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Application of He's Max-min Approach to a Generalized Nonlinear Oscillator

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Abstract: This paper applies He's max-min method for obtaining approximate periodic solution of a generalized nonlinear oscillator. The method is extremely simple but remarkably effective. Comparison of the obtained solution with the numerical one is given, showing high accuracy of the approximate solution.

Key words: Nonlinear oscillator . higher-order Duffing equation . frequency formulation

INTRODUCTION

Recently, in a review article [1], Ji-Huan He gave a heuristic introduction to ancient Chinese mathematics, elucidating that it can be powerfully applied to solving nonlinear differential equations! The article [1] has then sparked an intense interest in the application of ancient Chinese mathematical methods to nonlinear equations. The ancient Chinese mathematics mainly include the *Ying Buzu Shu* [2, 3], *He Chengtian* inequatity [4, 5] and ancient Chinese musical scales [6]. The *Ying Buzu Shu* [2, 3] was developed into a simple amplitude-frequency formulation [7-11]; *He Chengtian* inequatity was developed into the max-min approach [12, 13]; Tong [14] proved that the ancient Chinese musical scales, 2/3, 7/12, 24/41 and 31/53 used in 40 BC in ancient China are the best approximations. It is mysterious how ancient Chinese got such best results without using modern mathematics. Lan and Yang obtained another scale 179/306 (Lan-Yang Scale) [15]. It is really very exciting to see the revival of ancient Chinese mathematics.

In this paper, we will apply the max-min method [12, 16] to a generalized oscillator. The method is based on *He Chengtian's* inequality [4, 5]. Now it is easy to determine maximal and minimal solution thresholds of a nonlinear equation and the ancient technology can be powerfully applied.

MAX-MIN APPROACH

Consider the following Duffing equation with $2n+1$-order nonlinearity:

$$u'' + u + \varepsilon u^{2n+1} = 0, \quad u(0) = A, \ u'(0) = 0 \qquad (1)$$

where n is a positive integer and ε needs not to be small, i.e. $0 \le \varepsilon < \infty$.

This equation can be solved by the parameter-expansion method [17], the homotopy perturbation method [18-20], the equivalent linearization method [21] and others. A complete review on various analytical methods is available in Refs [1, 16]. Hereby we will apply the max-min approach suggested by Ji-Huan He [12].

We re-write Equation (1) in the following form:

$$u'' + (1 + \varepsilon u^{2n})u = 0 \qquad (2)$$

According to the max-min approach [12], we choose a trial function:

$$u(t) = A\cos \omega t \qquad (3)$$

where ω is a frequency to be determined later. With this trial function, the maximum and minimum values of $1+\varepsilon u^{2n}$ are $1+\varepsilon A^{2n}$ and 1, respectively and it follows that:

$$\frac{1}{1} < \omega^2 < \frac{1+\varepsilon A^{2n}}{1} \qquad (4)$$

Now, using the He Chentian interpolation [4, 5], we have:

$$\omega^2 = \frac{m + n(1+\varepsilon A^{2n})}{m+n} = 1 + k\varepsilon A^{2n} \qquad (5)$$

where m and n are weighting factors and $k=n/m+n$. The frequency can be approximated as:

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$$\omega = \sqrt{1 + k\varepsilon A^{2n}} \tag{6}$$

thus giving rise to an approximate solution:

$$u(t) = A\cos[\sqrt{1 + k\varepsilon A^{2n}}\, t] \tag{7}$$

In view of (7), we now re-write Equation (1) in the form

$$u'' + (1 + k\varepsilon A^{2n})u = k\varepsilon A^{2n}u - \varepsilon u^{2n+1} \tag{8}$$

If, by chance, Eq.(7) is the exact solution of Eq.(1), then the right hand side of Eq.(8) vanishes. On the other hand, if it is only an approximation to the exact solution, then we evaluate k by expanding the residuals, i.e., the right hand side of Eq.(8), into a Fourier series. More precisely, we have:

$$\begin{aligned} k\varepsilon A^{2n}u - \varepsilon u^{2n+1} &= \sum_{n=0}^{\infty} b_{2n+1}\cos[(2n+1)\omega t] \\ &= b_1\cos(\omega t) + \sum_{n=0}^{\infty} b_{2n+1}\cos[(2n+1)\omega t] \end{aligned}$$

We set

$$b_1 = \frac{4}{T}\int_0^{\frac{T}{4}} (k\varepsilon A^{2n}u - \varepsilon u^{2n+1})\cos(\omega t)dt = 0,\ T = \frac{2\pi}{\omega} \tag{9}$$

to approximately identify k, which reads:

$$k = \frac{2(2n+1)!!}{(2n+2)!!} \tag{10}$$

Finally, the frequency and the period are obtained as:

$$\omega = \sqrt{1 + \frac{2(2n+1)!!}{(2n+2)!!}\varepsilon A^{2n}} \tag{11}$$

and

$$T = \frac{2\pi}{\sqrt{1 + \frac{2(2n+1)!!}{(2n+2)!!}\varepsilon A^{2n}}} \tag{12}$$

The exact period of Eq.(1) reads:

$$T_{ex} = 4\int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{1 + \frac{\varepsilon A^{2n}}{n+1}(1 + \cos^2 t + \cdots + \cos^{2n} t)}} \tag{13}$$

The frequency is:

$$\omega_{ex} = \frac{2\pi}{T_{ex}} \tag{14}$$

Finally, by using equations (11) and (14), the approximate value of the frequency can be compared with the exact one for fixed values of A and ε. When n = 1, our result is same as those in Refs [1, 11].

CONCLUSION

The max-min method is of utter simplicity and effectiveness for nonlinear oscillators. The method can used by engineers with hand-and-pencil without the requirement of advanced calculus. Application of He Chengtian's inequality to other types of nonlinear equations was given by J.H. He in his recent publication [22].

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