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Variational Iteration Method for Evolution Equations

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Abstract: In this paper, we apply the Variational Iteration Method (VIM) for solving evolution equations. The method proved to be fully compatible for the solution of such equations. The higher accuracy of the numerical results shows complete reliability and efficiency of the suggested algorithm. Several examples are given to re-confirm the reliability of the VIM.

Key words: Variational iteration method . error estimates . Boussinesq equations . partial differential equations . evolution equations

INTRODUCTION

The evolution equations are of great significance in the diversified physical problems related to physics, astrophysics, magnetic dynamics, water surface, gravity waves, ion acoustic waves in plasma, electromagnetic radiation reactions, engineering and applied sciences [8, 10, 20, 31]. Several techniques including decomposition, homotopy perturbation and exp-function have been employed to solve such equations analytically and numerically, [8, 10, 20, 31] and the reference therein. Most of these used schemes are coupled with the inbuilt deficiencies like calculation of the so-called Adomian's polynomials, laborious calculation and non compatibility with the physical nature of the problems. He [11-18] developed the variational iteration method (VIM) which proved applicable for a wide class of physical problems, [1-7, 9, 11-18, 21-30] and the references therein. It is to be highlighted that the Variational Iteration Method (VIM) [1-7, 9, 11-18, 21-30] has certain advantages as compare to the decomposition method. Firstly, the use of Lagrange multiplier reduces the successive applications of the integral operator and hence minimizes the computational work to a tangible level while still maintaining a very high level of accuracy. Moreover, VIM is independent of the complexities arising in the calculation of the so-called Adomian's polynomials and this gives it a clear edge over the traditional decomposition method. The VIM is also independent of the small parameter assumption (which is either not there in the physical problems or difficult to locate) and hence is more convenient to apply as compare to the traditional perturbation method. It is worth mentioning that variational iteration method (VIM) is applied without any discretization, restrictive assumption or transformation and is free from round off errors. We apply the proposed VIM for all the nonlinear terms in the problem without discretizing either by finite difference or spline techniques at the nodes, involves laborious calculations coupled with a strong possibility of the ill-conditioned resultant equations which is a complicated problem to solve. Moreover, unlike the method of separation of variables that requires initial and boundary conditions, the VIM [1-7, 9, 11-18, 21-30] provides the solution by using the initial conditions only.

VARIATIONAL ITERATION METHOD (VIM)

To illustrate the basic concept of the He's VIM, we consider the following general differential equation

$$Lu + Nu = g(x) \quad (1)$$

where L is a linear operator, N a nonlinear operator and g(x) is the inhomogeneous term. According to variational iteration method [1-7, 9, 11-19, 21-30], we can construct a correction functional as follows

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda (Lu_n(s) + N\tilde{u}_n(s) - g(s))ds \quad (2)$$

where λ is a Lagrange multiplier [11-18], which can be identified optimally via variational iteration method. The subscripts n denote the nth approximation, $\tilde{u}_n$ is considered as a restricted variation. i.e. $\delta \tilde{u}_n = 0$; (2) is called a correction functional. The solution of the linear problems can be solved in a single iteration step due to the exact identification of the Lagrange multiplier. The principles of variational iteration method and its

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applicability for various kinds of differential equations are given in [11-18]. In this method, it is required first to determine the Lagrange multiplier λ optimally. The successive approximation u_{n+1}, $n\geq 0$ of the solution u will be readily obtained upon using the determined Lagrange multiplier and any selective function u_0, consequently, the solution is given by $u=\lim_{n\to\infty} u_n$.

NUMERICAL APPLICATIONS

In this section, we apply variational iteration method (VIM) for solving evolution equations. Numerical results are very encouraging.

Example 3.1: Consider the following (1+1)-dimensional new coupled modified KdV equation

$$u_t = \frac{1}{2}u_{xxx} - 3u^2u_x + \frac{3}{2}v_{xx} + 3uv_x + 3vu_x - 3\alpha u_x$$

$$v_t = -v_{xxx} - 3vu_x + 3u_xv_x + 3u^2v_x + 3\alpha u_x = 0$$

with initial conditions

$$u(x,0) = \frac{b}{2k} + k\tanh(kx)$$

$$v(x,0) = \frac{\alpha}{2}\left(1+\frac{k}{b}\right) + b\tanh(kx)$$

where k, b and α are arbitrary constants. The correction functional is given by

$$\begin{cases} u_{n+1}(x,t) = u_n + \int_0^t \lambda_1(s)\left(\frac{\partial u_n}{\partial s} - \frac{1}{2}(\tilde{u}_n)_{xxx} + 3(\tilde{u}_n)^2(\tilde{u}_n)_x - \frac{3}{2}(\tilde{v}_n)_{xx} - 3(\tilde{u}_n)(\tilde{v}_n)_x - 3(\tilde{v}_n)(\tilde{u}_n)_x + 3\alpha(\tilde{u}_n)_x\right)ds \\ v_{n+1}(x,t) = v_n + \int_0^t \lambda_2(s)\left(\frac{\partial v_n}{\partial s} + (\tilde{v}_n)_{xxx} + 3\tilde{v}_n(\tilde{u}_n)_x - 3(\tilde{u}_n)_x(\tilde{v}_n)_x + 3(\tilde{u}_n)^2(v_n)_x + 3\alpha(\tilde{u}_n)_x\right)ds \end{cases}$$

Making the above functional stationary, the Lagrange multiplier can be identified as $\lambda_1(s) = \lambda_2(s) = -1$, we get

$$\begin{cases} u_{n+1}(x,t) = u_n - \int_0^t \left(\frac{\partial u_n}{\partial s} - \frac{1}{2}(u_n)_{xxx} + 3(u_n)^2(u_n)_x - \frac{3}{2}(v_n)_{xx} - 3(u_n)(v_n)_x - 3(v_n)(u_n)_x + 3\alpha(u_n)_x\right)ds \\ v_{n+1}(x,t) = v_n - \int_0^t \left(\frac{\partial v_n}{\partial s} + (v_n)_{xxx} + 3v_n(u_n)_x - 3(u_n)_x(v_n)_x + 3(u_n)^2(v_n)_x + 3\alpha(u_n)_x\right)ds \end{cases}$$

Consequently, following approximants are obtained

$$\begin{cases} u_0(x,t) = \frac{b}{2k} + k\tanh(kx) \\ v_0(x,t) = \frac{\alpha}{2}\left(1+\frac{k}{b}\right) + b\tanh(kx) \end{cases}$$

$$\begin{cases} u_1(x,t) = \frac{b}{2k} + k\tanh(kx) + \frac{tk^2}{4}\left(-4k^2 - 6\alpha + \frac{6\alpha k}{b} + \frac{3b^2}{k^2}\right)\operatorname{sech}^2(kx) \\ v_1(x,t) = \frac{\alpha}{2}\left(1+\frac{k}{b}\right) + b\tanh(kx) + \frac{bkt}{4}\left(-4k^2 - 6\alpha + \frac{6\alpha k}{b} + \frac{3b^2}{k^2}\right)\operatorname{sech}^2(kx) \end{cases}$$

$$\begin{cases} u_2(x,t) = \frac{b}{2k} + k\tanh(kx) + \left(-4k^2 - 6\alpha + \frac{6\alpha k}{b} + \frac{3b^2}{k^2}\right)\left(\frac{tk^2}{4}\operatorname{sech}^2(kx) + \frac{k^3}{8}t^2\operatorname{sech}^2(kx)\tanh(kx)\right) \\ v_2(x,t) = \frac{\alpha}{2}\left(1+\frac{k}{b}\right) + b\tanh(kx) + \left(-4k^2 - 6\alpha + \frac{6\alpha k}{b} + \frac{3b^2}{k^2}\right)\left(\frac{bkt}{4}\operatorname{sech}^2(kx) + \frac{bk^2}{8}t^2\operatorname{sech}^2(kx)\tanh(kx)\right) \end{cases}$$

The series solution is given by

$$\begin{cases} u(x,t)=\frac{b}{2k}+k\tanh(kx)+\left(-4k^2-6\alpha+\frac{6\alpha k}{b}+\frac{3b^2}{k^2}\right)\left(\frac{tk^2}{4}\mathrm{sech}^2(kx)+\frac{k^3}{8}t^2\,\mathrm{sech}^2(kx)\tanh(kx)\right)+\cdots \\ v(x,t)=\frac{\alpha}{2}\left(1+\frac{k}{b}\right)+b\tanh(kx)+\left(-4k^2-6\alpha+\frac{6\alpha k}{b}+\frac{3b^2}{k^2}\right)\left(\frac{bkt}{4}\mathrm{sech}^2(kx)+\frac{bk^2}{8}t^2\,\mathrm{sech}^2(kx)\tanh(kx)\right)+\cdots \end{cases}$$

Example 3.2: Consider the following (1+1)-dimensional nonlinear Boussinesq equation

$$u_t+v_x+u\,u_x=0$$
$$v_t+(v\,u)_x+u_{xxx}=0$$

with initial conditions

$$u(x,0)=\alpha-2k\tanh(kx)$$

$$v(x,0)=2k^2\,\mathrm{sech}(kx)$$

where k and α are arbitrary constants. The correction functional is given by

$$\begin{cases} u_{n+1}(x,t)=u_n+\int_0^t \lambda_1(s)\left(\frac{\partial u_n}{\partial s}+(\tilde{v}_n)_x+\tilde{u}_n(\tilde{u}_n)_x\right)ds \\ v_{n+1}(x,t)=v_n+\int_0^t \lambda_2(s)\left(\frac{\partial v_n}{\partial s}+(\tilde{v}_n\tilde{u}_n)_x+(\tilde{u}_n)_{xxx}\right)ds \end{cases}$$

Making the above functional stationary, the Lagrange multiplier can be identified as $\lambda_1(s)=\lambda_2(s)=-1$, we get

$$\begin{cases} u_{n+1}(x,t)=\alpha-2k\tanh(kx)-\int_0^t \left(\frac{\partial u_n}{\partial s}+(v_n)_x+u_n(u_n)_x\right)ds \\ v_{n+1}(x,t)=2k^2\mathrm{sech}(kx)-\int_0^t \left(\frac{\partial v_n}{\partial s}+(v_n u_n)_x+(u_n)_{xxx}\right)ds \end{cases}$$

Table 1: Error Estimates, b = 0.1, t = 0.1, α = 0

x	u_{exact}	VIM	*Error
-50	0.400009	0.400009	9.991 E-10
-40	0.4000068	0.4000068	7.374 E-9
-30	0.400502	0.400502	5.402 E-8
-20	0.403650	0.403650	3.746 E-7
-10	0.424153	0.424151	1.755 E-6
00	0.500740	0.500740	1.350 E-8
10	0.576468	0.576470	1.747 E-6
20	0.596455	0.596455	3.712 E-7
30	0.599513	0.599513	5.340 E-8
40	0.599934	0.599934	7.302 E-9
50	0.599991	0.599991	9.893 E-10

*Error = Exact solution-Series solution

Table 2: Error Estimates, b = 0.1, t = 0.1, α = 0

x	v_{exact}	VIM	*Error
-50	-0.0999908	-0.0999908	9.99 E-10
-40	-0.0999319	-0.0999319	7.3744E-9
-30	-0.0994981	-0.0994982	5.4023E-8
-20	-0.0963501	-0.0963501	3.746 E-7
-10	-0.0758469	-0.0758486	1.755 E-6
00	0.000739986	0.0007400	1.350 E-8
10	0.0764684	0.0764702	1.747 E-6
20	0.0964547	0.0964550	3.712 E-7
30	0.0995127	0.0995128	5.340 E-8
40	0.0999339	0.0999339	7.302 E-9
50	0.0999911	0.0999911	9.893 E-10

*Error = Exact solution-Series solution

Table 3: Error Estimates, t = 1, k = α = 0.1

x	u_{exact}	VIM	*Error
-50	0.2999982	0.2999982	2.4080 E-11
-40	0.2998680	0.2998690	1.7750 E-10
-30	0.2990300	0.2990300	1.2890 E-9
-20	0.2929450	0.2929450	8.3861 E-9
-10	0.2531520	0.2531520	2.0775 E-8
00	0.1020000	0.1020000	6.6664 E-8
10	-0.0514725	-0.0514725	2.0664 E-8
20	-0.0926628	-0.0926629	8.4576 E-9
30	-0.0989910	-0.0989910	1.3023 E-9
40	-0.09986362	-0.09986362	1.7932 E-10
50	-0.0999815	-0.09998150	2.4325 E-11

Error = Exact solution-Series solution

Table 4: Error Estimates, t = 1, k = α = 0.1

x	v_{exact}	VIM	*Error
-50	3.355975924 E-6	3.35597592400	7.213200 E-10
-40	0.00026288310	0.00002628299	5.317599 E-9
-30	0.00019343247	0.00019339384	3.862728 E-8
-20	0.00138602398	0.00138577275	2.512270 E-7
-10	0.00827217062	0.00827154683	6.237975 E-7
00	0.01999800010	0.20000000000	1.999866 E-6
10	0.00852804619	0.00852742683	6.193631 E-7
20	0.00144051432	0.00144026000	2.540901 E-7
30	0.00020128677	0.00020124764	3.913538 E-8
40	0.00002736042	0.00002735503	5.388782 E-9
50	3.70502231 E-6	3.70429131 E-6	7.309980 E-10

*Error = Exact solution-Series solution

Consequently, following approximants are obtained

$$\begin{cases} u_0(x,t) = \alpha - 2k\tanh(kx) \\ v_0(x,t) = 2k^2 \operatorname{sech}(kx) \end{cases}$$

$$\begin{cases} u_1(x,t) = \alpha - 2k\tanh(kx) + 2\alpha t k^2 \operatorname{sech}^2(kx) \\ v_1(x,t) = 2k^2 \operatorname{sech}(kx) + 4\alpha t\, k^3 \operatorname{sech}^2(kx)\tanh(kx) \end{cases}$$

$$\begin{cases} u_2(x,t) = \alpha - 2k\tanh(kx) + 2\,\alpha t k^2 \operatorname{sech}^2(kx) \\ \qquad + 2\alpha^2 k^3 t^2 \operatorname{sech}^2(kx)\tanh(kx) \\ v_2(x,t) = 2k^2 \operatorname{sech}(kx) + 4\alpha t k^3 \operatorname{sech}^2(kx)\tanh(kx) \\ \qquad + 2\alpha^2 k^4 t^2 \operatorname{sech}^4(kx)\left(\cosh(2kx) - 2\right) \end{cases}$$

$$\vdots$$

The series solution is given by

$$\begin{cases} u(x,t) = \alpha - 2k\tanh(kx) + 2\alpha t k^2 \operatorname{sech}^2(kx) \\ \qquad + 2\alpha^2 k^3 t^2 \operatorname{sech}^2(kx)\tanh(kx) + \cdots \\ v(x,t) = 2k^2 \operatorname{sech}(kx) + 4\alpha t k^3 \operatorname{sech}^2(kx)\tanh(kx) \\ \qquad + 2\alpha^2 k^4 t^2 \operatorname{sech}^4(kx)\left(\cosh(2kx) - 2\right) + \cdots \end{cases}$$

Example 3.3: Consider the following (2+1)-dimensional dispersive long wave equation

$$u_t + v_x + \frac{1}{2}\left(u^2\right)_x = 0$$
$$v_t + (vu)_x\,(u)_{xxx} = 0$$

with initial conditions

$$u(x,y,0) = \alpha - 2k\tanh(kx),\quad v(x,y,0) = 2k^2 \operatorname{sech}(kx)$$

where α and α_0 are arbitrary constants. The correction functional is given by

$$\begin{cases} u_{n+1}(x,t) = u_n + \int_0^t \lambda_1(s)\left(\frac{\partial u_n}{\partial s} + (\tilde{v}_n)_x + \frac{1}{2}\left(\tilde{u}_n^2\right)_x\right) ds \\ v_{n+1}(x,t) = v_n + \int_0^t \lambda_2(s)\left(\frac{\partial v_n}{\partial s} + (\tilde{v}_n \tilde{u}_n)_x + (\tilde{u}_n)_{xxx}\right) ds \end{cases}$$

Making the above functional stationary, the Lagrange multiplier can be identified as $\lambda_1(s) = \lambda_2(s) = -1$, we get

$$\begin{cases} u_{n+1}(x,t) = \alpha - 2k\tanh(kx) - \int_0^t \left(\frac{\partial u_n}{\partial s} + (v_n)_x + \frac{1}{2}\left(u_n^2\right)_x\right) ds \\ v_{n+1}(x,t) = 2k^2 \operatorname{sech}(kx) - \int_0^t \left(\frac{\partial v_n}{\partial s} + (v_n u_n)_x + (u_n)_{xxx}\right) ds \end{cases}$$

Table 5: Error estimates, t = 0.3, α = 0.1, α_0 = 0.2, y = 10

X	u_{exact}	VIM	*Error
-50	0.00358666	0.00358664	1.531039 E-8
-40	0.00945811	0.00945807	3.677279 E-8
-30	0.02377770	0.02377760	7.193120 E-8
-20	0.05367040	0.05367030	8.180368 E-8
-10	0.099850	0.099850	1.124990 E-10
00	0.146094	0.146094	8.174000 E-8
10	0.176096	0.176096	7.200120 E-8
20	0.190488	0.190488	3.683206 E-8
30	0.196392	0.196392	1.533882 E-8
40	0.198657	0.198657	5.908909 E-9
50	0.199507	0.199507	2.211069 E-9

*Error = Exact solution-Series solution

Table 6: Error estimates, t = 0.3, α = 0.1, α_0 = 0.2, y = 10

x	v_{exact}	VIM	*Error
-50	-0.9999648	-0.9999648	1.419972 E-9
-40	-0.999099	-0.999099	2.962113 E-9
-30	-0.997905	-0.997905	3.498514 E-9
-20	-0.996073	-0.996073	3.168190 E-9
-10	-0.995000	-0.995000	3.168190 E-9
00	-0.996062	-0.996062	1.124998 E-8
10	-0.997895	-0.997895	3.190421 E-9
20	-0.999094	-0.999094	3.494773 E-9
30	-0.999646	-0.999646	2.965481 E-9
40	-0.999867	-0.999867	5.750020 E-10
50	-0.999951	-0.999951	2.189150 E-10

*Error = Exact solution-Series solution

Consequently, following approximants are obtained

$$\begin{cases} u_0(x,t) = \alpha - 2k\tanh(kx) \\ v_0(x,t) = 2k^2 \operatorname{sech}(kx) \end{cases}$$

$$\begin{cases} u_1(x,t) = \alpha - 2k\tanh(kx) + 2\alpha t k^2 \operatorname{sech}^2(kx) \\ v_1(x,t) = 2k^2 \operatorname{sech}(kx) + 4\alpha t\, k^3 \operatorname{sech}^2(kx)\tanh(kx) \end{cases}$$

$$\begin{cases} u_2(x,t) = \alpha - 2k\tanh(kx) + 2\,\alpha t k^2 \operatorname{sech}^2(kx) \\ \qquad + 2\alpha^2 k^3 t^2 \operatorname{sech}^2(kx)\tanh(kx) \\ v_2(x,t) = 2k^2 \operatorname{sech}(kx) + 4\alpha t k^3 \operatorname{sech}^2(kx)\tanh(kx) \\ \qquad + 2\alpha^2 k^4 t^2 \operatorname{sech}^4(kx)\left(\cosh(2kx) - 2\right) \end{cases}$$

$$\vdots$$

The series solution is given by

$$\begin{cases} u(x,t) = \alpha - 2k\tanh(kx) + 2\alpha t k^2 \operatorname{sech}^2(kx) \\ \qquad + 2\alpha^2 k^3 t^2 \operatorname{sech}^2(kx)\tanh(kx) + \cdots \\ v(x,t) = 2k^2 \operatorname{sech}(kx) + 4\alpha t k^3 \operatorname{sech}^2(kx)\tanh(kx) \\ \qquad + 2\alpha^2 k^4 t^2 \operatorname{sech}^4(kx)\left(\cosh(2kx) - 2\right) + \cdots \end{cases}$$

Table 7: Error estimates, t = 0.3, α = 0.1, α_0 = 0.2, y = 10

x	u_{exact}	VIM	*Error
-50	0.1018151	0.1018151	6.91529 E-8
-40	0.104783	0.104783	1.66013 E-7
-30	0.112015	0.112015	3.24319 E-7
-20	0.127071	0.127071	1.51873 E-9
-10	0.150225	0.150225	3.68399 E-7
00	0.173282	0.173282	3.23375 E-7
10	0.188174	0.188174	1.51873 E-9
20	0.195298	0.195298	1.65211 E-7
30	1.982170	1.982170	6.87692 E-8
40	0.199337	0.199337	2.64868 E-8
50	0.199755	0.199755	0.91050 E-9

*Error = Exact solution-Series solution

Table 8: Error Estimates, t = 0.3, α = 0.1, α_0 = 0.2, y = 10

x	v_{exact}	VIM	*Error
-50	0.00181459	0.00181452	6.95129 E-8
-40	0.00478341	0.00478325	1.66011 E-7
-30	0.0120151	0.0120148	3.34319 E-7
-20	0.0270715	0.0270711	3.67411 E-7
-10	0.0502250	0.0502250	1.51873 E-9
00	0.0732824	.07328240	3.68399 E-7
10	0.0881739	0.0881742	3.23375 E-7
20	0.0952979	0.0952981	1.65211 E-7
30	0.0982172	0.0982173	6.87692 E-8
40	0.9933670	0.0993367	2.64868 E-8
50	0.0997549	0.0997550	9.91050 E-9

*Error = Exact solution-Series solution

Example 3.4: Consider the following (2+1)-dimensional Boiti-Leon-Pempinelli equation

$$u_{ty} - \left(u^2 - u_x\right)_{xy} - 2\,v_{xxx} = 0$$
$$v_t - v_{xx} - 2\,u\,v_x = 0$$

with initial conditions

$$u(x,y,0) = \frac{1}{2}\alpha\,\tanh\left(\frac{\alpha}{2}(x+y)\right) + \alpha_0 - \frac{\alpha}{2}$$

$$v(x,y,0) = \frac{1}{2}\alpha\tanh\left(\frac{\alpha}{2}(x+y)\right)\beta_0 - \frac{\alpha}{2}$$

where α, α_0 and β_0 are arbitrary constants. Proceeding as before, the series solution is given by

$$\begin{cases} u(x,y,t) = \frac{1}{2}\alpha\,\tanh\left(\frac{\alpha}{2}(x+y)\right) + \left(\alpha_0 - \frac{\alpha}{2}\right) - \frac{\alpha^2 t}{4}(\alpha - 2\alpha_0)\,\mathrm{sech}^2\left(\frac{\alpha}{2}(x+y)\right) \\ \qquad \frac{1}{8}\alpha^3(\alpha - 2\alpha_0)^2 t^2\left(\mathrm{sech}^2\left(\frac{\alpha}{2}(x+y)\right)\tanh\left(\frac{\alpha}{2}(x+y)\right)\right) + \cdots \\ v(x,y,t) = \frac{1}{2}\alpha\,\tanh\left(\frac{\alpha}{2}(x+y)\right) + \left(\beta_0 - \frac{\alpha}{2}\right) - \frac{\alpha^2 t}{4}(\alpha - 2\alpha_0)\,\mathrm{sech}^2\left(\frac{\alpha}{2}(x+y)\right) \\ \qquad \frac{1}{8}\alpha^3(\alpha - 2\alpha_0)^2 t^2\left(\mathrm{sech}^2\left(\frac{\alpha}{2}(x+y)\right)\tanh\left(\frac{\alpha}{2}(x+y)\right)\right) + \cdots \end{cases}$$

CONCLUSION

In this paper, we applied the Variational Iteration Method (VIM) for solving evolution equations. The obtained results are of higher accuracy showing the complete reliability and efficiency of the proposed technique. The fact that the VIM is independent of the complexities arising in the calculation of the so-called Adomian's polynomials is a clear advantage of this technique over the decomposition method. Moreover, the suggested technique is independent of the small parameter assumption and hence is more convenient as compare to perturbation method.

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