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# Normal Filters in BL-Algebras

[1]A. Borumand Saeid and [2]S. Motamed

[1]Department of Mathematics, Shahid Bahonar University of Kerman, Iran
[2]Department of Mathematics, Islamic Azad University Kerman Branch, Iran

**Abstract:** In this paper we introduced the notion of normal filter in BL-algebras and we stated and proved some theorems which determine the relationship between this notion and other filters of BL-algebras and by some examples we show that this notion is different. Also we consider some relations between these filters and quotient algebras that are constructed via these filters.

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## INTRODUCTION

BL-algebra have been invented by P. Hajek [6, 7] in order to provide an algebraic proof of the completeness of “Basic Logic” (BL, for short) arising from the continuous triangular norms, familiar in the fuzzy Logic framework. The language of propositional Hajek basic logic [6] contains the binary connectives o and $\Rightarrow$ and the constant $\overline{O}$. Axioms of BL are:

$(A1)(\varphi \Rightarrow \psi) \Rightarrow ((\psi \Rightarrow \omega) \Rightarrow (\varphi \Rightarrow \omega))$
$(A2)(\varphi \circ \psi) \Rightarrow \varphi$
$(A3)(\varphi \circ \psi) \Rightarrow (\psi \circ \varphi)$
$(A4)(\varphi \circ (\varphi \Rightarrow \psi)) \Rightarrow (\psi \circ (\psi \Rightarrow \varphi))$
$(A5a)(\varphi \Rightarrow (\psi \Rightarrow \omega)) \Rightarrow ((\varphi \circ \psi) \Rightarrow \omega)$
$(A5b)((\varphi \circ \varphi) \Rightarrow w) \Rightarrow (\varphi \Rightarrow (\psi \Rightarrow \omega))$
$(A6)((\varphi \Rightarrow \psi) \Rightarrow w) \Rightarrow (((\psi \Rightarrow \varphi) \Rightarrow \omega) \Rightarrow \omega)$
$(A7)\overline{O} \Rightarrow \omega$

After Cignoli [3] proved that Hajek’s logic really is the logic of continuous t-norms as conjectured by Hajek. At the same time started a systematic study of BL-algebras, too. Indeed, Turunen [10, 12] published where BL-algebras were studied by deductive systems. In Turunen [11, 13], Boolean deductive systems and implicative deductive systems were introduced. Moreover, it was proved that these deductive systems coincide. In Haveshki [8] continued an algebraic analysis of BL-algebras and then introduced implicative filters of BL-algebras. Notice that implicative deductive systems and implicative filters are not, the same subsets. In Kondo [9], proved that positive implicative filters introduced in Haveshki [8] and the original implicative deductive systems introduced in Turunen [11] coincide and every positive implicative filter is a Boolean filter with out maximality condition. The above notions are generalized to an algebraic system in which the required conditions are fulfilled (Definition 1). Then we recall the basic definitions and theorems and we put in evidence many rules of calculus in BL-algebras which we need in the rest of the paper. Finally, we define the notion of normal filter and we state and prove some theorems that determine relationship between this notion and other filters of BL-algebras. We prove that if A/F is a Boolean algebra then F is a normal filter of A.

**Definition 1:** [6] A BL-algebra is an algebra $(A,\wedge,\vee,*,\rightarrow,0,1)$ with four binary operations $\wedge$, $\vee$, $*$, $\rightarrow$ and two constants 0, 1 such that:

(BL1) $(A,\wedge,\vee,0,1)$ is a bounded lattice,
(BL2) $(A, *, 1)$ is a commutative monoid,
(BL3) $*$ and $\rightarrow$ form an adjoint pair i.e, $c \le a \rightarrow b$ if and only $a*c \le b$ for all $a,b,c \in A$,
(BL4) $a \wedge b = a*(a \rightarrow b)$,
(BL5) $(a \rightarrow b) \vee (b \rightarrow a) = 1$

A BL-algebra is called a Godel algebra if $a^2 = a*a = a$, for all $a \in A$ and BL-algebra is called an MV-algebra if $x^{--} = x$, for all $x \in A$, where $x^- = x \rightarrow 0$.

**Lemma 1:** [1, 2, 4, 5, 6, 12] In each BL-algebra A, the following relations hold for all $x,y,z \in A$,

**Corresponding Author:** Dr. Arsham Borumand Saeid, Department of Mathematics, Shahid Bahonar University of Kerman, Kerman, Iran. E-mail:arsham@mail.uk.ac.ir

(1) $x*(x\to y)\le y$

(2) $x\le y\to (x*y)$

(3) $x\le y$ if only if $x\to y=1$

(4) $x\to(y\to z)=y\to(x\to z)$

(5) If $x\le y$, then $y\to z\le x\to z$ and $z\to x\le z\to y$

(6) $y\le(y\to x)\to x$

(7) $y\to x\le(z\to y)\to(z\to x)$

(8) $x\to y\le(y\to z)\to(x\to z)$

(9) $x*y\le x,y$, hence $x*y\le x\wedge y$ and $x*0=0$

(10) $x\le y$ implies $x*z\le y*z$

(11) $1\to x=x, x\to x=1, x\le y\to x, x\to 1=1$

(12) $x*x^{-}=0$

(13) $x*y=0$ iff $x\le y^{-}$

(14) $x\vee y=1$ implies $x*y=x\wedge y$

(15) $x\to(y\to z)=(x*y)\to z=y\to(x\to z)$

(16) $(x\to y)\to(x\to z)=(x\wedge y)\to z$

(17) $x\to(y\to z)\ge(x\to y)\to(x\to z)$

(18) $x\le y$ implies $y^{-}\le x^{-}$

(19) $((x\to y)\to y)\to y=x\to y$

(20) $x*(y\vee z)=(x*y)\vee(x*z)$

(21) $x*(y\wedge z)=(x*y)\wedge(x*z)$

(22) $(x\wedge y)^{n}=x^{n}\wedge y^{n},(x\vee y)^{n}=x^{n}\vee y^{n}$, such that n is positive integer,

(23) $x\to(y\wedge z)=(x\to y)\wedge(x\to z)$

(24) $(y\wedge z)\to x=(y\to x)\vee(z\to x)$

(25) $(x\vee y)\to z=(x\to z)\wedge(y\to z)$

(26) $x\to y\le(x*z)\to(y*z)$

(27) $x*(y\to z)\le y\to(x*z)$

(28) $(y\to z)*(x\to y)\le x\to z$

(29) $x,y\le z$ and $z\to x=z\to y$ implies $x=y$

(30) $x\vee(y*z)\ge(x\vee y)*(x\vee z)$

(31) $x^{m}\vee y^{n}\ge(x\vee y)^{mn}$, such that m, n are positive integers,

(32) $(x\to y)\to z\le((y\to x)\to z)\to z$

(33) $x\le x^{--}, 1^{-}=0, 0^{-}=1, x^{---}=x^{-}, x^{--}\le x^{-}\to x$

(34) $(x\wedge y)^{-}=x^{-}\vee y^{-}$ and $(x\vee y)^{-}=x^{-}\wedge y^{-}$

(35) $(x\wedge y)^{--}=x^{--}\wedge y^{--}$

$(x\vee y)^{--}=x^{--}\vee y^{--},(x\to y)^{--}=x^{--}\to y^{--}$

(36) If $x^{--}\le x^{--}\to x$, then $x^{--}=x$

(37) $x=x^{--}*(x^{--}\to x)$

(38) $x\to y^{-}=y\to x^{-}=x^{--}\to y^{-}=(x*y)^{-}$

(39) $(x^{--}\to x)^{-}=0$, $(x^{--}\to x)\vee x^{--}=1$

(40) $y^{-}\le x$ implies $x\to(x*y)^{--}=y^{--}$

(41) $x\vee y=[(x\to y)\to y]\wedge[(y\to x)\to x]$

Where

$$x^{n}=\underbrace{x*\ldots*x}_{n-\text{times}}$$

and n is a positive integer.

**Definition 2:** [6] A filter of a BL-algebra A is a non-empty subset F of A such that for all $x,y\in A$ we have

$$x,y\in F \text{ implies } x*y\in F \quad (1)$$

$$x\in F \text{ and } x\le y \text{ imply } y\in F \quad (2)$$

**Definition 3:** [12] A non-empty subset D of BL-algebra A is called a deductive system if

(1) $1 \in D$

(2) If $x \in D$ and $x \to y \in D$ imply $y \in D$.

**Proposition 1:** [11] A non-empty subset F of BL-algebra is a deductive system if and only if F is a filter.

**Theorem 1:** [6] Let F be a filter of a BL-algebra A. Define: $x \equiv_F y$ if and only if $x \to y \in F$ and $y \to x \in F$. Then $\equiv_F$ is a congruence relation on A. The set of all congruence classes is denoted by A/F, i.e,

$$A/F := \{[x] \mid x \in A\}, \text{ where } [x] = \{y \in A \mid x \equiv_F y\}$$

Define $\bullet, \to, \sqcap, \sqcup$ on A/F as follows

$$[x] \bullet [y] = [x * y], [x] \to [y] = [x \to y]$$

$$[x] \sqcap [y] = [x \wedge y], [x] \sqcup [y] = [x \vee y]$$

Therefore $(A/F, \sqcap, \sqcup, \bullet, \to, [1], [0])$

is a BL-algebra with respect to F.

**Definition 4:** [11] Let A be a BL-algebra and F be a filter of A. F is called a Boolean filter if

$x \vee \bar{x} \in F$, for all $x \in A$

**Theorem 2:** [11] Let D be a subset of the BL-algebra A. Then the following are equivalent:

D is a Boolean filter, (1)

A/D is a Boolean algebra. (2)

**Definition 5:** [8] A non-empty subset F of A is called an implicative filter of A of it satisfies:

(1) $1 \in F$

(2) $x \to (y \to z) \in F$ and $x \to y \in F$ imply that $x \to z \in F$

for all x, y, $z \in A$.

**Definition 6:** [8] A non-empty subset F of A is called a positive implicative filter if it satisfies:

(1) $1 \in F$

(2) $x \to ((y \to z) \to y) \in F$ and $x \in F$ imply $y \in F$

for all x, y, $z \in A$.

**Definition 7:** [8] A nonempty subset F of A is called a fantastic filter if it satisfies:

$$1 \in F \quad (1)$$

$$z \to (y \to x) \in F \text{ and } z \in F \text{ imply } ((x \to y) \to y) \to x \in F \quad (2)$$

for all x, y, x, $z \in A$

**Definition 8:** [4, 6, 12] Let A and B are BL-algebras. A function f: $A \to B$ is called homomorphism of BL-algebras if and only if:

(1) $f(0) = 0$

(2) $f(x * y) = f(x) * f(y)$

(3) $f(x \to y) = f(x) \to f(y)$

for all x, $y \in A$

**Remark 1:** [4, 6, 12] Let f: $A \to B$ be a BL-homomorphism. Then

$$f(1) = 1, f(\bar{x}) = [f(x)]^-$$

$$f(x \wedge y) = f(x) \wedge f(y), f(x \vee y) = f(x) \vee f(y)$$

## NORMAL FILTERS

From now $(A, \wedge, \vee, *, \to, 0, 1)$ or simply A is a BL-algebra.

**Definition 9:** Let F be a filter of A. F is called a normal filter of A if it satisfies:

$$z \to ((y \to x) \to x) \in F \text{ and } z \in F \text{ imply that}$$
$$(x \to y) \to y \in F, \text{ for all } x, y \in A$$

In the following we show that any filter need not be a normal filter.

**Example 1:** Let A= [0, 1]. Define * and $\to$ as follow:

$$x * y = m,in \{x, y\}$$

and

$$x \to y = \begin{cases} 1 & \text{if } x \le y \\ y & \text{if } x > y \end{cases}$$

Then $(A, \wedge, \vee, *, \to, 0, 1)$ is a BL-algebra and F = [1/2,1] is a filter. But is not a normal filter, since $1 \in F$ and

$$1 \to \left(\left(\frac{1}{4} \to \frac{1}{5}\right) \to \frac{1}{5}\right) = 1 \in F$$

but

$$\left(\frac{1}{5} \to \frac{1}{4}\right) \to \frac{1}{4} = \frac{1}{4} \notin F$$

**Theorem 3:** Let F be a filter of A. Then F is a normal filter if only if $(y \to x) \to x \in F$ implies $(x \to y) \to y \in F$, for all $x, y \in A$.

**Proof:** Let F be a normal filter of A and $(y \to x) \to x \in F$, for all $x, y \in A$. Since $1 \to ((y \to x) \to x) = (y \to x) \to x$, thus

$1 \to ((y \to x) \to x) \in F$. $1 \in F$ implies that $(x \to y) \to y \in F$.

Conversely, let $z \to ((y \to x) \to x) \in F$ and $z \in F$, for all $x,y,z \in A$. Since F is a filter, then $(y \to x) \to x \in F$, by hypothesis we get that $(x \to y) \to y \in F$. Hence F is a normal filter of A.

**Theorem 4:** Let F be an implicative filter of A. Then F is a positive implicative filter of A if and only if is a normal filter.

**Proof:** By Theorems 3.14[8] and 3 is clear.

**Corollary 1:** Let F be a positive implicative filter of A. Then F is a normal filter, but the converse is not true.

**Example 2:** Let B={0,a,b,1}. Define * and → as follow:

| * | 0 | a | b | 1 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| a | 0 | 0 | 0 | a |
| b | 0 | 0 | a | b |
| 1 | 0 | a | b | 1 |

| → | 0 | a | b | 1 |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| a | b | 1 | 1 | 1 |
| b | a | b | 1 | 1 |
| 1 | 0 | a | b | 1 |

Then $(B,\wedge,\vee,*,\to,0,1)$ is a BL-algebra and it is clear that F={1} is a normal filter, while it is not a positive implicative filter, since

$$(b \to 0) \to b = a \to b = 1 \in F, \text{ but } b \notin F.$$

**Corollary 2:** Let F be an implicative filter of A. Then F is normal filter if and only if $(x \to y) \to x \in F$ implies $x \in F$.

**Proof:** Let $(x \to y) \to y \in F$. It is enough to show that $(y \to x) \to x \in F$. By Lemma 1, we have

$$\begin{aligned}(x \to y) \to y &\le (y \to x) \to ((x \to y) \to x) \\ &= (x \to y) \to ((y \to x) \to x)\end{aligned}$$

Since F is a filter, then

$$(x \to y) \to (y \to x) \to x \in F \qquad (1)$$

By Lemma 1, we have

$$\begin{aligned}x \le y \vee x &= ((y \to x) \to x) \wedge ((x \to y) \to y) \\ &\le (y \to x) \to x\end{aligned}$$

Therefore $x \le (y \to x) \to x$ and so $((y \to x) \to x) \to y \le x \to y$, thus $(x \to y) \to ((y \to x) \to x) \le (((y \to x) \to x) \to y) \to ((y \to x) \to x)$. Then by (1) and filter property, we get that

$$(((y \to x) \to x) \to y) \to ((y \to x) \to x) \in F$$

By hypothesis we can conclude $(y \to x) \to x \in F$. Therefore F is a normal filter.

For the converse see Corollary 1 and Theorem 3.13 [8].

**Corollary 3:** Let F be a filer of A. If $((x \to y) \to x) \in F$, implies $x \in F$, then F is a normal filter.

**Proof:** By Theorem 3.13[8] and Corollary 1, is clear.

**Corollary 4:** Let F be a filter of A. If $z \to ((x \to y) \to x) \in F$ and $z \in F$ implies $x \in F$, then F is a normal filter.

**Corollary 5:** Let F be a filter of A. If $(a \to 0) \to a \in F$ implies $a \in F$, then F is a normal filter.

**Proof:** By Proposition 6 [9], we have F is a positive implicative filter, by Corollary 1, we get that F is a normal filter.

**Proposition 2:** Let F be a filter of A. If $(x^{-} \to x) \to x \in F$, then F is a normal filter.

**Proof:** By Proposition 7 [9], we have F is a positive implicative filter. Thus F is a normal filter.

By above proposition and corollaries we have:

**Theorem 5:** Let F be an implicative filter of A. Then the following conditions are equivalent:

1. F is a normal filter,
2. F is a filter and $(x\to y)\to x\in F$ implies that $x\in F$ for all $x,y\in A$,
3. F is a filter and $(x\to 0)\to x\in F$ implies that $x\in F$ for all $x,y\in A$,
4. F is a filter and $(x^{-}\to x)\to x\in F$, for all $x\in A$.

**Remark 2:** Any normal filter need not be an implicative filter. For example, let B= {0,a,b,1}. Define * and → as follow:

| * | 0 | a | b | 1 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| a | 0 | 0 | a | a |
| b | 0 | a | b | b |
| 1 | 0 | a | b | 1 |

| → | 0 | a | b | 1 |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| a | a | 1 | 1 | 1 |
| b | 0 | a | 1 | 1 |
| 1 | 0 | a | b | 1 |

Then $(B,\wedge,\vee,*,\to,0,1)$ is a BL-algebra and it is clear that F={b,1} is a normal filter, while it is not an implicative filter, since $a\to(a\to 0)=1\in F$ and $a\to a=1\in F$ but $a\to 0=a\notin F$.

In the following example, we show that an implicative filter need not be a normal filter.

**Example 3:** Let B= {0, a, b, c, 1}. Define * and → as follow:

| * | 1 | 0 | a | b | c |
|---|---|---|---|---|---|
| 1 | 1 | 0 | a | b | c |
| 0 | 0 | 0 | 0 | 0 | 0 |
| a | a | 0 | a | a | a |
| b | b | 0 | a | b | a |
| c | c | 0 | a | a | c |

| → | 1 | 0 | a | b | c |
|---|---|---|---|---|---|
| 1 | 1 | 1 | a | b | c |
| 0 | 1 | 0 | 1 | 1 | 1 |
| a | 1 | 0 | 1 | 1 | 1 |
| b | 1 | 0 | c | 1 | c |
| c | 1 | 0 | b | b | 1 |

Then $(B,\wedge,\vee,*,\to,0,1)$ is a BL-algebra and it is clear that F={b,1} is an implicative filter, while it is not a normal filter since.

$$(a\to 0)\to 0 = 1\in F$$

but

$$(0\to a)\to a = a\notin F$$

**Proposition 3:** Let F be a filter of A. If $x\vee y\in F$, for all $x,y\in A$ where $x\neq y$ and $(x,y\neq 0,1)$ then F is a normal filter of A.

**Proof:**Let $(x\to y)\to y\in F$. Since $x\vee y\leq(y\to x)\to x$ and F is a filter, then we that $(y\to x)\to x\in F$. Hence F is a normal filter of A.

**Theorem 6:** Let f be a BL-homomorphism from A into B and G be a normal filter of B. Then the inverse image of G is a normal filter of A.

**Proof:** Let$(x\to y)\to y\in f^{-1}(G)$. Then $f((x\to y)\to y)\in G$. Hence.

$$(f(x)\to f(y))\to f(y)\in G,$$

Since f is a BL-homomorphism. Then

$$(f(y)\to f(x))\to f(x)\in G,$$

Since G is a normal filter of B. Hence

$$f((y\to x)\to x)\in G.$$

Therefore

$$(y\to x)\to x\in f^{-1}(G).$$

Hence $f^{-1}(G)$ is a normal filter of B.

In the following example we show that image of normal filter need not be a normal filter.

**Example 4:** Let A= {0,a,b,1} be in Remark 2 and B= {0,a,b,1} be in Example 2. Define f as follow: f: A→B, where f (0)=0, f(1)=1, f(a)=0, f(b)=b. Then f is a BL-homomorphism from A into B. Let F= {b,1}, then F is a normal filter of A but f (F) ={1,b} is not filter of B, since $b\in f(F)$ and $b\to a=b\in f(F)$ but $a\notin f(F)$.Clearly, f(F) is not a normal filter.

**Lemma 2:** Let F be a Boolean filter of A. Then $x^{-}\to x\in F$ implies $x\in F$ for all $x\in A$.

**Proof:** Let F be a Boolean filter of A. Therefore $x\vee x^{-}\in F$, for all $x\in A$ Let $x^{-}\to x\in F$. By Lemma 1, we have

$$\left(x\vee x^{-}\right)\to x=\left(x\to x\right)\wedge\left(x^{-}\to x\right)$$
$$=1\wedge\left(x^{-}\to x\right)=x^{-}\to x\in F$$

Then $(x\vee x^{-})\to x\in F$, We have $x\vee x^{-}\in F$. Thus $x\in F$.

**Proposition 4:** Let F be a filter of A. Then F is a Boolean filter of A if and only if $(x\to y)\to x\in F$ implies $x\in F$, for all $x,y\in A$.

**Proof:** Let F be a Boolean filter and $(x\to y)\to x\in F$. By Lemma 1, we know that $x^{-}\le x\to y$. Then by Lemma 1, we have $(x\to y)\to x\le x^{-}\to x$.
Since F is a filter thus

$$x^{-}\to x\in F \tag{1}$$

We have

$$\begin{aligned}(x\vee x^{-})\to x &= (x\to x)\wedge(x^{-}\to x)\\ &= 1\wedge(x^{-}\to x)\\ &= x^{-}\to x\in F\end{aligned}$$

Therefore $(x\vee x^{-})\to x\in F$. Since F is a Boolean filter so $x\vee x^{-}\in F$. Then $x\in F$.

Conversely, see Theorems 2[9] and 3.13[8].
By Theorem 2[9] and Corollary 1, we get that the following corollary:

**Corollary 6: L**et F be a Boolean filter of A. Then F is a normal filter.

In the following example we show that converse of above corollary is not true.

**Example 5:** Let B={0,a,b,c,d,1}. Define * and → as follow:

| * | 1 | a | b | c | d | 0 |
|---|---|---|---|---|---|---|
| 1 | 1 | a | b | c | d | 0 |
| a | a | b | b | d | 0 | 0 |
| b | b | b | b | 0 | 0 | 0 |
| c | c | d | 0 | c | d | 0 |
| d | d | 0 | 0 | d | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| → | 1 | a | b | c | d | 0 |
| 1 | 1 | a | b | c | d | 0 |
| a | 1 | 1 | a | c | c | d |
| b | 1 | 1 | 1 | c | c | c |
| c | 1 | a | b | 1 | a | b |
| d | 1 | 1 | a | 1 | 1 | a |
| 0 | 1 | 1 | 1 | 1 | 1 | 1 |

Then $(B,\wedge,\vee,*,\to,0,1)$ is a BL-algebra and F={1,c} is a normal filter but is not a Boolean filter, since $a\vee a^{-}\notin F$.

**Corollary 7:** Let F be a filter of A. If A/F is a Boolean algebra, then F is a normal filter.

**Proof:** By Theorem 2 and Corollary 1 is clear.

**Theorem 7:** Let F be a filter of A. Then F is a normal filter if $(x^{-}\to y)\to y\in F$, for all x, $y\in A$, such that x, $y\neq 0,1$.

**Proof:** Let F be a filter of A and $(x\to y)\to y\in F$. It is sufficient to show that $(y\to x)\to x\in F$, for all x, $y\in A$. Since $0\le y$, for all $y\in A$, hence by Lemma 1, we have $x\to 0\le x\to y$. Thus

$$(x\to y)\to y\le(x\to 0)\to y=x^{-}\to y\in F$$

Hence it follows from the assumption

$$(x^{-}\to y)\to y\in F$$

That $y\in F$. By Lemma 1, we have $y\le(y\to x)\to x$.
This means that $(y\to x)\to x\in F$.

**Remark 3:** If x = y = 0 then the above theorem is not hold. In Example 3, we have $(0^{-}\to 0)\to\in F$ but F is not normal filter.

In the following example we show that converse of above theorem is not true.

**Example 6:** Let B = {0,a,b,c,d,e,f,g,1}. Define * and → as follow:

| * | 0 | a | b | c | d | e | f | g | 1 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| a | 0 | 0 | a | 0 | 0 | a | 0 | 0 | a |
| b | 0 | a | b | 0 | a | b | 0 | a | b |
| c | 0 | 0 | 0 | 0 | 0 | 0 | c | c | c |
| d | 0 | 0 | a | 0 | 0 | a | c | c | d |
| e | 0 | a | b | 0 | a | b | c | d | e |
| f | 0 | 0 | 0 | c | c | c | f | f | f |
| g | 0 | 0 | a | c | c | d | f | f | g |
| 1 | 0 | a | b | c | d | e | f | g | 1 |
| → | 0 | a | b | c | d | e | f | g | 1 |
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| a | g | 1 | 1 | g | 1 | 1 | g | 1 | 1 |
| b | f | g | 1 | f | g | 1 | f | g | 1 |
| c | e | e | e | 1 | 1 | 1 | 1 | 1 | 1 |
| d | d | e | e | g | 1 | 1 | g | 1 | 1 |
| e | c | d | e | f | g | 1 | f | g | 1 |
| f | b | b | b | e | e | e | 1 | 1 | 1 |
| g | a | b | b | d | e | e | g | 1 | 1 |
| 1 | 0 | a | b | c | d | e | f | g | 1 |

Then $(B,\wedge,\vee,*,\to,0,1)$ is a BL-algebra and F = {1,g} is a normal filter but

$$\begin{aligned}(c^{-}\to a)\to a &= ((c\to 0)\to a)\to a\\ &= (e\to a)\to a=d\to a=e\notin F\end{aligned}$$

**Theorem 8:** Let F be a filter of A. Then F is a normal filter if $x^{-}\in F$ implies $x\in F$, for all $x\in A$.

**Proof:** Let F be a normal filter and $x^{-}\in F$, we know that

$$(x \to 0) \to 0 = x^{--} \in F$$

Since F is a normal filter then we get that

$$(0 \to x) \to x = 1 \to x = x \in F$$

Conversely, let $(x\to y)\to y\in F$, for all $x,y\in A$. Take $y = 0$, therefore

$$x^{--} = (x \to 0) \to 0 \in F$$

Then by hypothesis, we get that $x\in F$. By Lemma 1, we have $x*(y\to x)\leq x$. Thus $x\leq(y\to x)\to x$. Therefore $(y\to x)\to x\in F$, for all $x,y\in A$. Hence F is a normal filter of A.

**Theorem 9:** Let F be a fantastic filter of A. Then F is a normal filter of A.

**Proof:** Let F be a fantastic filter of A. By Lemma 1 [9], we have $x^{-}\to x\in F$, for all $x\in A$. Let $x^{-}\in F$. Hence $x\in F$, since F is a filter. By Theorem 8, we get that F is a normal filter.

**Remark 4:** Let F be a filter of A. Then F is not a subalgebra of A, since $0\notin F$. Hence every normal filter of A is not a subalgebra of A.

Converse of above remark is not true. In Example 3, $F = \{b,1\}$ is a subalgebra of B, but is not a normal filter.

## CONCLUSION AND FUTURE RESEARCH

In this paper we introduced the notion of normal filter in BL-algebra.

We have provided condition for a filter to be a normal filter. In addition, we proved that every Boolean filter is a normal filter, but the converse may not be true. We also gave the conditions under which a normal filter is a Boolean filter.

There are still some open problems:

1. Under what suitable condition a normal filter becomes a fantastic filter?
2. (Extension property for a normal filter) Under what suitable condition extension property for normal filter holds?

Some important issues for future research are trying to define the notion of n-fold normal basic logic as schematic extensions [6] of basic logic BL. We hope above work would serve as a foundation for further on study the structure of BL-algebra and develop corresponding many-valued logical system.

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