional order systems.

Theorem 3: [15]. The following autonomous system:

$$\frac{d^{\alpha}x}{dt^{\alpha}}=Ax, \quad x(0)=x_0 \quad (11)$$

with $0<\alpha<1$, $x\in R^n$ and $A\in R^{n\times n}$, is asymptotically stable if and only if $|\arg(\lambda)|>\alpha\pi/2$ is satisfied for all eigenvalues λ of matrix A. Furthermore, the component of state decays towards 0 like $t^{-\alpha}$. Also, this system is stable if and only if $|\arg(\lambda)|\ge\alpha\pi/2$ is satisfied for all eigenvalues (λ) of matrix A with those critical eigenvalues satisfying $|\arg(\lambda)|=\alpha\pi/2$ have geometric multiplicity of one.

The following theorem considers stability in the incommensurate fractional order systems.

Theorem 4: Consider the following n-dimensional nonlinear fractional order system:

$$\begin{cases} D^{\alpha_1}x_1=f_1(x_1,x_2,...,x_n) \\ D^{\alpha_2}x_2=f_2(x_1,x_2,...,x_n) \\ . \\ . \\ . \\ D^{\alpha_n}x_n=f_3(x_1,x_2,...,x_n) \end{cases} \quad (12)$$

where all α_i s are rational numbers between 0 and 1. i = 1, 2,…,n. Let $X^{eq}=(x_1^{eq},x_2^{eq},...,x_n^{eq})$ is the equilibrium of system (12), i.e. $f_i(x_1^{eq},x_2^{eq},...,x_2^{eq})=0$ for i = 1, 2,…,n and $A=\frac{\partial f}{\partial X}|_{X^{eq}}, f=[f_1,f_2,...,f_n]$ is the Jacobian matrix at the point X^{eq}, then the point X^{eq} is asymptotically stable when $|\arg(\mathrm{eig}(A))| > \frac{\alpha_m \pi}{2}$ where

$$\alpha_m = \max(\alpha_i), i=1,2,\ldots,n.$$

Proof: To evaluate the asymptotic stability of this point, we define:

$$x_i(t) = x_i^{eq} + \varepsilon_i(t), i=1,2,\ldots,n$$

which implies that

$$D^{\alpha_i}\varepsilon_i(t) = f_i(x_1^{eq}+\varepsilon_1, x_2^{eq}+\varepsilon_2,\ldots,x_n^{eq}+\varepsilon_n)$$

but

$$f_i(x_1^{eq}+e_1, x_2^{eq}+e_2,\ldots,x_n^{eq}+e_n) \simeq f_i(x_1^{eq}, x_2^{eq},\ldots,x_n^{eq}+e_n)$$

$$+\frac{\partial f_i}{\partial x_1}|_{eq}.\varepsilon_1 + \frac{\partial f_i}{\partial x_2}|_{eq}.\varepsilon_2 + \ldots \frac{\partial f_i}{\partial x_n}|_{eq}.\varepsilon_n$$

$$\Rightarrow f_i(x_1^{eq}+\varepsilon_1, x_2^{eq}+\varepsilon_2,\ldots,x_n^{eq}+\varepsilon_n) \approx \frac{\partial f_i}{\partial x_1}|_{eq}\varepsilon_1 + \frac{\partial f_i}{\partial x_2}|_{eq}\varepsilon_2 + \ldots + \frac{\partial f_i}{\partial x_n}|_{eq}\varepsilon_n \quad (13)$$

where $f_i(x_1^{eq}, x_2^{eq},\ldots,x_n^{eq}) = 0$, then

$$D^{\alpha_i}\varepsilon_i(t) \approx \frac{\partial f_i}{\partial x_1}|_{eq}\varepsilon_1 + \frac{\partial f_i}{\partial x_2}|_{eq}\varepsilon_2 + \ldots \frac{\partial f_i}{\partial x_2}|_{eq}\varepsilon_n, \quad i=1,2,\ldots,n \quad (14)$$

and we obtain the system

$$\begin{bmatrix} \frac{d^{\alpha_1}x_1}{dt^{\alpha_1}} \\ \frac{d^{\alpha_2}x_2}{dt^{\alpha_2}} \\ \cdot \\ \cdot \\ \cdot \\ \frac{d^{\alpha_n}x_n}{dt^{\alpha_n}} \end{bmatrix} = J\varepsilon \quad (15)$$

where $\varepsilon = [\varepsilon_1, \varepsilon_2,\ldots,\varepsilon_n]^T$

$$J = \begin{pmatrix} a_{11} & a_{11} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ . & . & . & \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} \quad (16)$$

and

$$a_{ij} = \frac{\partial f_i}{\partial x_j}|_{eq}, i,j=1,2,\ldots n$$

By applying theorem3.2 like $t^{-\alpha_m}$, $\varepsilon(t)$ is decreasing if

$$|\arg(\mathrm{eig}(A))| > \frac{\alpha_m \pi}{2}$$

which implies that the equilibrium X^{eq} of the FODE (12) is as asymptotically stable as their integer order counterpart.

Proposition 1

1. If a>2c, then system (9) only has one equilibrium O(0,0,0);
2. If a<2c then system (9) has three equilibria, O(0,0,0), $C^+ = (\sqrt{(2c-a)b}, \sqrt{(2c-a)b}, 2c-a)$ and $C^- = (-\sqrt{(2c-a)b}, -\sqrt{(2c-a)b}, 2c-a)$. system (9) displays a bifurcation when c = 2c. Notice that we omit the proof here, which is obvious from the previous corollary. As can be observed, the results of Proposition3.2 is similar to the integer-order Chen system.

Proposition 2: With respect to the system (9), we have

1. The equilibrium O is asymptotically stable when $\begin{cases} b>0 \\ a>2c>0 \end{cases}$ and O is unstable when $\begin{cases} b<0 \\ a<0 \text{ or } a<2c \end{cases}$
2. The equilibria C^+ and C^- are all asymptotically stable if $\begin{cases} c > \frac{a}{2} > 0 \\ (a+b-c)c - 2a(2c-a) > 0 \end{cases}$

Proof: (1). $A_O = \begin{pmatrix} -a & a & 0 \\ c-a & c & 0 \\ 0 & 0 & -b \end{pmatrix}$ is the Jacobian matrix of the system (9) at the equilibrium O, whose corresponding eigenvalues are

$$\lambda_{1,2} = \frac{-(a-c) \pm \sqrt{c^2 - 3a^2 + 6ac}}{2}, \lambda_3 = -b \quad (17)$$

One has $\lambda_{1,2}\leq 0$, $\lambda_3\leq 0$ when $a>2c>0$ and $b>0$, so $|\arg(\text{spec}(A))|=\pi>\frac{\alpha_M\pi}{2}$, where $\alpha_M=\max(\alpha_i\leq 1)$, $1\leq i\leq 3$ Therefore the equilibrium O is asymptotically stable. Furthermore, one has $\lambda_3>0$ when $b<0$ or if $a<0$ or $a<2c$ the point O is unstable.

Proof: (2) The Jacobian matrix of system (9) at the points C^+ and C^- are

$$A_{C^+}=\begin{bmatrix} -a & a & 0 \\ -c & c & -\sqrt{(2c-a)b} \\ \sqrt{(2c-a)b} & \sqrt{(2c-a)b} & -b \end{bmatrix} \quad (18)$$

and

$$A_{C^-}=\begin{bmatrix} -a & a & 0 \\ -c & c & \sqrt{(2c-a)b} \\ -\sqrt{(2c-a)b} & -\sqrt{(2c-a)b} & -b \end{bmatrix} \quad (19)$$

respectively. However, they share the same characteristic equation

$$f(\lambda)=\lambda^3+U\lambda^2+V\lambda+W=0 \quad (20)$$

where $U = a+b-c$, $V = bc$ and $W = 2ab(2c-a)$ From Routh-Hurwitz criteria, the eigenvalues of Eq. (13) are all negative if $U>0$, $V>0$, $UV>W>0$. Through simplifying computation, these inequalities can be reduced as:

$$\begin{cases} c>\dfrac{a}{2}>0 \\ b>c+3a-\dfrac{2a^2}{c}>0 \end{cases}$$

Remark 1: The integer-order Chen system displays a Hopf bifurcation when $c>[(\sqrt{17}-3)a/2]$ and $b=\frac{c^2+3ac-2a^2}{c}$. But the corresponding fractional-order system will not produce bifurcation under the same conditions. The numerical simulations in the next section will support the result.

NUMERICAL METHODS AND SIMULATIONS

According to the Adams predictor-corrector scheme shown in [16, 17], the numerical solution of the initial value problem for fractional-order Chen system (9) will be yielded as below: Set $h=\frac{T}{N}$, $t_n=nh, n=0,1,\ldots,N\in Z^+$, the system (9) can be discredited as follows:

$$\begin{cases} x_{1n+1}=x_{1,0}+a\dfrac{h^{\alpha_1}}{\Gamma(\alpha_1+2)}(x^p_{2n+1}-x^p_{1n+1} \\ \qquad +x^q_{1n+1}) \\ x_{2n+1}=x_{20}+\dfrac{h^{\alpha_2}}{\Gamma(\alpha_2+2)}((c-a)x^p_{1n+1}- \\ x^p_{1n+1}x^p_{3n+1}+cx^p_{2n+1}+x^q_{2n+1}) \\ x_{3n+1}=x_{30}+\dfrac{h^{\alpha_3}}{\Gamma(\alpha_3+2)}(x^p_{1n+1}x^p_{2n+1}- \\ \qquad bx^p_{3n+1}+x^q_{3n+1}) \end{cases} \quad (21)$$

where

$$x^p_{1n+1}=x_{10}+\frac{a}{\Gamma(\alpha_1)}\sum_{j=0}^{n}\beta_{1,j,n+1}(x_{2j}-x_{1j})$$

$$x^p_{2n+1}=x_{20}+\frac{1}{\Gamma(\alpha_2)}\sum_{j=0}^{n}\beta_{2,j,n+1}((c-a)x_{1j}-x_{2j}-x_{1j}x_{3j})$$

$$x^p_{3n+1}=x_{30}+\frac{1}{\Gamma(\alpha_3)}\sum_{j=0}^{n}\beta_{3,j,n+1}(x_{1j}x_{2j}-bx_{3j})$$

$$x^q_{1n+1}=\sum_{j=0}^{n}\gamma_{1,j,n+1}(x_{2j}-x_{1j})$$

$$x^q_{2n+1}=\sum_{j=0}^{n}\gamma_{2,j,n+1}((c-a)x_{1j}-x_{2j}-x_{1j}x_{3j})$$

$$x^q_{3n+1}=\sum_{j=0}^{n}\gamma_{3,j,n+1}(x_{1j}x_{2j}-bx_{3j})$$

and

$$\beta_{i,j,n+1}=\frac{h^{\alpha_i}}{\alpha_i}((n-j-1)^{\alpha_i}-(n-j)^{\alpha_i}), \qquad 0\leq j\leq n$$

$$\gamma_{i,j,n+1}=$$
$$\begin{cases} n^{\alpha_i+1}-(n-\alpha_i)(n+1)^{\alpha_i} & j=0 \\ (n-j+2)^{\alpha_i+1}+(n-j)^{\alpha_i+1}-2(n-j+1)^{\alpha_i+1} & 1\leq j\leq n, \\ & i=1,2,3. \\ 1 & j=n+1 \end{cases}$$

To verify the success of the obtained results, some numerical simulations for the fractional-order Chen system have been conducted. All the differential equations are solved by using the above-mentioned method (21). In the following simulations, let α = (0.985, 0.99, 0.98) and h = 0.01.

Figure 2 shows that the states $x_1(t)x_2(t)$ and $x_3(t)$ of the system (9) are asymptotically decreasing towards zero, where a = 35, b = 3 and c = 10 and the corresponding initial states are set as x_{10} = 12, x_{20} = -5, x_{30} = -13. In particular, as depicted in Fig. 3, when

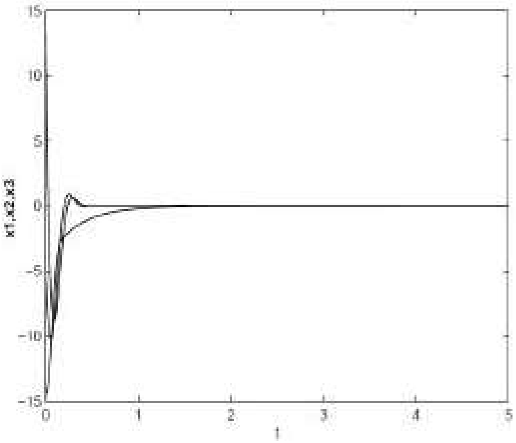

Fig. 2: The states of fractional-order Chen system when $c<a/2$

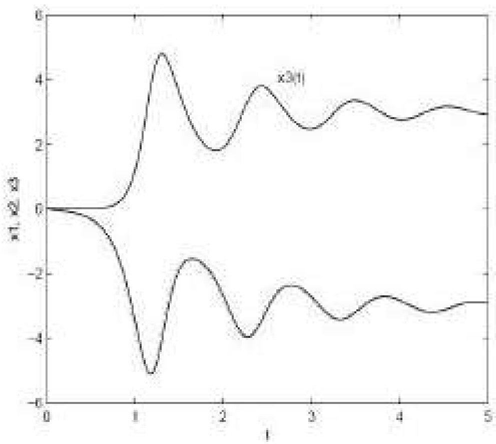

Fig. 3: The zero point of fractional-order Chen system is unstable when $c>a/2$

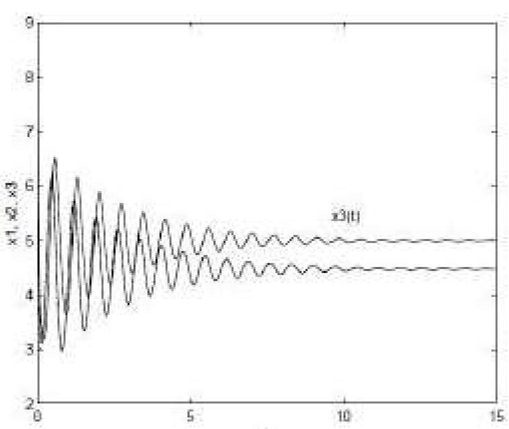

Fig. 4: The equilibrium C^+ of fractional-order Chen system is locally asymptotically stable when

$$c > \frac{a}{2} > 0\ ,\ b > c + 3a - \frac{2a^2}{c} > 0$$

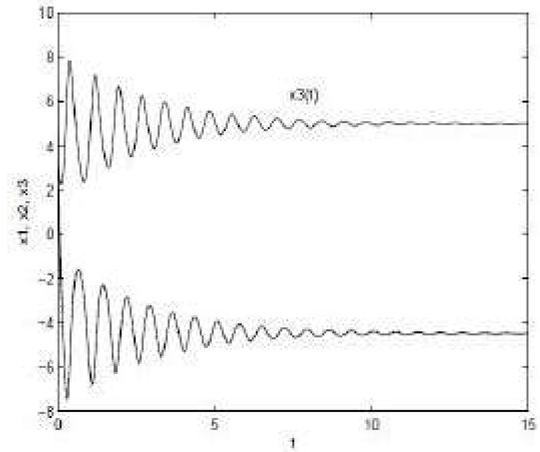

Fig. 5: The equilibrium C^- of fractional-order Chen system is locally asymptotically stable when

$$c > \frac{a}{2} > 0\ , b > c + 3a - \frac{2a^2}{c} > 0$$

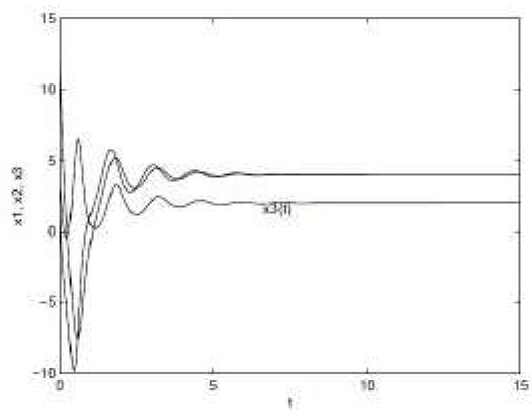

Fig. 6: The C^+ point of fractional-order Chen system is still locally asymptotically stable when

$$b > c + 3a - \frac{2a^2}{c} > 0$$

$c = 19$, the point O is still unstable even though the initial value is small enough, where $x_{10} = 0.02$, $x_{20} = -0.01$, $x_{30} = +0.01$.

Figure 4 and 5 demonstrate that equilibria C^+ and C^- are locally asymptotically stable when

$$c > \frac{a}{2} > 0$$

$$b > c + 3a - \frac{2a^2}{c} > 0$$

where $a = 35$, $b = 4$, $c = 20$ and the initial values are $x_{10} = 8$, $x_{20} = 5$, $x_{30} = 3$ and $x_{10} = 8$, $x_{20} = 2$, $x_{30} = 3$, respectively. As illustrated in Fig. 6, when $a = 6$, $c = 4$ and

$$b = 8 > c + 3a - \frac{2a^2}{c} = 4$$

the equilibria C^+ and C^- still display their locally asymptotical stability. This phenomenon is significantly different from the corresponding integer-order Chen system which exhibits a Hopf bifurcation when

$$c > \frac{a}{2} > 0$$

$$b = c + 3a - \frac{2a^2}{c}$$

i.e. the equilibria C^+ and C^- will lose stability once

$$b < c + 3a - \frac{2a^2}{c}$$

CONCLUSION

The dynamics of fractional-order Chen chaotic system has been investigated in this paper. A strict proof of existence and uniqueness of solutions for the fractional-order Chen chaotic system has been provided, as well as its stability in contrast with the integer-order Chen system. It has been shown that the fractional-order Chen system has the similar equilibria and stability with the integer-order counterpart. However, the bifurcation conditions is fundamentally different between these two systems. Numerical solutions and simulations confirm the given analytic results.

REFERENCES

1. Engheta, N.B., 1996. On fractional calculus and fractional multipoles in electromagnetism. IEEE Transactions on Antennas and Propagation, 44 (4): 554-566.
2. Laskin, N., 2000. Fractional market dynamics, Physica A, 287: 482-492.
3. Hilfer, R., 2000. Applications of Fractional Calculus in Physics. World Scientificc Pub Co. Singapore.
4. Hartley, T., C. Lorenzo and H. Qammer, 1995. Chaos in a fractional order Chua's system. IEEE Trans Circuit System I., 42: 485-490.
5. Li, C. and G. Chen, 2004. Chaos in the fractional order Chen system and its control. Chaos Solitons and Fractals, 22: 549-554.
6. Grigorenko, I. and E. Grigorenko, 2003. Chaos dynamics of fractional Lorenz system. Phys. Rev. Lett., 91: 34-101.
7. Wang, J. and Y. Zhang, 2006. Designing synchronization schemes for chaotic fractional order unified systems. Chaos Solitons and Fractals, 30: 1265-1272.
8. Diethelm, K. and J. Ford, 2002. Analysis of fractional differential equations. J. Math. Anal. Appl., 265: 229-248.
9. Podlubny, I., 1999. Fractional Differential Equations, Academic Press, San Diego.
10. Chen, G. and T. Ueta, 1999. Yet another attractor. Internat J. Bifurc. Chaos, 9: 1465-1466.
11. Li, C.P. and G.J. Peng, 2004. Chaos in Chen's system with a fractional order. Chaos. Solitons and Fractals, 20: 443-450.
12. Peng, G., 2007. Synchronization of fractional order chaotic systems. Phys. Lett. A, 363: 414-419.
13. Lu, J., 2006. Nonlinear observer design to synchronize fractional-order chaotic systems via a scalar transmitted signal. Physica A, 359: 107-118.
14. Sweilam, N., M. Khader and R. Al-Bar, 2007. Numerical studies for a multi-order fractional differential equation. Phys. Lett. A, 1-2: 26-33.
15. Matignon, D., 1996. Stability results of fractional differential equations with applications to control processing. In: IMACS. IEEE-SMC: Lille. France, pp: 963-968.
16. Diethelm, K., J. Ford and A. Freed, 2004. Detailed error analysis for a fractional Adams method. Numer. Algorithms, 36: 31-52.
17. Diethelm, K. and J. Ford, 2004. Multi-order fractional differential equations and their numerical solution. Appl. Math. Comput., 154: 621-640.