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Modified Variational Iteration Method for Solving Sine-Gordon Equations

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Abstract: In this paper, we apply the Modified Variational Iteration Method (MVIM) for solving Sine-Gordon equations which arise in differential geometry, propagation of magnetic flux, stability of fluid motions, nonlinear physics and applied sciences. The proposed modification is made by introducing Adomian's polynomials in the correction functional of VIM. The use of Lagrange multiplier is a clear advantage of this technique over the decomposition method. Numerical results show the efficiency of the suggested algorithm.

Key words: Variational iteration method . Lagrange multiplier . Sine-Gordon equations . Adomian's polynomials

INTRODUCTION

The Sine-Gordon equations appear in differential geometry, propagation of magnetic flux, stability of fluid motions, nonlinear physics and applied sciences [1-4]. Several techniques including, Backlund transformations, inverse scattering, similarity, variational iteration, homotopy analysis, tanh and decomposition [1-4] have been used for the solution of these equations. It is worth mentioning that Yücel [4] applied Homotopy Analysis Method (HAM) for solving these problems and also proved the compatibility of the obtained results with VIM [2] for h =-1. The standard form of such equations is given by

$$u_{tt}(x,t)-c^2u_{xx}(x,t)+\alpha\sin u=0$$

with initial conditions

$$u(x,0)=f(x),\qquad u_t(x,0)=g(x)$$

He [5-12] developed and formulated the Variational Iteration Method (VIM) for solving various physical problems. The method has been extremely useful for diversified physical problems [1-31]. In a later work, Abbasbandy [1, 13] used Adomian's' polynomials in the correction functional for solving Riccati differential and Klein-Gordon equations. Most recently, Noor and Mohyud-Din developed the elegant coupling of Adomian's polynomials and the correctional functional of VIM calling it as modified variational iteration method(MVIM) and applied this reliable version for solving various singular and nonsingular initial and boundary value problems [14-17]. Inspired and motivated by the ongoing research in this area, we apply the Modified Variational Iteration Method (MVIM) which is formulated by the elegant coupling of Adomian's polynomials and the correctional functional for solving Sine-Gordon equations. The use of Lagrange multiplier in the MVIM gives it a clear advantage over the decomposition method since it avoids the successive application of the integral operator and hence reduces the computational work to a tangible level. Moreover, the coupling of Adomian's polynomials makes the technique more compatible with the nonlinearity of the physical problems [14-17]. Numerical results show the complete reliability of the proposed technique.

MODIFIED VARIATIONAL ITERATION METHOD (MVIM)

To illustrate the basic concept of the MVIM, we consider the following general differential equation

$$Lu+Nu=g(x) \tag{1}$$

where L is a linear operator, N a nonlinear operator and g (x) is the inhomogeneous term. According to variational iteration method [1-31], we can construct a correction functional as follows

$$u_{n+1}(x)=u_n(x)+\int_0^x\lambda(Lu_n(s)+N\tilde{u}_n(s)-g(s))ds \tag{2}$$

where λ is a Lagrange multiplier [5-12], which can be identified optimally via variational iteration method. The subscripts n denote the nth approximation, $\tilde{u}_n$ is considered as a restricted variation. i.e. $\delta\tilde{u}_n=0$; (2) is

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called a correction functional. The modified version of VIM is obtained by the coupling of correction functional (2) of variational iteration method with Adomian's polynomials [18, 3] and is given by

$$u_{n+1}(x) = u_n(x) + \int_0^t \lambda \left(L u_n(x) + \sum_{n=0}^{\infty} A_n - g(x) \right) dx \quad (3)$$

where A_n are the so-called Adomian's polynomials [3, 18] and the above combination (3) is called the Modified Variational Iteration Method (MVIM) [14-17].

NUMERICAL APPLICATIONS

In this section, we apply the modified variational iteration method (MVIM) for solving Sine-Gordon equations. Numerical results are very encouraging.

Example 3.1: Consider the following Sine-Gordon equation

$$u_{tt}(x,t) - u_{xx}(x,t) = \sin u$$

with initial conditions

$$u(x,0) = \frac{\pi}{2}, \qquad u_t(x,0) = 0$$

The correction functional is given by

$$u_{n+1}(x,t) = \frac{\pi}{2} + \int_0^t \lambda(s) \left(\frac{\partial^2 u_n}{\partial s^2} - \frac{\partial^2 \tilde{u}_n}{\partial x^2} - \sin \tilde{u}_n \right) ds$$

Making the correction functional stationary, the Lagrange multiplier can be identified as $\lambda(s) = (s-t)$ we obtain the following iterative formula

$$u_{n+1}(x,t) = \frac{\pi}{2} + \int_0^t (s-t) \left(\frac{\partial^2 u_n}{\partial s^2} - \frac{\partial^2 u_n}{\partial x^2} - \sin u_n \right) ds$$

Applying the modified Variational Iteration Method (MVIM), we get the following iterative scheme

$$u_{n+1}(x,t) = \frac{\pi}{2} + \int_0^t (s-t) \left(\frac{\partial^2 u_n}{\partial s^2} - \sum_{n=0}^{\infty} \frac{\partial^2 u_n}{\partial x^2} - \sum_{n=0}^{\infty} A_n \right) ds$$

where A_n are the so-called Adomian's polynomials which can be generated for all types of nonlinearities according to algorithm defined in [3, 18]. First few Adomian's polynomials are as follows:

$$A_0 = \sin u_0$$

$$A_1 = u_1 \cos u_0$$

$$A_2 = u_2 \cos u_0 - \frac{1}{2!} u_1^2 \sin u_0$$

$$A_3 = u_3 \cos u_0 - u_2 u_1 \sin u_0 - \frac{1}{3!} u_1^3 \cos u_0$$

$$\vdots$$

Consequently, following approximants are obtained

$$u_0(x,t) = \frac{\pi}{2}$$

$$u_1(x,t) = \frac{\pi}{2} + \frac{1}{2} t^2$$

$$u_2(x,t) = \frac{\pi}{2} + \frac{1}{2} t^2 + 0$$

$$u_3(x,t) = \frac{\pi}{2} + \frac{1}{2} t^2 + \frac{1}{240} t^6$$

$$\vdots$$

The series solution is given by

$$u(x,t) = \frac{\pi}{2} + \frac{1}{2} t^2 + \frac{1}{240} t^6 + \cdots$$

which is in full agreement with [3] where the same problem was solved by using decomposition method.

Example 3.2: Consider the following Sine-Gordon equation

$$u_{tt}(x,t) - u_{xx}(x,t) = \sin u$$

with initial conditions

$$u(x,0) = \frac{\pi}{2}, \qquad u_t(x,0) = 1$$

The correction functional is given by

$$u_{n+1}(x,t) = \frac{\pi}{2} + t + \int_0^t \lambda(s) \left(\frac{\partial^2 u_n}{\partial s^2} - \frac{\partial^2 \tilde{u}_n}{\partial x^2} - \sin \tilde{u}_n \right) ds$$

Making the correction functional stationary, the Lagrange multiplier can be identified as $\lambda(s) = (s-t)$ we obtain the following iterative formula

$$u_{n+1}(x,t)=\frac{\pi}{2}+t+\int_0^t (s-t)\left(\frac{\partial^2 u_n}{\partial s^2}-\frac{\partial^2 u_n}{\partial x^2}-\sin u_n\right)ds$$

Applying the modified Variational Iteration Method (MVIM), we get the following iterative scheme:

$$u_{n+1}(x,t)=\frac{\pi}{2}+t+\int_0^t (s-t)\left(\frac{\partial^2 u_n}{\partial s^2}-\sum_{n=0}^{\infty}\frac{\partial^2 u_n}{\partial x^2}-\sum_{n=0}^{\infty}A_n\right)ds$$

where A_n are the so-called Adomian's polynomials which can be generated for all types of nonlinearities according to algorithm defined in [3, 18]. Proceeding as before, consequently, following approximants are obtained

$$u_0(x,t)=\frac{\pi}{2}+t$$

$$u_1(x,t)=\frac{\pi}{2}+t+1-\cos t$$

$$u_2(x,t)=\frac{\pi}{2}+t+1-\cos t+\sin t-\frac{3}{4}t-\frac{1}{8}\sin 2t$$

$$\vdots$$

The series solution is given by

$$u(x,t)=\frac{\pi}{2}+t+\frac{1}{2!}t^2-\frac{1}{4!}t^4+\cdots$$

which is in full agreement with [3] where the same problem was solved by using decomposition method.

Example 3.3: Consider the following Sine-Gordon equation

$$u_{tt}(x,t)-u_{xx}(x,t)=\sin u$$

with initial conditions

$$u(x,0)=\pi, \qquad u_t(x,0)=1$$

The correction functional is given by

$$u_{n+1}(x,t)=\pi+t+\int_0^t \lambda(s)\left(\frac{\partial^2 u_n}{\partial s^2}-\frac{\partial^2 \tilde{u}_n}{\partial x^2}-\sin\tilde{u}_n\right)ds$$

Making the correction functional stationary, the Lagrange multiplier can be identified as λ (s) = (s-t), we obtain the following iterative formula

$$u_{n+1}(x,t)=\pi+t+\int_0^t (s-t)\left(\frac{\partial^2 u_n}{\partial s^2}-\frac{\partial^2 u_n}{\partial x^2}-\sin u_n\right)ds$$

Applying the modified variational iteration method (MVIM), we get the following iterative scheme:

$$u_{n+1}(x,t)=\pi+t+\int_0^t (s-t)\left(\frac{\partial^2 u_n}{\partial s^2}-\sum_{n=0}^{\infty}\frac{\partial^2 u_n}{\partial x^2}-\sum_{n=0}^{\infty}A_n\right)ds$$

where A_n are the so-called Adomian's polynomials [3, 18]. Proceeding as before, consequently, following approximants are obtained

$$u_0(x,t)=\pi+t$$

$$\vdots$$

The series solution is given by

$$u(x,t)=\pi+t-\frac{1}{3}t^3+\cdots$$

which is in full agreement with [3] where the same problem was solved by using decomposition method.

Example 3.4: Consider the following Sine-Gordon equation

$$u_{tt}(x,t)-u_{xx}(x,t)=\sin u$$

with initial conditions

$$u(x,0)=\frac{3\pi}{2}, \qquad u_t(x,0)=1$$

The correction functional is given by

$$u_{n+1}(x,t)=\frac{3\pi}{2}+t+\int_0^t \lambda(s)\left(\frac{\partial^2 u_n}{\partial s^2}-\frac{\partial^2 \tilde{u}_n}{\partial x^2}-\sin\tilde{u}_n\right)ds$$

Making the correction functional stationary, the Lagrange multiplier can be identified as λ (s) = (s-t), we obtain the following iterative formula

$$u_{n+1}(x,t)=\frac{3\pi}{2}+t+\int_0^t (s-t)\left(\frac{\partial^2 u_n}{\partial s^2}-\frac{\partial^2 u_n}{\partial x^2}-\sin u_n\right)ds$$

Applying the modified Variational Iteration Method (MVIM), we get the following iterative scheme:

$$u_{n+1}(x,t)=\frac{3\pi}{2}+t+\int_0^t (s-t)\left(\frac{\partial^2 u_n}{\partial s^2}-\sum_{n=0}^{\infty}\frac{\partial^2 u_n}{\partial x^2}-\sum_{n=0}^{\infty}A_n\right)ds$$

where A_n are the so-called Adomian's polynomials [3, 18]. Proceeding as before, the series solution is given by

$$u(x,t)=\frac{3\pi}{2}+t-\frac{1}{4}t^2+\cdots$$

which is in full agreement with [3] where the same problem was solved by using decomposition method.

Example 3.5: Consider the pendulum like equation [2, 4] which arises from the following Sine-Gordon equation

$$\frac{d^2u}{dt^2}-\sin u=0$$

with initial conditions

$$u(0)=\pi,\ \frac{du}{dt}(0)=-2$$

The correction functional is given by

$$u_{n+1}(x,t)=\pi-2t+\int_0^t \lambda(s)\left(\frac{d^2u_n}{\partial s^2}-\sin\tilde{u}_n\right)ds$$

Making the correction functional stationary, the Lagrange multiplier can be identified as λ (s) = (s-t), we obtain the following iterative formula

$$u_{n+1}(x,t)=\pi-2t+\int_0^t (s-t)\left(\frac{d^2u_n}{\partial s^2}-\sin u_n\right)ds$$

Applying the Modified Variational Iteration Method (MVIM), we get the following iterative scheme:

$$u_{n+1}(x,t)=\pi-2t+\int_0^t (s-t)\left(\frac{d^2u_n}{\partial s^2}-\sum_{n=0}^{\infty} A_n\right)ds$$

where A_n are the so-called Adomian's polynomials [3, 18]. Proceeding as before, following approximants are obtained

$$p^{(0)}: u_0(t)=\pi-2t$$

$$p^{(1)}: u_1(t)=\pi-\frac{3}{2}t-\frac{\sin(2t)}{4}$$

$$\vdots$$

The series solution is given by

$$u(t)=\pi-\frac{3}{2}t-\frac{\sin(2t)}{4}+\cdots$$

which is in full agreement with [2] where the same problem was solved by using the Variational Iteration Method (VIM) and with [4] for h =-1 where it was tackled by Homotopy Analysis Method (HAM).

Example 3.6: Consider the following Sine-Gordon equation

$$u_{tt}(x,t)-u_{xx}(x,t)+\sin u=0$$

with initial conditions

$$u(x,0)=\pi+\varepsilon\cos(\beta x),\qquad u_t(x,0)=0$$

The correction functional is given by

$$u_{n+1}(x,t)=\pi+\varepsilon\cos(\beta x)+\int_0^t \lambda(s)\left(\frac{\partial^2u_n}{\partial s^2}-\frac{\partial^2\tilde{u}_n}{\partial x^2}+\sin\tilde{u}_n\right)ds$$

Making the correction functional stationary, the Lagrange multiplier can be identified as λ (s) = (s-t), we obtain the following iterative formula

$$u_{n+1}(x,t)=\pi+\varepsilon\cos(\beta x)+\int_0^t (s-t)\left(\frac{\partial^2u_n}{\partial s^2}-\frac{\partial^2u_n}{\partial x^2}+\sin u_n\right)ds$$

Applying the modified Variational Iteration Method (MVIM), we get the following iterative scheme:

$$u_{n+1}(x,t)=\pi+\varepsilon\cos(\beta x)+\int_0^t (s-t)\left(\frac{\partial^2u_n}{\partial s^2}-\frac{\partial^2u_n}{\partial x^2}+\sum_{n=0}^{\infty} A_n\right)ds$$

where A_n are the so-called Adomian's polynomials [3, 18]. Proceeding as before, following approximants are obtained

$$p^{(0)}: u_0(x,t)=\pi+\varepsilon\cos(\beta x)$$

$$p^{(1)}: u_1(t)=\pi+\varepsilon\cos(\beta x)-\frac{t^2}{2}\left(\beta^2\varepsilon\cos(\beta x)-\sin(\varepsilon\cos(\beta x))\right)$$

$$\vdots$$

The series solution is given by

$$u(x,t)=\pi+\varepsilon\cos(\beta x)-\frac{t^2}{2}\left(\beta^2\varepsilon\cos(\beta x)-\sin(\varepsilon\cos(\beta x))\right)+\cdots$$

which is in full agreement with [3] where the same problem was solved by variational iteration method (VIM) and with [4] for h =-1 where it was tackled by Homotopy Analysis Method (HAM).

CONCLUSION

In this paper, we apply the modified variational iteration method (MVIM) for solving Sine-Gordon equations. The method is applied in a direct way without using linearization, transformation, perturbation, discretization or restrictive assumptions. The use of Lagrange multiplier gives this method a clear advantage over the decomposition method because it avoids the successive application of integral operator.

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REFERENCES

1. Abbasbandy, S., 2007. Numerical solutions of nonlinear Klein-Gordon equation by variational iteration method. Internat. J. Numer. Meth. Engrg., 70: 876-881.
2. Batiha, B., M.S.M. Noorani and I. Hashim, 2007. Numerical solution of sine-Gordon equation by variational iteration method. Phys. Lett., A 370: 437-440.
3. Wazwaz, A.M., 2002. Partial differential equations, methods and applications, A. A. Balkema Publishers.
4. Yücel, U., 2008. Homotopy analysis method for the sine-Gordon equation with initial conditions. Appl. Math. Comput., 203: 387-395.
5. He, J.H., 1999. Variational iteration method, A kind of non-linear analytical technique, some examples, Internat. J. Nonlin. Mech., 34 (4): 699-708.
6. He, J.H., 2000. Variational iteration method for autonomous ordinary differential systems. Comput. Math. Appl. 114 (2-3): 115-123.
7. He. J.H. and X.H. Wu, 2006. Construction of solitary solution and compaction-like solution by variational iteration method. Chaos, Solitons & Fractals, 29 (1): 108-113.
8. He, J.H., 2007. Variational iteration method-some recent results and new interpretations. J. Comput. Appl. Math., 207: 3-17.
9. He, J.H. and X.H. Wu, 2007. Variational iteration method: New development and applications. Comput. Math. Appl., 54: 881-894.
10. He, J.H., 2007. The variational iteration method for eighth-order initial boundary value problems. Phys. Scr. 76 (6): 680-682.
11. He, J.H., 2008. An elementary introduction of recently developed asymptotic methods and nanomechanics in textile engineering. Int. J. Mod. Phys., B 22 (21): 3487-4578.
12. He, J.H., 2006. Some asymptotic methods for strongly nonlinear equation. Int. J. Mod. Phys., 20 (20)10: 1144-1199.
13. Abbasbandy, S., 2007. A new application of He's variational iteration method for quadratic Riccati differential equation by using Adomian's polynomials. J. Comput. Appl. Math., 207: 59-63.
14. Mohyud-Din, S.T., M.A. Noor and K.I. Noor, 2008. Travelling wave solutions of seventh-order generalized KdV equations using Adomian's polynomials. Int. J. Mod. Phys. B.
15. Noor, M.A. and S.T. Mohyud-Din, 2007. Variational iteration decomposition method for solving eighth-order boundary value problems. Diff. Eqns. Nonlin. Mech. 2007 (2007), Article ID, 19529, doi: 10.1155/2007/19529.
16. Noor, M.A. and S.T. Mohyud-Din, 2008. Solution of singular and nonsingular initial and boundary value problems by modified variational iteration method. Math. Prob. Engg. 2008 (2008), Article ID 917407, 23 pages, doi:10.1155/2008/917407.
17. Noor, M.A., S.T. Mohyud-Din and M. Tahir, 2008. Modified variational iteration methods for Thomas-Fermi equation. World Appl. Sci. J., 4 (4): 479-498.
18. Saadati, R., B. Rafbari, H Abibi, S.M. Maezpour and S. Shakeri, 2008. A comparison between variational iteration method and trapezoidal rule for solving linear integro-differential equations. World Appl. Sci. J., 4 (3): 321-325.
19. Wazwaz, A.M., 2006. The modified decomposition method for analytic treatment of differential equations. Appl. Math. Comput., pp: 165-176.
20. Abdou, M.A. and A.A. Soliman, 2005. Variational iteration method for solving Burger's and coupled Burger's equations. J. Comput. Appl. Math., 181: 245-251.

21. Abdou, M.A. and A.A. Soliman, 2005. New applications of variational iteration method, Phys., D211 (1-2): 1-8.
22. Esmailpour, M., and D.D. Ganji, 2008. A study on generalized coquette flow by He's methods and comparison with the numerical solution. World Appl. Sci. J., 4 (4): 470-478.
23. Noor, M.A. and S.T. Mohyud-Din, 2008. Variational iteration method for solving higher-order nonlinear boundary value problems using He's polynomials. Int. J. Nonlin. Sci. Num. Simul., 9 (2): 141-157.
24. Noor, M.A. and S.T. Mohyud-Din, 2007. Variational iteration technique for solving higher order boundary value problems. Appl. Math. Comput., 189: 1929-1942.
25. Noor, M.A. and S.T. Mohyud-Din, 2007. An efficient method for fourth order boundary value problems. Comput. Math. Appl., 54: 1101-1111.
26. Noor, M.A. and S.T. Mohyud-Din, 2008. Variational iteration method for solving fifth-order boundary value problems using He's polynomials, Math. Prob. Engg. 2008 (2008), Article ID 954794, doi: 10:1155/2008/954794.
27. Noor, M.A. and S.T. Mohyud-Din, 2008. Modified variational iteration method for heat and wave-like equations. Acta Applnda. Mathmtce. (2008), DOI: 10.1007/s10440-008-9255-x.
28. Noor, M.A. and S.T. Mohyud-Din, 2008. Modified variational iteration method for solving fourth-order boundary value problems, J. Appl. Math. Computg. 2008 (2008), Article ID: 90, DOI: 10.1007/s12190-008-0090-z.
29. Noor, M.A. and S.T. Mohyud-Din, 2008. Variational homotopy perturbation method for solving higher dimensional initial boundary value problems. Math. Prob. Engg. 2008. Article ID 696734, doi:10.1155/2008/696734.
30. Noor, M.A. and S.T. Mohyud-Din, 2008. Variational iteration method for unsteady flow of gas through a porous medium using He's polynomials and Pade´ approximants, Comput. Math. Appl.
31. Noor, M.A. and S.T. Mohyud-Din, 2008. Modified variational iteration method for Goursat and Laplace problems. World Appl. Sci. J., 4 (4): 487-498.