World Applied Sciences Journal 6 (2): 191-195, 2009
ISSN 1818-4952
© IDOSI Publications, 2009

# A New Wavelet Linear Survival Function Estimator

*Hassan Doosti*

Department of Statistics, School of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad, Iran

**Abstract:** Let $\{X_n, n=1\}$ be a sequences of *i.i.d.* random variables with survival function $\overline{F}(x) = P[X_1 > x]$. A wavelet linear survival function $\overline{F}_n(x)$ based on $X_1, X_2, \ldots, X_n$ is introduced as an estimator for $\overline{F}_n(x)$. We establish that the $L_{p'}$-loss $(2=r=p'=\infty)$ of the linear wavelet survival function estimator for a stochastic processes convergence at the rate $n^{\frac{-s'}{2s'+1}}(s' = s - 1/p + 1/p')$ when the survival function, $\overline{F}_n(x)$ belongs to the Besov space $B^s_{p,q}$. Strong consistency and pointwise as well as uniform of $\overline{F}_n(x)$ are discussed.

**Key words**: Besov space . Wavelets . Survival function

## INTRODUCTION

Suppose that $\{X_n, n=1\}$ is a sequences of *i.i.d.* random variables with a common one-dimensional marginal probability density function f and survival function $\overline{F}(x) = P[X_1 > x]$. Bagai and Prakasa Rao [1] proposed empirical distribution and studied strong consistensy of it for sequence of associated random variables. Doosti and Zarei [2] extended their results to negatively associated case. shirazi and Doosti [3] and Doosti and Niroumand [4] extended the results for m-dependent and Mixing sequences of random variables. Antoniadis *et al.* [5] describes a wavelet method for the estimation of density and hazard rate functions from randomly right-censored data. Their method is based on dividing the time axis into a dyadic number of intervals and then counting the number of events within each interval. Wu and Wells [6] studied hazard rate estimation by non-linear wavelet methods in the setting of the counting process intensity model. An asymptotic formula for the mean integrated squared error (MISE) was provided.Rodriguez-casal and Una-alvarez [7] assume the Koziol-Green model of random censorship, under which the survival function of the censoring variable is a power of the survival function of interest.In this paper, a wavelet linear survival function $\overline{F}_n(x)$ based on $X_1, X_2, \ldots, X_n$ is introduced as an estimator for $\overline{F}_n(x)$. We establish that the $L_{p'}$-loss $(2=r=p'=\infty)$ of the linear wavelet survival function estimator for a stochastic processes convergence at the rate $n^{\frac{-s'}{2s'+1}}(s' = s - 1/p + 1/p')$ when the survival function, $\overline{F}_n(x)$ belongs to the Besov space $B^s_{p,q}$. Strong consistency and pointwise as well as uniform of $\overline{F}_n(x)$ are discussed.

Some preliminaries of the linear wavelet estimator of a survival function are given in section 2 and section 3 provides its asymptotic properties. Chapter 4 provides some conclusions and suggestions for future works.

## PRELIMINARY

Let $\{X\}_{i=0}$ be a sequence of real-valued random variables on the probability space $(\Omega, \aleph, P)$. We suppose that $X_i$ has a bounded and compactly supported marginal density f(.), with respect to Lebesgue measure, which does not depend on i We are interested in estimating this density from n observations $X_i$, i = 1,…,n. The motivation behind wavelet based linear estimator of the survival function comes from a formal expansion (Daubechies [8, 9]) for any function $\overline{F}(x) \in \mathbf{L}_2(\mathbf{R})$,

$$\overline{F}(x) = \sum_{k\in Z} \alpha_{j_0,k}\phi_{j_0,k} + \sum_{j\ge j_0}\sum_{k\in Z} \delta_{j,k}\psi_{j,k} = P_{j_0}\overline{F}(x) + \sum_{j\ge j_0} D_j\overline{F}(x)$$

where the functions

$$\phi_{j_0,k}(x) = 2^{j_0/2}\phi(2^{j_0}x - k)$$

and

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$$

constitute an (inhomogeneous) orthonormal basis of $L_2(R)$. Here $\phi(x)$ and $\psi(x)$ are the scale function and the orthogonal wavelet, respectively.Define

$$\overline{\phi}_{j_0,k}(x) = \int_{-\infty}^{x}\phi_{j_0,k}(t)dt \quad \overline{\psi}_{j,k}(x) = \int_{-\infty}^{x}\psi_{j,k}(t)dt$$

**Corresponding Author:** Dr. Hassan Doosti, Department of Statistics, School of Mathematical Sciences, Ferdowsi University, P.O. Box 91775, Mashhad, Iran

Wavelet coefficients are given by the integrals

$$\begin{aligned}\alpha_{j_0,k} &= \int \overline{F}(x)\phi_{j_0,k}(x)dx \\ &= \int_{-\infty}^{\infty} \overline{F}(x)\phi_{j_0,k}(x)dx \\ &= \int_{-\infty}^{\infty}\int_{x}^{\infty} f(t)\phi_{j_0,k}(x)dtdx \qquad (2.1)\\ &= \int_{-\infty}^{\infty}[\int_{-\infty}^{t} \phi_{j_0,k}(x)dx]f(t)dt \\ &= \mathbf{E}\overline{\phi}_{j_0,k}(X)\end{aligned}$$

similarly we could show $\delta_{j,k} = \mathbf{E}\overline{\psi}_{j,k}(X)$.
We could estimate coefficients as follows

$$\hat{\alpha}_{j_0,k} = \frac{1}{n}\sum_{i=1}^{n}\overline{\phi}_{j_0,k}(X_i)$$

$$\hat{\delta}_{j,k} = \frac{1}{n}\sum_{i=1}^{n}\overline{\psi}_{j,k}(X_i)$$

We suppose that both $\phi$ and $\psi \in C^{r+1}$, $r \in N$, have compact supports included in $[-\delta,\delta]$. Note that, by corollary 5.5.2 in Daubechies [8], $\psi$ is orthogonal to polynomials of degree =r, *i.e.*

$$\int \psi(x)x^l dx = 0, \forall l = 0,1,...,r$$

We suppose that $\overline{F}_n(x)$ belongs to the Besov class (see Meyer [10], §VI.10),

$$F_{s,p,q} = \{f \in B^s_{p,q}, ||f||_{B^s_{p,q}} \le M\}$$

for some 0<s<r+1, p=1 and q=1, where

$$||f||_{B^s_{p,q}} = ||P_{j_0}f||_p + (\sum_{j\ge j_0}(||D_j f||_p 2^{js})^q)^{1/q}$$

We may also say $\overline{F}(x) \in B^s_{p,q}$ if and only if

$$\| \alpha_{j_0,.} \|_{p(Z)} < \infty$$

and

$$(\sum_{j\ge j_0}(\| \delta_{j,.} \|_{p(Z)} 2^{j(s+1/2-1/p)})^q)^{1/q} < \infty \qquad (2.2)$$

where $\| \gamma_{j,.} \|_{l_p(Z)} = (\sum_{k\in Z}\gamma^p_{j,k})^{1/p}$. We consider Besov spaces essentially because of their executional expressive power [Triebel [11] and the discussion in Donoho *et al.* [12]]. We construct the survival function estimator

$$\overline{F}_n = \sum_{k\in K_{j0}} \hat{\alpha}_{j_0,k}\phi_{j_0,k} \qquad (2.3)$$

where $K_{j0}$ is the set of k such that $supp(\overline{F}) \cap supp(\phi_{j_0,k}) \ne \phi$

The fact that $\phi$ has a compact support implies that $K_{j0}$ is finite and Card $K_{j0} = O(2^{j0})$.

## MAIN RESULTS

**Theorem 1:** Let $\overline{F} \in F_{s,p,q}$ with s=1/p, p=1 and q=1 then for some r=2 and p′=max(r,p), there exists a constant C such that

$$\mathbf{E}||\overline{F}_n - \overline{F}||^r_{p'} \le Cn^{-\frac{s'}{1+2s'}}$$

where $s' = s + 1/p' - 1/p$ and $2^{j_0} = n^{\frac{1}{1+2s'}}$.

**Proof:** First, we decompose $\mathbf{E}||\overline{F}_n - \overline{F}||^r_{p'}$ into a bias term and stochastic term

$$\mathbf{E}||\overline{F}_n - \overline{F}||^r_{p'} \le 2\begin{pmatrix} ||\overline{F} - P_{j_0}\overline{F}||^r_{p'} \\ +\mathbf{E}||\overline{F}_n - P_{j_0}\overline{F}||^r_{p'}\end{pmatrix} = 2(T_1 + T_2) \qquad (3.1)$$

Now, we want to find upper bounds for $T_1$ and $T_2$.

$$T_1^{1/r} = ||\sum_{j\ge j_0} D_j\overline{F}||_{p'} \le \sum_{j\ge j_0}(||D_j\overline{F}||_{p'} 2^{js'})2^{-js'}$$

$$\le \{\sum_{j\ge j_0}(||D_j\overline{F}||_{p'} 2^{js'})^q\}^{1/q} \{\sum_{j\ge j_0} 2^{-js'q'}\}^{1/q'}$$

By Holder's inequality, with 1/q+1/q′ = 1, from the above equation, we have

$$T_1^{1/r} \le C||\overline{F}||_{B^{s'}_{p',q}} 2^{-s'j_0} \le C||\overline{F}||_{B^s_{p,q}} 2^{-s'j_0}.$$

The last inequality holds,because of the continuous Sobolev injection [see Triebel [11] and the discussion in Donoho *et al.* [12]] which implies that for $B^s_{p,q} \subset B^{s'}_{p',q}$, one gets,

$$||\overline{F}||_{B^{s'}_{p',q}} \le ||\overline{F}||_{B^s_{p,q}}$$

Therefore,

$$T_1 \le K2^{-rs'j_0} \qquad (3.2)$$

Next, we have

$$T_2 = \mathbf{E}\|\bar{F}_n - P_{j_0}\bar{F}\|_{p'}^2 = \mathbf{E}\|\sum_{k\in K_{j_0}}(\hat{\alpha}_{j_0,k} - \alpha_{j_0,k})\phi_{j_0,k}(x)\|_{p'}^2$$

Now the use of Lemma 1 in Leblanc [13], p. 82 (using Meyer [10]) gives

$$T_2 \le C\mathbf{E}\{\|\hat{\alpha}_{j_0,k} - \alpha_{j_0,k}\|_{l_{p'}}^r\}2^{rj_0(1/2-1/p')}$$

Further, by using Jensen's inequality the above equation implies,

$$T_2 \le C2^{rj_0(1/2-1/p')}\{\sum_{k\in K_{j_0}}\mathbf{E}|\hat{\alpha}_{j_0,k} - \alpha_{j_0,k}|^{p'}\}^{r/p'} \qquad (3.3)$$

To complete the proof, it is sufficient to estimate $\mathbf{E}|\hat{\alpha}_{j_0,k} - \alpha_{j_0,k}|^{p'}$. We know that

$$\hat{\alpha}_{j_0,k} - \alpha_{j_0,k} = \frac{1}{n}\sum_{i=1}^{n}\{[\bar{\phi}_{j_0,k}(X_i) - \alpha_{j_0,k}]\}.$$

Denote

$$\xi_i = [\bar{\phi}_{j_0,k}(X_i) - \alpha_{j_0,k}].$$

We have

$$\|\xi_i\|_\infty \le K\,2^{j_0/2},\quad \mathbf{E}\xi_i = 0, \mathbf{E}\xi_i^2 \le 2$$

and

$$|\hat{\alpha}_{j_0,k} - \alpha_{j_0,k}| = \tfrac{1}{n}|\textstyle\sum_{i=1}^n \xi_i|.$$

Using the Rosental's inequality and the fact that card $K_{j0} = O(2^{j0})$ we have

$$\{\sum_{k\in K_{j_0}}\mathbf{E}|\hat{\alpha}_{j_0,k} - \alpha_{j_0,k}|^{p'}\}^{r/p'} \le \{C2^{j_0}\frac{1}{n^{p'}}(n2^{j_0/2(p'-2)}c_1 + n^{p'/2}c_2)\}^{r/p'}$$
$$\le K_1\{\frac{2^{rj_0/2}}{n^{2(1-1/p')}} + \frac{2^{j_0}}{n^{-r/2}}\}.$$

Now by substituting above inequality in (3.3), we get

$$T_2 \le K_1 2^{rj_0(1/2-1/p')}\{\frac{2^{rj_0/2}}{n^{r(1-1/p')}} + \frac{2^{rj_0/p'}}{n^{r/2}}\} = K_1\{\frac{2^{rj_0 - rj_0/p'}}{n^{r-r/p'}} + \frac{2^{rj_0/2}}{n^{r/2}}\}$$
$$= K_1\{\frac{2^{rj_0/2}}{n^{r/2}}(\frac{2^{j_0}}{n})^{r/2(1-2/p')} + \frac{2^{rj_0/2}}{n^{r/2}}\},$$

since $n=2^{j0}$ and $r/2(1-2/p') \ge 0$ zmply

$$(\tfrac{2^{j_0}}{n})^{r/2(1-2/p')} \le 1$$

Hence

$$T_2 \le \frac{K_2 2^{rj_0/2}}{n^{r/2}} \qquad (3.4)$$

By substituting (3.2), (3.4) and $2^{j_0} = n^{\frac{1}{1+2s'}}$ in (3.1) theorem is proved.

**Theorem 2:** Let $\bar{F} \in F_{s,p,q}$ with s=1/p and p=1. Then for some r>1, there exists a constant C>0 such that, for every ε>0,

$$\sup_x P[|\bar{F}_n(x) - \bar{F}(x)| > \varepsilon] \le C\varepsilon^{-2r}n^{\frac{-2rs'}{1+2s'}} \quad \text{for every } n \ge 1,$$

where $s' = s - 1/p$ and $2^{j_0} = n^{\frac{1}{1+2s'}}$.

**Proof:** By using Markov inequality, we get that for every ε>0,

$$\begin{aligned}\sup_x P[|\bar{F}_n(x) - \bar{F}(x)| > \varepsilon] &= \sup_x P[(\bar{F}_n(x) - \bar{F}(x))^{2r} > \varepsilon^{2r}]\\ &\le \sup_x\{(\varepsilon)^{-2r}E|\bar{F}_n(x) - \bar{F}(x)|^{2r}\}\\ &\le (\varepsilon)^{-2r}E\|\bar{F}_n(x) - \bar{F}(x)\|_\infty^{2r}\\ &= (\varepsilon)^{-2r}n^{\frac{-2rs'}{1+2s'}}.\end{aligned} \qquad (3.5)$$

**Corollary 1:** Under the conditions of Theorem 3.2 for every x, if r>1+1/2s′, then

$$\bar{F}_n(x) \to \bar{F}(x) \quad \text{a.s.} \quad \text{as} \quad n \to \infty.$$

**Proof:** For r>1+1/2s′, observe that

$$\sum_{n=1}^{\infty}P[|\bar{F}_n(x) - \bar{F}(x)| > \varepsilon] \le C\varepsilon^{-2r}\sum_{n=1}^{\infty}n^{\frac{-2rs'}{1+2s'}} < \infty \qquad (3.6)$$

The result then follows by using the Borel-Contelli Lemma.

Next we ontaiend a version of Glivenko-Cantelli Theorem. The proof follows along the lines of analogous result in Bagai and Prakasa Rao [1].

Theorem Let $\{X_n, n=1\}$ be a stationary sequence of *i.i.d.* random variables satisfying the conditions of Theorem 3.2 Then for any compact subset J⊂R,

$$\sup[|\bar{F}_n(x) - \bar{F}(x)|: \quad x \in J] \to 0 \quad \text{a.s} \quad \text{as} \quad n \to \infty$$

**Proof:** Let $K_1$ and $K_2$ be chosen such that $J \subset [K_1, K_2]$ into $b_n$ sub-intervals of length $\delta_n \to 0$ where $\{\delta_n\}$ is chosen such that

$$\sum_n \delta_n^{-1} n^{\frac{-2rs'}{1+2s'}} < \infty \qquad (3.7)$$

such a choice of $\{\delta_n\}$ is possible. For instance, choose $\delta_n = n^{-\theta}$ where $0 < \theta < \frac{2rs'}{1+2s'} - 1$. Note that $b_n \le C\delta_n^{-1}$. Let

$$I_{nj} = (x_{n,j}, x_{n,j+1}), \quad j = 1,\ldots,b_n = N,$$

where

$$K_1 = x_{n,1} < x_{n,2} < \ldots < x_{n,N+1} = K_2,$$

with

$$x_{n,j+1} - x_{n,j} \le \delta_n$$

for $1 \le j \le N$. Then for $x \in I_{nj}$, $j = 1,2,\ldots,N$ we have

$$\bar{F}(x_{n,j+1}) \le \bar{F}(x) \le \bar{F}(x_{n,j}),$$

and

$$\bar{F}_n(x_{n,j+1}) \le \bar{F}_n(x) \le \bar{F}_n(x_{n,j}).$$

Hence

$$[\bar{F}_n(x_{n,j+1}) - \bar{F}(x_{n,j+1})] + [\bar{F}(x_{n,j+1}) - \bar{F}(x)]$$

$$\le \bar{F}_n(x) - \bar{F}(x) \le [\bar{F}_n(x_{n,j}) - \bar{F}(x_{n,j})] + [\bar{F}(x_{n,j}) - \bar{F}(x)].$$

Therefore

$$\begin{aligned}
\sup[|\bar{F}_n(x) - \bar{F}(x)| : x \in J] &\le \sup[|\bar{F}_n(x) - \bar{F}(x)| : K_1 \le x \le K_2] \\
&\le \max_{1\le j\le N} |\bar{F}_n(x_{n,j}) - \bar{F}(x_{n,j})| \\
&\quad + \max_{1\le j\le N} |\bar{F}(x_{n,j+1}) - \bar{F}(x_{n,j+1})| \\
&\quad + \max_{1\le j\le N} \sup_{x\in I_{nj}} |\bar{F}_n(x_{n,j}) - \bar{F}(x)| \\
&\quad + \max_{1\le j\le N} \sup_{x\in I_{nj}} |\bar{F}(x_{n,j+1}) - \bar{F}(x)|.
\end{aligned}$$

Now by the mean value theorem for $X_{n,j} < u^* < x$ we have

$$\bar{F}(x_{n,j}) - \bar{F}(x) = F(x) - F(x_{n,j}) = (x - x_{n,j}) f(u^*)$$

Since f, the density of $X_1$ is bounded by the hypothesis, it follows that there exists a constant C>0 such that

$$|\bar{F}(x_{n,j}) - \bar{F}(x)| \le C\delta_n, \qquad |\bar{F}(x_{n,j+1}) - \bar{F}(x)| \le C\delta_n$$

for $1 \le j = N$ and $x \in I_{nj}$. Then for $\varepsilon > 0$, choose $n = n(\varepsilon)$ such that

$$2C\delta_n \le \frac{1}{3}\varepsilon.$$

From (3.5) and (3.6), we get, for $n = n(\varepsilon)$,

$$\begin{aligned}
P[\sup_{x\in J} |\bar{F}_n(x) - \bar{F}(x)| > \varepsilon] &\le P[\max_{1\le j\le N} |\bar{F}_n(x_{n,j}) - \bar{F}(x_{n,j})| > \frac{1}{3}\varepsilon] \\
&\quad + P[\max_{1\le j\le N} |\bar{F}(x_{n,j+1}) - \bar{F}(x_{n,j+1})| > \frac{1}{3}\varepsilon] \\
&\le \sum_{j=1}^{N} P|\bar{F}_n(x_{n,j+1}) - \bar{F}(x_{n,j})| > \frac{1}{3}\varepsilon] \\
&\quad + \sum_{j=1}^{N} P[|\bar{F}_n(x_{n,j+1}) - \bar{F}(x_{n,j})| > \frac{1}{3}\varepsilon] \\
&\le CN\varepsilon^{-2r} n^{\frac{-2rs'}{1+2s'}} \quad \text{(by Theorem 3.2.)} \\
&\le C\varepsilon^{-2r} \delta_n^{-1} n^{\frac{-2rs'}{1+2s'}}
\end{aligned}$$

The result follows by using (3.7) and Borel-Cantelli Lemma.

**Remark 1:** Suppose $1<p'<2$. One can get upper bounds similar to those as the theorem 3.1 for the expected loss $\mathbf{E}\,||\bar{F}_n(x) - \bar{F}||_{p'}^{p'}$.

Observing that

$$\mathbf{E}\,||\bar{F}_n(x) - \bar{F}||_{p'}^{p'} \le 2^{p'-1}(||\bar{F} - P_{j_0}\bar{F}\,||_{p'}^{p'} + \mathbf{E}\,||\bar{F}_n(x) - P_{j_0}\bar{F}||_{p'}^{p'})$$

$$||\bar{F} - P_{j_0}\bar{F}\,||_{p'}^{p'} \le C_1 2^{-p's j_0}$$

$$\begin{aligned}
\mathbf{E}\,||\bar{F}_n(x) - P_{j_0}\bar{F}||_{p'}^{p'} &\le C_2 2^{2j_0(p'/2+)} \{\sum_{k\in K_{j_0}} \mathbf{E}\,|\hat{\alpha}_{j_0,k} - \alpha_{j_0 k}|^{p'}\} \\
&\le C_2 2^{2j_0(p'/2+)} \{\sum_{k\in K_{j_0}} \sqrt{\mathbf{E}\,|\hat{\alpha}_{j_0,k} - \alpha_{j_0 k}|^{2p'}}\}.
\end{aligned}$$

for some positive constant $C_1, C_2$.

**Remark 2:** By using some other version of Rosental's inequality, The above results could be easily extended for some classes of dependent random variables such as Negatively dependent (ND), Positive dependent (PA), m- dependent and so on.

## CONCLUSION AND SUGGESTIONS

The survival function, also known as a survivor function or reliability function, is a property of any random variable that maps a set of events, usually associated with mortality or failure of some system, onto time. It captures the probability that the system will survive beyond a specified time. The term reliability function is common in engineering while the term survival function is used in a broader range of applications, including human mortality. In this paper a wavelet linear survival function $\overline{F}_n(x)$ based on $X_1, X_2, \ldots, X_n$ is introduced as an estimator for $\overline{F}_n(x)$. We establish that the $L_{p'}$-loss $(2=r=p'=\infty)$ of the linear wavelet survival function estimator for a stochastic processes convergence at the rate $n^{\frac{-rs'}{2s'+1}} (s' = s - 1/p + 1/p')$ when the survival function, $\overline{F}_n(x)$ belongs to the Besov space $B^s_{p,q}$. Strong consistency and pointwise as well as uniform of $\overline{F}_n(x)$ are discussed. Some open problems which could be done in future works might be finding distribution of the estimator, asymptotic biased and variance of the estimator and comparing with old versions of survival function.

## ACKNOWLEDGEMENTS

Partial support from Ordered and Spatial Data Center of Excellence of Ferdowsi University of Mashhad is acknowledged.

## REFERENCES

1. Bagai, I. and B.L.S. Prakasa Rao, 1991. Estimation of the survival function for stationary associated processes. Statistics and Probability Letters, 12: 385-391.
2. Doosti, H. and Zarei, 2006. Estimation of the survival function for negatively associated processes. J. Sci. Islam. Repub. Iran, 17 (3): 249-251.
3. Shirazi, E. and H. Doosti, 2008. Estimation of the survival function for m-dependent random variables. Far East Journal of Theoretical Statistics, 25 (1): 135-144.
4. Doosti, H. and H.A. Niroumand, 2009. Estimation of the survival function for a discrete-time stochastic process. Pakistan Journal of Statistics, To Appear.
5. Antoniadis, A., G. Gregoire and G. Nason, 1999. Density and hazard rate estimation for right-censored data by using wavelet methods. Journal of the Royal Statistical Society. Series B. Methodological, 61: 63-84.
6. Wu, S. and M. Wells, 2003. Nonparametric Estimation of Hazard Function by Wavelet Methods. Journal of Nonparametric Statistics, 5 (2): 187-203.
7. Rodriguez-casal, A. and J. Una-alvarez, 2004. Nonlinear wavelet density estimation under the Koziol-Green model. Journal of Nonparametric Statistics, 16: 91-109.
8. Daubechies, I., 1988. Orthogonal bases of compactly supported wavelets. Communication in Pure and Applied Mathematics, 41: 909-996.
9. Daubechies, I., 1992. Ten Lectures on Wavelets, CBMS-NSF regional conferences series in applied mathematics. SIAM, Philadelphia.
10. Meyer, Y., 1990. Ondelettes et Operateurs, Hermann, Paris.
11. Tribiel, H., 1992. Theory of Function Space II. Birkhairkhauser Verlag, Berlin.
12. Donoho, D.L., I.M. Johnstone, G. Kerkyacharian and D. Picard, 1996. Density estimation by wavelet thresholding. The Annals of Statistics, 2: 508-539.
13. Leblanc, F. (1996). Wavelet linear density estimator for a discrete-time stochastic process: $L_p$-losses. Statistics and Probability Letters, 27: 71-84.