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On n-fold Ideals in BCH-algebras and Computation Algorithms

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Abstract: In this paper, we introduce the notion of n-fold (P, implicative, normal and fantastic) ideals in BCH-algebras which is a natural generalization of notion of (P, implicative, normal and fantastic) ideals in BCH-algebras and we stated and proved some theorems which determine the relationship between these notions.

Key words: BCH-algebras . (Implicative, P, normal and fantastic) ideal . n-fold (implicative, P, normal and fantastic) ideals

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INTRODUCTION

In 1966, Imai and Iseki introduced two classes of abstract algebras, BCK-algebras and BCI-algebras [9-12]. BCI-algebras are generalizations of BCK-algebras which were studied by many researchers [7, 13-15].

In 1983, Hu and Li [5, 6] introduced the notion of a BCH-algebra, which is a generalization of the notions of BCK and BCI-algebras. They have studied a few properties of these algebras. Certain other properties have been studied by Chaudhry [2, 3] and Dudek and Thomys [4]. In [16], X. H. Zhang, H. Jiang and S.A. Bhatti studied P-ideals of BCI-algebras. In [8], Y. Huang and Z. Chen introduced the foldness of some ideals in BCK-algebra.

Authors introduced the notions of P, normal, implicative and fantastic ideals in BCH-algebra [1]. In this paper, we introduced the notions of n-fold (P, implicative, normal and fantastic) ideals in BCH-algebra. Finally, we establish the extension property for n-folds (P, implicative, normal and fantastic) ideals in BCH-algebra. Afterwards, we construct some algorithms for computation in finite BCH-algebras.

PRELIMINARIES

By a BCH-algebra we shall mean algebra $(X,*,0)$ of type (2, 0) satisfying the following axioms: for every $x, y, z \in X$, [5]

(I1) $x * x = 0$,

(I2) $x * y = 0$ and $y * x = 0$ imply $x = y$,

(I3) $(x * y) * z = (x * z) * y$.

In a BCH-algebra X, the following holds for all $x, y \in X$

(1) $x * 0 = x$,

(2) $(x * (x * y)) * y = 0$,

(3) $0 * (x * y) = (0 * x) * (0 * y)$,

(4) $0 * (0 * (0 * x)) = 0 * x$,

(5) $x \leq y$ implies $0 * x = 0 * y$.

It is known that every BCI-algebra is a BCH-algebra but not conversely.

A BCH-algebra X is called proper if it is not a BCI-algebra. It is known that proper BCH-algebras exist.

In any BCH/BCI-algebra X we can define a partial order $\leq$ by putting $x \leq y$ if and only if $x*y=0$.

It is known that in a BCH-algebra, $x \leq y$ implies $x*z \leq y*z$ and $x \leq y$ implies $z*y \leq z*x$ does not hold.

A BCH-algebra X is said to be medial if $(x*y)*(z*u) = (x*z)(y*u)$ for all $x, y, z, u \in X$.

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For any $n \geq 1$, $x,y \in X$ we denote respectively $(\ldots((x*y)*y)*\ldots)*y$ by $x*y^n$, where y occur n times.

Definition 1: A nonempty subset I of a BCH-algebra X is called:

(a) [5] ideal of X if it satisfies:
  (i) $0 \in I$
  (ii) $x*y \in I$ and $y \in I$ imply $x \in I$,
(b) [1] P-ideal if:
  (i) $0 \in I$
  (ii) $(x*z)*(y*z) \in I$ and $y \in I$ imply $x \in I$,
(c) [1] implicative ideal of X if:
  (i) $0 \in I$
  (ii) $(x*(y*x))*z \in I$ and $z \in I$ imply $x \in I$,
(d) [1] normal ideal if $x*(x*y) \in I$ implies $y*(y*x) \in I$,
(e) [1] fantastic ideal if:
  (i) $0 \in I$
  (ii) $(x*y)*z \in I$ and $z \in I$ imply $(x*(y*(y*x))) \in I$,

for all $x,y,z \in X$.

An ideal I of X is called closed if for any $x \in I$, $0*x \in I$.

N-FOLD THEORY APPILED TO SOME TYPES OF IDEALS OF BCH-ALGEBRAS

From now on X is a BCH-algebra, unless otherwise is stated.

Definition 1: Let I be a subset of X. If $0 \in I$ and there exists a fixed $n \in N$ such that $(x*z^n)*(y*z^n) \in I$ and $y \in I$ imply that $x \in I$, for all $x,y,z \in X$, then I is said to be an n-fold P-ideal.

Now we describe the relation between ideals and n-fold P-ideal.

Proposition 2: Every n-fold P-ideal is an ideal.

The following example shows that any ideal need not be an n-fold P-ideal.

Example 3: Let X= {0, a, b, c, d}. The following table shows the BCH-algebra structure on X

*	0	a	b	c	d
0	0	0	d	c	b
a	a	0	d	c	b
b	b	b	0	d	c
c	c	c	b	0	d
d	d	d	c	b	0

Then I= {0} is an ideal which is not a 1-fold P-ideal.

Proposition 4: If X is a medial BCH-algebra, then every ideal is an n-fold P-ideal.

Definition 5: Let I be a subset of X. if $0 \in I$ and there exists a fixed $n \in N$ such that $(x*(y^n*x))*z \in I$ and $z \in I$ imply that $x \in I$, for all $x,y,z \in X$, then I is said to be an n-fold implicative ideal..

Proposition 6: Every n-fold implicative ideal is an ideal.

The following example shows that any ideal need not be an n-fold implicative ideal.

Example 7: Let X = {0, a, b, c, d, e, f, g, h, i, j, k, l, m, n}. The following table shows the BCH-algebra structure on X.

*	0	a	b	c	d	e	f	g	h	i	j	k	l	m	n
0	0	0	0	0	0	0	0	0	h	h	h	h	l	l	n
a	a	0	a	0	a	0	a	0	h	h	h	h	m	l	n
b	b	b	0	0	f	f	f	f	i	h	k	k	l	l	n
c	c	b	a	0	g	f	g	f	i	h	k	k	m	l	n
d	d	d	0	0	0	0	d	d	j	h	h	j	l	l	n
e	e	e	a	0	a	0	e	d	j	h	h	j	m	l	n
f	f	f	0	0	0	0	0	0	k	h	h	h	l	l	n
g	g	f	a	0	a	0	a	0	k	h	h	h	a	l	n
h	h	h	h	h	h	h	h	h	0	0	0	0	n	n	l
i	i	i	h	h	k	k	k	k	b	0	f	f	n	n	l
j	j	j	h	h	h	h	j	j	d	0	0	d	n	n	l
k	k	k	h	h	h	h	h	h	f	0	0	0	n	n	l
l	l	l	l	l	l	l	l	l	n	n	n	n	0	0	h
m	m	l	m	l	m	l	m	l	n	n	n	n	a	0	h
n	n	n	n	n	n	n	n	n	l	l	h	l	h	h	0

Then I = {0, a, b, c, d, e, f, g} is an ideal, but is not a 1-fold implicative ideal.

Theorem 8: Let I be an ideal of X. Then I is an n-fold implicative ideal if and only if $(x*(y^n*x)) \in I$ imply that $x \in I$.

Proof: Assume that I is an n-fold implicative ideal and $(x*(y^n*x)) \in I$. Consider $(x*(y^n*x))*(x*(y^n*x)) = 0 \in I$, by hypothesis we get that $x \in I$.

Conversely, let I be an ideal. If $(x*(y^n*x))*z \in I$ and $z \in I$, then $(x*(y^n*x)) \in I$ by hypothesis $x \in I$, therefore I is an n-fold implicative ideal.

By the following examples we show that notions of an n-fold implicative ideal and an n-fold P-ideal are independent.

Example 9: Let X= {0, a, b, c}. The following table shows the BCH-algebra structure on X.

*	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Then I= {0, a} is a 1-fold P-ideal, also $(b*(0*b))*a = (b*b)*a=a \in I$ and $a \in I$, but $b \notin I$. So I is not a 1-fold implicative ideal.

Example 10: Let X= {0, a, b}. The following table shows the BCH-algebra structure on X.

*	0	a	b
0	0	0	0
a	a	0	a
b	b	b	0

Then I= {0, a} is a 1-fold implicative ideal which is not a 1-fold P-ideal, because $(b*b)*(0*b)=0 \in I$ and $0 \in I$, but $b \notin I$.

In the above example I= {0, a} is a 1-fold implicative ideal, but is not a 2-fold implicative ideal because $(b*(b*(b*b)))*0=0 \in I$ and $0 \in I$, but $b \notin I$. Then every n-fold implicative ideal is not (n+1)-fold implicative ideal.

Open problem: Under what suitable condition an n-fold implicative ideal becomes an (n+1)-fold implicative ideal?

Theorem 11: Let I be an n-fold implicative ideal of X, satisfying $(x*y)*(z^n*x) = (x*z^n)*(y*z^n)$, for all $x,y,z \in X$. Then I is an n-fold P-ideal.

Proof: If $(x*z^n)*(y*z^n) \in I$ and $y \in I$, for all $x, y, z \in X$, then $(x*z^n)*(y*z^n) = (x*y)*(z^n*x) = ((x*(z^n*x))*y)$. Since I is an n-fold implicative ideal, we conclude that $x \in I$, therefore I is an n-fold P-ideal.

Definition 12: An ideal I of X is called an n-fold normal ideal if $x*(x*y^n) \in I$ implies that $y*(y*x^n) \in I$, for all $x, y \in X$.

The following example shows that any ideal need not be an n-fold normal ideal.

Example 13: Let X= {0, a, b, c}. The following table shows the BCH-algebra structure on X.

*	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	c	0	c
c	c	0	0	0

Then I= {0, c} is an ideal of X, but I is not a 1-fold normal ideal. Since $(b*(b*a)) = c \in I$, but $(a*(a*b)) = a \notin I$.

In the following example, we show that every n-fold normal ideal may not be an n-fold P-ideal or an n-fold implicative ideal.

Example 14: Let X= {0, a, b, c}. The following table shows the BCH-algebra structure on X.

*	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	0
c	c	c	b	0

Then I= {0, a} is a 1-fold normal ideal of X, but I is not a 1-fold p-ideal, since $(b*b)*(0*b)=0 \in I$ and $0 \in I$, but $b \notin I$.

Also I is not a 1-fold implicative ideal, since $(b*(c*b))*0 = (b*b)*0 = 0 \in I$ and $0 \in I$, but $b \notin I$.

Proposition 15: Let I be an ideal of X. If $x*(x*y^n) \in I$ implies $(y*(x*(x*y^n)))*(y*x^n) \in I$, for all $x,y \in X$, then I is an n-fold normal ideal.

Proof: Let $x,y \in X$ be such that $x*(x*y^n) \in I$. By hypothesis

$$(y*(y*x^n))*(x*(x*y^n)) = (y*(x*(x*y^n)))*(y*x^n) \in I,$$

since I is an ideal and $x*(x*y^n) \in I$, then $y*(y*x^n) \in I$. Therefore I is an n-fold normal ideal.

Definition 16: A nonempty subset I of X is called an n-fold fantastic ideal if

(i) $0 \in I$
(ii) $(x*y^n)*z \in I$ and $z \in I$ imply $x*y^n*(y^n*x)) \in I$, for all $x,y,z \in X$.

By the following example, we show the relation between n-fold fantastic ideals and ideals in BCH-algebras.

Example 17: Let X= {0, 1, 2, 3, 4}. The following table shows the BCH-algebra structure on X.

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	1	3	0	3
4	4	4	2	4	0

Then I= {0, 1, 2} is a 1-fold fantastic ideal which is not an ideal. Because $(3*2) = 2 \in I$ and $2 \in I$, but $3 \notin I$.

Example 18: Let X= {0, 1, 2, 3, 4}. The following table shows the BCH-algebra structure on X.

*	0	1	2	3	4
0	0	0	0	0	4
1	1	0	0	1	4
2	2	2	0	0	4
3	3	3	3	0	4
4	4	4	4	4	0

Then I= {0} is an ideal. We have $(1*2)*0=0 \in I$ and $0 \in I$, but $(1*(2*(2*1))) =1 \notin I$, so I is not a 1-fold fantastic ideal.

Proposition 19: Let I be an ideal of X. Then I is an n-fold fantastic ideal if and only if $x*y^n \in I$ imply that $x*y^n*(y^n*x)) \in I$, for all $x,y \in X$.

Proof: Let I be an n-fold fantastic ideal and $x*(y^n=(x*y^n)*0 \in I$. Since $0 \in I$, then $x*(y^n*(y^n*x)) \in I$.
Conversely, let $(x*y^n)*z \in I$ and $z \in I$. Then $x*y^n \in I$, by hypothesis $x*(y^n*(y^n*x)) \in I$. Therefore I is an n-fold fantastic ideal.

CONCLUSION

We introduced the notion of n-fold (P, implicative, normal and fantastic) ideals in BCH-algebras and gave some relations between them. There is still an open problem:

Under what suitable condition an n-fold implicative ideal becomes an (n+1)-fold implicative ideal?
This could be topics of further research:

- In general, study the relationship between n-fold and (n+1)-fold ideals that defined in this paper.
- What are the relations between n-fold (normal, P, fantastic and normal) BCH-algebra and other logical algebraic structures?

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ALGHORITHMS

Here we give some algorithms for studying the structure of finite BCH-algebras and n-fold ideals in finite BCH-algebras.

Algorithm for finite BCH-algebras

```
Input(X: set, *: binary operation)
Output ("X is a BCH-algebra or not")
Begin
  If X≠ϕ then
    go to (1);
  If 0∉X then
    go to (1);
    Stop: = false;
  i := 1;
  While i≤|X| and not (stop) do
    If xi*xi≠0 then
      stop := true;
    j := 1
    While j≤|X| and not (stop) do
      If xi*(xi*yi)≠0 then
        Stop: = true;
      If (xi*yi=0) and (yi*xi=0)
        If xi≠yi then
          Stop: = true;
      EndIf
    EndWhile
  EndWhile
```

```
    If Stop then
        (1) Output ("X is not a BCH-algebra")
    Else
        Output ("X is a BCH-algebra")
    EndIf
End
```

Algorithm for n-fold P-ideals in finite BCH-algebras

```
Input (X: BCH-algebra, I: subset of X, n ∈N);
Output ("I is an n-fold P-ideal of X or not");
Begin
    If I =ϕ then
        go to (1);
    If 0∉I then
        go to (1);
    Stop: = false;
    i := 1;
    While i≤|X| and not (Stop) do
        j := 1;
        While j≤|X| and not (Stop) do
            k := 1;
            While k≤|X| and not (Stop) do
                If (x_i * z_k^n)*(y_j * z_k^n )∈ I and y_j∈I then
                    If x_i∉I then
                        Stop: = ture;
                EndIf
            EndWhile
        EndWhile
    EndWhile
    If Stop then
        Output ("I is an n-fold P-ideal of X")
    Else
        (1) Output ("I is not an n-fold P-ideal of X")
    EndIf
End
```

Algorithm for n-fold implicative ideals in finite BCH-algebras

```
Input (X : BCH-algebra, I : subset of X, n ∈N);
Output ("I is an n-fold implicative ideal of X or not");
Begin
    If I =ϕ then
        go to (1);
    If 0∉I then
        go to (1);
    Stop: = false;
    i := 1;
    While i≤|X| and not (Stop) do
        j := 1;
        While j≤|X| and not (Stop) do
            k := 1;
            While k≤|X| and not (Stop) do
                If (x_i *(y_k^n * x_i))*z_j ∈I and z_j∈I then
                    If x_i∉I then
                        Stop: = ture;
                EndIf
            EndWhile
        EndWhile
    EndWhile
    If Stop then
        Output ("I is an n-fold implicative ideal of X")
    Else
        (1) Output ("I is not an n-fold implicative ideal of X")
    EndIf
End
```

Algorithm for n-fold normal ideals in finite BCH-algebras

```
Input (X : BCH-algebra, I : subset of X, n ∈N);
Output ("I is an n-fold normal ideal of X or not");
Begin
    If I =ϕ then
        go to (1);
    If 0∉I then
        go to (1);
    Stop: = false;
    i := 1;
    While i≤|X| and not (Stop) do
        j := 1;
        While j≤|X| and not (Stop) do
            If x_i * (x_i * y_j^n) ∈ I then
                If y_j *(y_j * x_i^n )∉I then
                    Stop: = ture;
            EndIf
```

```
        EndWhile
    EndWhile
    If Stop then
        Output ("I is an n-fold normal ideal of X")
    Else
        (1) Output ("I is not an n-fold normal ideal of X")
    EndIf
End
```

Algorithm for n-fold fantastic ideals in finite BCH-algebras

```
Input (X: BCH-algebra, I: subset of X, n ∈ N);
Output("I is an n-fold fantastic ideal of X or not");
Begin
    If I =ϕ then
        go to (1);
    If 0∉I then
        go to (1);
    Stop := false;
    i := 1;
    While i≤|X| and not (Stop) do
        j := 1;
        While j≤|X| and not (Stop) do
            k := 1;
            While k≤|X| and not (Stop) do
                If (x_i * y_k^n)*z_j ∈I  and z_j∈I then
                    If x_i * (y_k^n * (y_k^n * x_i)) ∉I  then
                        stop := ture;
                EndIf
            EndWhile
        EndWhile
    EndWhile
    If Stop then
        Output ("I is an n-fold fantastic ideal of X")
    Else
        (1) Output ("I is not an n-fold fantastic ideal of X")
    EndIf
End
```

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