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Recurrence Relation for Single and Product Moments of Record Values from Erlang-truncated Exponential Distribution

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Abstract: Consider X_1, X_2, X_3,... be a sequence of independently and identically distributed random variables with continuous cumulative distribution function F(x). In this paper, some recurrence relations for single and product moments are derived for Erlang-truncated exponential distribution that are helpful in finding the higher order moments from that of lower order moments.

Key words: Erlang-truncated exponential distribution . upper record values . cumulative distribution function . recurrence relation

INTRODUCTION

A random variable X is said to have Erlang-truncated exponential distribution [1], if its probability density function is of the form

$$f(x) = \beta\left(1-e^{-\lambda}\right)e^{-\beta x\left(1-e^{-\lambda}\right)}, 0 \le x \le \infty, \beta > 0, \lambda > 0 \quad (1.1)$$

The distribution function is

$$F(x) = 1 - e^{-\beta x\left(1-e^{-\lambda}\right)} \quad (1.2)$$

Let X_1, X_2, X_3,... be a sequence of independently and identically distributed random variables with cdf F(x). Set $Y_i = max(min)\{X_1, X_2, ...X_i\}$ for i≥1, then X_j is called an upper (lower) record value of $\{X_i, i\geq 1\}$, if $Y_j > Y_{j-1}, j>1, \left(Y_j < Y_{j-1}, j>1\right)$

It obvious from the definition that X_1 is an upper as well as lower record.

Ahsanullah [2] have also derived the distributional properties of the records by using the Lomax distribution. Balakrishnan [5-7] have obtained the recurrence relations for moments of record values for Gumbel distribution. Some moment properties of the records have been given by Ahsanullah [3, 4]. Nevzorov [8] have given a comprehensive review of the mathematical foundation of the records.

Ahsanullah [4] has given the distribution of k-th upper record; $X_{U(k)}$ as:

$$f_{k:n}\left(x_k\right) = \frac{1}{\Gamma(k)} f\left(x_k\right)\left[R\left(x_k\right)\right]^{k-1} \quad (1.3)$$

where

$$R(x) = -\ln\{1 - F(x)\}.$$

The joint distribution of k-th and m-th upper records; $X_{U(k)}$ and $X_{U(m)}$ can be obtained by using the following expression given by Ahsanullah [4]:

$$f_{k,m:n}\left(x_k, x_m\right) = \frac{1}{\Gamma(k)\Gamma(m-k-1)} r\left(x_k\right) f\left(x_m\right)\left[R\left(x_k\right)\right]^{k-1} \left[R\left(x_m\right) - R\left(x_k\right)\right]^{m-k-1}, \quad (1.4)$$

with $r(x) = R'(x)$.

In this paper, some recurrence relations for single and product moments are derived for Erlang-truncated exponential distribution in section 2 and 3 respectively. Some concluding remarks are given in section 4.

RECURRENCE RELATION FOR SINGLE MOMENTS

It is easy to note from (1.1) and (1.2) that

$$x f(x) = \left[-ln\left(1 - F(x)\right)\right]\left(1 - F(x)\right) \quad (2.1)$$

The relation in (2.1) will be used to establish recurrence relations for moments of the upper record value from Erlang-truncated exponential distribution.

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Theorem 2.1: For $n \geq 1$, $k = 0, 1, 2, \ldots\ldots\ldots$ and $k < n$,

$$E\left(X_{U(n+1)}^{k}\right) = (n-k)\, E\left(X_{U(n)}^{k}\right)$$

Proof: The k-th moment of the n-th upper record is defined as:

$$E\left(X_{U(n)}^{k}\right) = \frac{1}{\Gamma n}\int_{0}^{\infty} x^{k}\left[R(x)\right]^{n-1} f(x)\, dx. \qquad (2.2)$$

Using (2.1) in (2.2), we have

$$E\left(X_{U(n)}^{k}\right) = \frac{1}{\Gamma(n)}\int_{0}^{\infty} x^{k-1}\left[-\ln\left(1-F(x)\right)\right]^{n}\left(1-F(x)\right) dx. \quad (2.3)$$

Integrating (2.3) by parts taking X^{K-1} as integrating and rest of integrand for differentiation, we obtain:

$$E\left(X_{U(n+1)}^{k}\right) = (n-k)\, E\left(X_{U(n)}^{k}\right) \;\; ; k < n$$

RECURRENCE RELATIONS FOR PRODUCT MOMENTS

In this section some recurrence relation for product moments for Erlang-truncated exponential distribution have been developed.

Theorem 3.1: For $1 \leq m < n$, r, s = 0, 1, 2,... and $m > (r+1)$,

$$E\left(X_{U(m+1)}^{s+r+1}\right) = \frac{m-(r+1)}{m} E\left(X_{U(m)}^{r+1} X_{U(n)}^{s}\right)$$

And

For $n \geq m+2$, r, s =0, 1,2,……… and $m > (r+1)$

$$E\left(X_{U(m)}^{r+1} X_{u(n)}^{s}\right) = \frac{m-(r+1)}{m}\; E\left(X_{U(m+1)}^{r+1} X_{U(n-1)}^{s}\right)$$

Proof: We have

$$E\left(X_{U(m)}^{r+1} X_{U(n)}^{s}\right) = \int_{0}^{\infty}\int_{y}^{\infty} x^{r+1} y^{s} f_{m,n}(x,y)\, dx\, dy.$$

$$= \frac{1}{\Gamma(m)\Gamma(n-m)}\int_{0}^{\infty}\int_{y}^{\infty} x^{r+1} y^{s}\left[-\ln\left(1-f(x)\right)\right]^{m-1}\left[-\ln\left(1-F(y)\right)+\ln\left(1-F(x)\right)\right]^{n-m-1}\frac{f(x)}{\left[1-f(x)\right]} f(y)\, dx dy. \quad (3.1)$$

$$= \frac{1}{\Gamma(m)\Gamma(n-m)}\int_{0}^{\infty}\int_{y}^{\infty} x^{r+1} y^{s}\left[-\ln\left(1-f(x)\right)\right]^{m-1}\left[-\ln\left(1-F(y)\right)+\ln\left(1-F(x)\right)\right]^{n-m-1}\frac{f(x)}{\left[1-f(x)\right]} f(y)\, dx dy.$$

Substituting (2.1) in (3.1) we get

$$E\left(X_{U(m)}^{r+1} X_{U(n)}^{s}\right) = \frac{1}{\Gamma(m)\,\Gamma(n-m)}\int_{0}^{\infty}\int_{y}^{\infty} x^{r} y^{s}\left[-\ln\left(1-F(x)\right)\right]^{m}\left[-\ln\left(1-F(y)\right)+\ln\left(1-F(x)\right)\right]^{n-m-1} f(y)\, dx dy.$$

$$= \frac{1}{\Gamma(m)\,\Gamma(n-m)}\int_{0}^{\infty} y^{s}\, f(y)\, I(y)\, dy. \qquad (3.2)$$

Where

$$I(y) = \int_{y}^{\infty} x^{r}\left[-\ln\left(1-F(x)\right)\right]^{m}\left[-\ln\left(1-F(y)\right)+\ln\left(1-F(x)\right)\right]^{n-m-1} dx. \qquad (3.3)$$

Case-1. If $n = m + 1$.

Integrating (3.3) by parts taking X^{r} as integrating and rest of integrand for differentiation, we obtain:

$$I(y) = \left[-\left\{-\ln\left(1-F(y)\right)\right\}^{m}\frac{y^{r+1}}{r+1}\right] + \frac{m}{r+1}\int_{y}^{\infty} x^{r+1}\left[-\ln\left(1-F(x)\right)\right]^{m-1}\frac{f(x)}{1-F(x)}\, dx.$$

Substituting the value of I(y) in (3.2) and simplifying we get

$$E\left(X_{U(m)}^{r+1}X_{U(n)}^{s}\right)=\frac{1}{r+1}\left[\begin{array}{l}\frac{m}{\Gamma(m)}\int_0^\infty\int_y^\infty x^{r+1}y^s\left[-\ln\left(1-F(x)\right)\right]^{m-1}\frac{f(x)f(y)}{1-F(x)}\,dx\,dy\\ -\frac{m}{\Gamma(m+1)}\int_0^\infty y^{s+r+1}\left[-\ln\left(1-F(x)\right)\right]^{m}f(y)dy\end{array}\right]$$

$$E\left(X_{U(m)}^{r+1}X_{U(n)}^{s}\right)=\frac{m}{(r+1)}\left[E\left(X_{U(m)}^{r+1}X_{U(n)}^{s}\right)-E\left(X_{U(m+1)}^{s+r+1}\right)\right]$$

or

$$E\left(X_{U(m+1)}^{s+r+1}\right)=\frac{m-(r+1)}{m}E\left(X_{U(m)}^{r+1}X_{U(n)}^{s}\right) \tag{3.4}$$

Case-2: When n≥m+2,

Again, integrating (3.3) by parts taking X^r as integrating and rest of integrand for differentiation, we obtain:

$$\begin{aligned}I(y)=&\frac{m}{r+1}\int_y^\infty x^{r+1}\left[-\ln\left(1-F(x)\right)\right]^{m-1}\left[-\ln\left(1-F(y)\right)+\ln\left(1-F(x)\right)\right]^{n-m-1}\\ &\frac{f(x)}{1-F(x)}dx-\frac{(n-m-1)}{r+1}\int_y^\infty x^{r+1}\left[-\ln\left(1-f(x)\right)\right]^{m}\left[-\ln\left(1-F(y)\right)+\ln\left(1-F(x)\right)\right]^{n-m-2}\frac{f(x)}{1-F(x)}dx.\end{aligned} \tag{3.5}$$

Putting (3.5) in (3.2), we get

$$\begin{aligned}E\left(X_{U(m)}^{r+1}X_{U(n)}^{s}\right)=&\frac{1}{\Gamma(n)\Gamma(n-m)}\Bigg[\frac{m}{r+1}\int_0^\infty\int_y^\infty x^{r+1}y^s\left[-\ln\left(1-F(x)\right)\right]^{m-1}\left[-\ln\left(1-F(y)\right)+\ln\left(1-F(x)\right)\right]^{n-m-1}\\ &\frac{f(x)}{1-F(x)}f(y)\,dx\,dy-\frac{(n-m-1)}{r+1}\int_0^\infty\int_y^\infty x^{r+1}y^s\left[-\ln\left(1-F(x)\right)\right]^{m}\\ &\left[-\ln\left(1-F(y)\right)+\ln\left(1-F(x)\right)\right]^{n-m-2}\frac{f(x)}{1-F(x)}f(y)\,dxdy.\Bigg]\end{aligned} \tag{3.6}$$

$$\begin{aligned}E\left(X_{U(m)}^{r+1}X_{U(n)}^{S}\right)&=\frac{m}{(r+1)}E\left(X_{U(m)}^{r+1}X_{U(n)}^{s}\right)-\frac{m}{(r+1)}E\left(X_{U(m+1)}^{r+1}X_{U(n-1)}^{s}\right).\\ E\left(X_{U(m)}^{r+1}X_{U(n)}^{s}\right)&=\frac{m-(r+1)}{m}E\left(X_{U(m+1)}^{r+1}X_{U(n-1)}^{s}\right),\qquad r\,,s>0.\end{aligned} \tag{3.7}$$

Corollary: From (3.4) substituting the value of $E\left(X_{U(m)}^{r+1}X_{U(n)}^{s}\right)$ in (3.7), we get

$$E\left(X_{U(m+1)}^{s+r+1}\right)=\left[\frac{m-(r+1)}{m}\right]^2E\left(X_{U(m+1)}^{r+1}X_{U(n-1)}^{s}\right) \tag{3.8}$$

CONCLUSION

The recurrence relations for single and product moments of the record statistics for the Erlang-truncated exponential distribution are derived in this paper. Recurrence relations are useful to characterize the distribution and to reduce the number of operation necessary to obtain a general form for the function under consideration. It can be hoped that following the same procedure researcher may derived the recurrence relation for other continuous distributions.

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