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Speed of Thermoelastic Rayleigh Wave in a Transversely Isotropic Heat-conducting Elastic Material

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Abstract: Thermal effects on Rayleigh wave speed in transversely isotropic medium are studied. A formula for the speed is derived first time in the said material. The speed of waves in some model transversely isotropic materials is calculated and is compared with the speed of the waves which propagate without thermal effects. It is observed that two Rayleigh waves propagate in the material under thermal effect. One wave propagates with the speed of the wave which propagates without thermal effect and the other one propagates with some higher speed.

Key words: Rayleigh waves . transversely isotropic . orthotropic . strain energy

INTRODUCTION

The theory of irreversible thermodynamics for an elastic material was established by Biot [1]. His coupled equations for thermo-elastic waves in an isotropic material were solved by Deresiewicz [2], Lessen [3], Chadwick and Sneddon [4] and Chadwick [5] and in a transversely isotropic medium by Chadwick and Seet [6]. Thermo-elastic Rayleigh waves were studied by Chadwick [5] in isotropic material. In this article we have studied the thermo-elastic Rayleigh waves in transversely isotropic conducting material using the uncoupled theory of thermodynamics in non-steady temperature field case only.

We have observed that two Rayleigh waves propagate in transversely isotropic material under thermal effect. One wave propagates with the same speed as that of the wave which propagates without thermal effect and the other one propagates with some higher speed.

BASIC EQUATIONS AND INEQUALITIES

Consider a semi-infinite stress-free surface of a homogeneous heat-conducting elastic material which is transversely isotropic in both elastic and thermal response and we choose a system of rectangular Cartesian coordinates x_1, x_2, x_3 in such a way that x_3-axis (axis of symmetry) is normal to the boundary and the body occupies the region $x_3 \le 0$.

We consider a plane harmonic wave in x_1-direction in x_1x_3-plane with displacement components (u_1, u_2, u_3) such that

$$u_i = u_i(x_1, x_3, t),\ i = 1, 3,\ u_2 = 0 \tag{1}$$

Then by using the relations [7]

$$\begin{aligned}\sigma_{11} &= c_{11}u_{1,1} + c_{13}u_{3,3} - \beta_1\theta \\ \sigma_{33} &= c_{13}u_{1,1} + c_{33}u_{3,3} - \beta_2\theta \\ \sigma_{13} &= c_{44}(u_{1,3} + u_{3,1})\end{aligned} \tag{2}$$

and by following Chadwick and Seet [6] the linearized equations of motion in the absence of body forces and heat supply are

$$\begin{aligned}c_{11}u_{1,11} + c_{13}u_{3,31} + c_{44}(u_{1,33} + u_{3,13}) - \beta_1\theta_{,1} &= \rho\ddot{u}_1 \\ c_{44}(u_{1,31} + u_{3,11}) + c_{13}u_{1,13} + c_{33}u_{3,33} - \beta_2\theta_{,3} &= \rho\ddot{u}_3\end{aligned} \tag{3}$$

The non-steady temperature field is governed by the following equation of heat conduction [7]

$$\kappa_1\theta_{,11} + \kappa_2\theta_{,33} = \rho c_v\dot{\theta} \tag{4}$$

where θ is the temperature change measured from a natural reference configuration of the material in which the density and temperature have the uniform values ρ and T_0 respectively. The comma notation is used for partial derivatives with respect to x_1, x_3 and a dot denotes a partial differentiation with respect to the time variable t. The material constants appearing in the equations (1) and (2) are five independent isothermal linear elasticities c_{11}, c_{12}, c_{13}, c_{33}, c_{44}, the specific heat at constant deformation c_v, the two independent first temperature coefficients of stress,

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β_1, β_2 and the two independent linear thermal conductivities κ_1, κ_2,; where

$$\beta_1 = \alpha_1(c_{11}+c_{12}) + \alpha_2 c_{13},\ \beta_2 = 2\alpha_1 c_{13} + \alpha_2 c_{33}$$

α_1, α_2, are the two independent linear coefficients of thermal expansion and it can be proved that $\kappa_1 \geq 0$, $\kappa_2 \geq 0$ and of course, $\rho > 0$, $T_0 > 0$. We assume in addition that $c_v > 0$ and that the isothermal linear elasticities are components of a positive definite fourth order tensor. Necessary and sufficient conditions for the latter requirements are [6]

$$c_{11} > 0,\ c_{11}^2 > c_{12}^2,\ c_{44} > 0,\ c_{33}(c_{11}+c_{12}) > 2c_{13}^2 \qquad (5)$$

The boundary conditions of zero traction are

$$\sigma_{3i} = 0 \quad , i = 1,3 \quad \text{on the plane} \quad x_3 = 0 \qquad (6)$$

Usual requirements that the displacement and the stress components decay away from the boundary implies

$$u_i \to 0\,, \quad \sigma_{ij} \to 0 \ \ (i,j = 1,3) \ \text{as} \ \ x_3 \to -\infty \qquad (7)$$

SECULAR EQUATION

By following Pham and Ogden [8] and Chadwick [5] we may assume;

$$\begin{aligned} u_j &= \varphi_j(y)\exp[ik(x_1 - ct)]\ ,\ j = 1,3 \\ \theta &= k\psi(y)\exp[ik(x_1 - ct)] \ \ \text{where}\ y = kx_3 \end{aligned} \qquad (8)$$

where k is the wave number, c is the wave speed, φ_j, ψ are the functions to be determined.
Substituting Eqs.(8) in (3) and (4) we have

$$\begin{aligned} &(\rho c^2 - c_{11})\varphi_1 + i(c_{13}+c_{44})\varphi_3' + c_{44}\varphi_1'' - \beta_1\psi = 0 \\ &(\rho c^2 - c_{44})\varphi_3 + i(c_{13}+c_{44})\varphi_1' + c_{33}\varphi_3'' - \beta_2\psi' = 0 \\ &\kappa_2 k\psi'' - (\kappa_1 k - i\rho c_V c)\psi = 0 \end{aligned} \qquad (9)$$

The boundary conditions (6) and (7) may be written as

$$\left.\begin{aligned} &ic_{13}\varphi_1 + c_{33}\varphi_3' - \beta_2 k\psi(y) = 0 \\ &\varphi_1' + i\varphi_3 = 0 \end{aligned}\right\} \text{on the plane, } y=0 \qquad (10)$$

$$\text{and} \quad \varphi_j, \varphi_j', \psi, \psi' \to 0,\ j = 1,3 \ \ \text{as}\ y \to -\infty \qquad (11)$$

Imposing the conditions (11), we assume from (9)

$$\begin{aligned} \varphi_1(y) &= A\exp(s_1 y) + B\exp(s_2 y) + C\exp(s_3 y) \\ \varphi_3(y) &= \alpha_1 A\exp(s_1 y) + \alpha_2 B\exp(s_2 y) + \alpha_3 C\exp(s_3 y) \\ \psi(y) &= \gamma_3 C\exp(s_3 y) \end{aligned} \qquad (12)$$

where s_1, s_2, s_3 have positive real parts and

$$a_j = -\frac{i[\{\beta_2(\rho c^2 - c_{11}) + \beta_1(c_{13}+c_{44})\}s_j + c_{44}\beta_2 s_j^3]}{\{\beta_1 c_{33} - \beta_2(c_{13}+c_{44})\}s_j^2 + \beta_1(\rho c^2 - c_{44})};\ j = 1,2,3$$

$$\gamma_3 = -\frac{c_{33}c_{44}s_3^4 + [c_{33}(\rho c^2 - c_{11}) + c_{44}(\rho c^2 - c_{44}) + (c_{13}+c_{44})^2]s_3^2 + (\rho c^2 - c_{11})(\rho c^2 - c_{44})}{\{\beta_1 c_{33} - \beta_2(c_{13}+c_{44})\}s_j^2 + \beta_1(\rho c^2 - c_{44})}$$

In Eqs. (12) s_1^2, s_2^2 are the roots of the equation

$$\begin{aligned} &c_{33}c_{44}s^4 + [c_{33}(\rho c^2 - c_{11}) + c_{44}(\rho c^2 - c_{44}) + (c_{13}+c_{44})^2]s^2 \\ &\quad + (\rho c^2 - c_{11})(\rho c^2 - c_{44}) = 0 \end{aligned} \qquad (13)$$

having the following relations

$$\begin{aligned} s_1^2 + s_2^2 &= -\frac{c_{33}(\rho c^2 - c_{11}) + c_{44}(\rho c^2 - c_{44}) + (c_{13}+c_{44})^2}{c_{33}c_{44}} \\ s_1^2 s_2^2 &= \frac{(\rho c^2 - c_{11})(\rho c^2 - c_{44})}{c_{33}c_{44}} \end{aligned} \qquad (14)$$

and

$$s_3^2 = \frac{\kappa_1 k - i\rho c_v c}{\kappa_2 k} \qquad (15)$$

Substituting the solutions (12) into the boundary conditions (10) and the thermal boundary condition [5]

$$\frac{\partial\theta}{\partial y} + h\theta = 0 \qquad (16)$$

on the plane, y=0
where h is non-negative thermal constant, we obtain

$$\begin{aligned} &(ic_{13} + c_{33}s_1\alpha_1)A + (ic_{13} + c_{33}s_2\alpha_2)B - \beta_2\gamma_3 kC\exp(s_3 y) = 0 \\ &(s_1 + i\alpha_1)A + (s_2 + i\alpha_2)B = 0 \\ &k(s_3 + h)C = 0 \end{aligned} \qquad (17)$$

The last equation of Eqs. (17) implies C=0 (because if $C \neq 0$, then absurd values of h and c are obtained.) Therefore, from (17) we obtain

$$\begin{aligned} &(ic_{13} + c_{33}s_1\alpha_1)A + (ic_{13} + c_{33}s_2\alpha_2)B = 0 \\ &(s_1 + i\alpha_1)A + (s_2 + i\alpha_2)B = 0 \end{aligned} \qquad (18)$$

The condition of consistency between these two homogeneous equations is

$$\left.\begin{matrix}\{\beta_1 c_{33}-\beta_2(c_{13}+c_{44})\}[c_{13}\{\beta_1 c_{33}-\beta_2 c_{13}\}- \\ c_{33}\{\beta_2(\rho c^2-c_{11})+\beta_1(c_{13}+c_{44})\}]+c_{33}c_{44}\beta_1\beta_2(\rho c^2-c_{44})\end{matrix}\right\} s_1^2 s_2^2+\beta_1 c_{13}(\rho c^2-c_{44})(\beta_1 c_{33}-\beta_2 c_{13})(s_1^2+s_2^2)+$$
$$\left\{\{\beta_2(\rho c^2-c_{11})+\beta_1(c_{13}+c_{44})\}[c_{33}\{\beta_1(\rho c^2+c_{13})+\beta_2(\rho c^2-c_{11})\}-c_{13}\{\beta_1 c_{33}-\beta_2(c_{13}+c_{44})\}]+\beta_1\beta_2 c_{13}c_{44}(\rho c^2-c_{44})\right\}s_1 s_2+ \tag{19}$$
$$\beta_2 c_{33}c_{44}(\beta_1 c_{33}-\beta_2 c_{13})s_1^3 s_2^3+\beta_2 c_{33}c_{44}\{\beta_1(\rho c^2+c_{13})+\beta_2(\rho c^2-c_{11})\}s_1 s_2(s_1^2+s_2^2)+\beta_1 c_{13}(\rho c^2-c_{44})\{\beta_1(\rho c^2+c_{13})+\beta_2(\rho c^2-c_{11})]=0$$

Table 1: Basic data for single crystals of three metals

Quantity	Units	Cobalt	Magnesium	Zinc
ρ_0	Kg m^{-3}	8.836×10^3	1.74×10^3	7.14×10^3
T_0	^{0}K	298	298	296
c_{11}	N m^{-2}	3.071×10^{11}	5.974×10^{10}	1.628×10^{11}
c_{12}	N m^{-2}	1.650×10^{11}	2.64×10^{10}	0.362×10^{11}
c_{13}	N m^{-2}	1.027×10^{11}	2.17×10^{10}	0.508×10^{11}
c_{33}	N m^{-2}	3.581×10^{11}	6.17×10^{10}	0.627×10^{11}
C_{44}	N m^{-2}	0.755×10^{11}	1.639×10^{10}	0.385×10^{11}
β_1	N m^{-2} deg^{-1}	7.04×10^6	2.68×10^6	5.75×10^6
β_2	N m^{-2} deg^{-1}	6.90×10^6	2.68×10^6	5.17×10^6
c_v	J kg^{-1} deg^{-1}	4.27×10^2	1.04×10^3	3.9×10^2
κ_1	W m^{-1} deg^{-1}	0.690×10^2	1.7×10^2	1.24×10^2
κ_2	W m^{-1} deg^{-1}	0.690×10^2	1.7×10^2	1.24×10^2

Table 2: Rayleigh wave speed under thermal effect

Materials	Rayleigh wave speed (m/s)
Cobalt	2811.58, 2923.11
Magnesium	2894.65, 3069.13
Zinc	2045.01, 2322.1

Table 3: Rayleigh wave speed without thermal effect

Materials	Rayleigh wave speed (m/s)
Cobalt	2811.58
Magnesium	2894.65
Zinc	2045.01

where the values of $s_1^2 s_2^2, s_1^2+s_2^2, s_1 s_2, s_1^3 s_2^3$ and $s_1 s_2(s_1^2+s_2^2)$ can be obtained from Eqs. (14). It is evident from Eqs. (12) and (14) that if s_1^2, s_2^2 are real, they must be positive in order to ensure that s_1, s_2 should have positive real parts. But if s_1^2, s_2^2 are complex, they must be complex conjugate. In both the cases $s_1^2 s_2^2$ must be positive. Therefore, from Eq. (14b)

$$(\rho c^2-c_{11})(\rho c^2-c_{44})>0 \tag{20}$$

Therefore, either $0<\rho c^2<\min\{c_{11}, c_{44}\}$ or $\rho c^2>\max\{c_{11}, c_{44}\}$. But if the latter inequality holds, then it is evident that right-hand side of Eq. (14a) will be negative and so Eq. (!3) will have two negative real roots s_1^2, s_2^2. This contradicts the requirement that s_1, s_2 should have positive real parts. Therefore, the Rayleigh wave speed must satisfy the following inequality

$$0<\rho c^2<\min\{c_{11}, c_{44}\} \tag{21}$$

The inequality (21) is the same necessary condition for Rayleigh wave propagation in transversely isotropic material as that of the wave without thermal effect [9].

MODEL WORK

Here we derive the Rayleigh wave speed for some of the transversely isotropic materials Cobalt, Magnesium and Zinc. We take the elastic and thermal constants from the Table 1 [6].

Now substituting the values of $s_1^2 s_2^2$, $s_1^2+s_2^2$, $s_1 s_2$, $s_1^3 s_2^3$ and $s_1 s_2(s_1^2+s_2^2)$ from Eqs. (14) and elastic and thermal constants from Table 1 into the Eq. (19) and by using the computer software Mathematica we obtain the values of the Rayleigh wave speeds for the three materials. We accept only those values of speeds which satisfy the inequality (21) and are shown in Table 2.

Rayleigh wave speed without thermal effect for the said materials is shown in Table 3 (see,[9])

Comparing Table 2 and 3 we observe that one extra Rayleigh wave with some higher speed propagates under thermal effect.

CONCLUSION

Rayleigh wave speed in some model transversely isotropic materials, under thermal and without thermal effect, is calculated. It is observed that two Rayleigh waves propagate under thermal effect. One wave propagates with the same speed as that of the wave which propagates without thermal effect and the other one propagates with some higher speed. It is also observed that the necessary condition for Rayleigh wave propagation in the said material does not change under thermal effect.

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