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Some Results on Clean Rings and Modules

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Abstract: An element in a ring R is said to be strongly n-clean for n≥1 if it is a sum of a nonzero Idempotent and n invertible elements. The ring R is called strongly n-clean if each element of R is strongly n-clean. We shall extend strongly 2-clean and then we shall show that, if R is strongly 2-clean then the endomorphism ring of any free R-module is also strongly 2-clean.

Key words: Strongly n-clean . idempotent element . invertible element

INTRODUCTION

Throughout this paper, R denotes a ring with Identity and all modules will be unitary.

An element in R is said to be (strongly) n-clean, if it is a sum of an (nonzero) idempotent element and n invertible elements.

The ring R is called (strongly) n-clean if every element of R is (strongly) n-clean. Let R be a ring. R is called a potent ring if every ideal $I\not\subset J(R)$ contains a nonzero idempotent. A ring R is called I-finite if it contains no infinite set of orthogonal idempotents.

1-clean rings were introduced for the first time by W.K Nicholson [1].we use U(R) for the group of units, Id (R) for the set of all idempotents and J(R) for the Jacobson radical of R also $M_n(R)$ for n×n matrics over R and $End_R(M)$ for the set of all endomorphisms of the R-module M.

Strongly 2-clean rings

Definition 2.1: An element x of a ring R is called strongly 2-clean if $x = e + u_1 + u_2$ where $0\neq e\in Id(R)$ and $u_i\in U(R)$ for i = 1, 2. The ring R is called strongly 2-clean if every element of R is strongly 2-clean.

Example: Since in Z_3 we have

$$\overline{0}=\overline{1}+\overline{1}+\overline{1}, \overline{1}=\overline{1}+\overline{1}+\overline{2} \text{ and } \overline{2}=\overline{1}+\overline{2}+\overline{2}$$

Hence $Z_3=\{\overline{0},\overline{1},\overline{2}\}$ is a strongly 2- clean, also it is clear that $Z_3\times Z_2$ is a strongly 2-clean but Z_2 is not strongly 2-clean. It is observed in [1] that the ring $\prod_{\alpha\in I} R_\alpha$ is clean if and only if R_α is clean for every $\alpha\in I$. In the following we shall try to prove the above result for strongly n-clean rings.

Proposition 2.2: Let $\{R_\alpha\}_{\alpha\in I}$ be a family of rings such that at least one of them is strongly n-clean and the others are n-clean, then $\prod_{\alpha\in I} R_\alpha$ is strongly n-clean.

Proof: For example let $R_\beta(\beta\in I)$ be strongly 1-clean and let $(r_\alpha)_{\alpha\in I}\in\prod_{\alpha\in I} R_\alpha$, since for all $\alpha\in I$, R_α is 1-clean so $r_\alpha = e_\alpha+u_\alpha$ where $e_\alpha(\alpha\in I)$ are idempotents and u_α are units and since R_β is strongly1-clean hence $e_\beta\neq 0$.So $r_\alpha = e_\alpha + u_\alpha$ where clearly (e_α) is a nonzero idempotent and (u_α) is invertible element of $\prod_{\alpha\in I} R_\alpha$, hence $\prod_{\alpha\in I} R_\alpha$ is strongly 1-clean ring, now by simple argument result for every n is clear.Now by example 1, The reverse of proposition 2 is not true.

Corollary 2.3: Let R be a strongly n-clean, then R[[x]] is strongly n-clean.

Proof: Since we have

$$R[[x]]=\prod R$$

(R[[x]] = {(a, b, c, …) | a, b, c,…∈ R }) hence by proposition 2 it is clear.

It is clear that if ϕ: R→S is a monomorphism ring, $0\neq\phi(e)\in Id(R)$ and $u\in U(R)$ Then $0\neq\phi(e)\in Id(S)$ and $\phi(u)$ is belong to U(S) In fact, we have The following proposition.

Proposition 2.4: Let *f*: R→S be homomorphism rings and $Id(R)\cap\ker f = \{0\}$. If R is strongly n-clean then Im *f* are strongly n-clean rings.

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Proof: Let $s \in \operatorname{Im} f$ be an arbitrary element, so there exist $r \in R$ such that $f(r) = s$. But as R is strongly n-clean so, we have $r = e + u_1 + u_2 + ... + u_n$, where $0 \neq e \in Id(R)$ and $u_1, \dots, u_n$ are belong to U(R). Therefore $S = f(r) = f(e) + f(u_1) + ... + f(u_n)$ since $f(e)$ is a nonzero idempotent of $f(R)$ and $f(u_1), \dots, f(u_n)$ are invertible element of $f(R)$ hence $\operatorname{Im} f$ is strongly n-clean rings.

Corollary 2.5: Let R be a strongly n-clean ring and I an ideal of R such that $I \cap Id(R) = \{0\}$, then R/I is a strongly n-clean ring.

Proof: It follows by proposition 2.4.

Lemma 2.6: Let R be a ring such that *e*Re and (1-*e*)R(1-*e*) are both strongly n-clean where e is an idempotent in R. Then R is strongly n-clean.

Proof: By using of pierce decomposition and writing $\bar{r}$ for 1 – r we have:

$$R = \begin{pmatrix} eRe & eR\bar{e} \\ \bar{e}Re & \bar{e}R\bar{e} \end{pmatrix} . \text{Let} A = \begin{pmatrix} a & x \\ y & b \end{pmatrix} \in R,$$

and by hypothesis $a = f + u_1 + u_2 + ... + u_n$ where $0 \neq f = f^2 \in eRe$ and $u_1, u_2, \dots, u_n$ are units in *e*Re with inverses $u'_1, u'_2, \dots, u'_n$ respectively. Then $b - y u'_1 x \in \bar{e}R\bar{e}$ so write $b - y u'_1 x = g + v_1 + v_2 + ... + v_n$ where $0 \neq g = g^2 \in \bar{e}R\bar{e}$ and $v_1, \dots, v_n$ are units in $\bar{e}R\bar{e}$ with inverses $v'_1, \dots, v'_n$ respectively. Hence

$$A = \begin{pmatrix} f + u_1 + u_2 + ... + u_n & x \\ y & g + v_1 + v_2 + ... + v_n + yu'_1 x \end{pmatrix} = \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}$$
$$+ \begin{pmatrix} u_1 + u_2 + ... + u_n & x \\ y & v_1 + v_2 + ... + v_n + yu'_1 x \end{pmatrix} = \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} +$$
$$\begin{pmatrix} u_1 & 0 \\ 0 & v_n + y u'_1 x \end{pmatrix} + \begin{pmatrix} u_2 & 0 \\ 0 & v_1 \end{pmatrix} + ... + \begin{pmatrix} u_n & 0 \\ 0 & v_{n-1} \end{pmatrix}$$

Clearly $\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}$

is a nonzero idempotent and

$$\begin{pmatrix} u_2 & 0 \\ 0 & v_1 \end{pmatrix}, \dots, \begin{pmatrix} u_n & 0 \\ 0 & v_{n-1} \end{pmatrix}$$

are invertible. Also by using of pierce decomposition [2] it is simple to show that:

$$\begin{pmatrix} u_1 & 0 \\ 0 & v_n + yu'_1 x \end{pmatrix}$$

is a unit in R. so A is a strongly n-clean element and the proof is complete.

Corollary 2.7: Let R be a Ring and $e_1 + e_2 = 1$ where e_1 and e_2 are orthogonal Idempotent and each $e_i R e_i$ is strongly n-clean.then R is strongly n-clean. Using the lemma 2.6 and corollary 2.7, an inductive argument gives immediately.

Theorem 2.8: Let $e_1 + e_2 + .. + e_n = 1$ in a ring R where e_i are orthogonal idempotents and each $e_i R e_i$ is strongly n-clean, Then R is Strongly n - clean.

The following results are direct consequences of the previous theorem.

Corollary 2.9: If R is a strongly n-clean ring so also is the matrix ring $M_n(R)$

Corollary 2.10: If $M = M_1 \oplus M_2 \oplus ... \oplus M_n$ are modules and $End_R(M_i)$ is strongly n-clean for each i, then $End_R(M)$ is strongly n-clean for each i, then $End_R(M)$ is strongly n-clean.

Proposition 2.11: Let $0 \neq e \in R$ be an Idempotent and the ring End(Re) has exactly one nonzero Idempotent e also let R be a strongly 1-clean then R is a potent ring and *e*Re is a local ring.

Proof: Since 0 and e are the only Idempotents in End(Re), hence e is primitive Idempotent. Now, let $a \in R$ and suppose that aR contains only the zero Idempotent. Given $r \in R$, write ar = e+u in R where $e^2 = e$ and u is a unit In and

$$u(1-e)u^{-1} = (ar - e)(1-e)u^{-1} = ar(1-e)u^{-1} \in aR.$$

It follows that $e = 1$, that is I – ar = -u is a uint for all $r \in R$.

This shows that $a \in J(R)$, so R is a potent ring.Now for the second section let $f^2 = f \in R$ be primitive,and choose $a \in fRf$ with $a \notin J(fRf) = J(R) \cap fRf$. Then first section gives $0 \neq g = g^2 \in aR \subset fR$, so since f is primitive, $gR = fR$ Hence $aR = fR$ so, if $f = ab$, $b \in R$,we see that a has right inverse *fbf* in *fRf*. Since $fbf \notin J(Jrf)$.it too has a right inverse in *fRf* and this follows that eRe is local.

Corollary 2.12: A ring R is semiprefect if and only if it is strongly 1- clean and I-finite.

Proof: If R is semiprefect it is well known that R is I-finite and that $1 = e_1 + e_2 + ... + e_n$ where $e_i R e_i$ is a local ring for each i. since the local rings $e_i Re_i$ are strongly 1-clean, R is clean.

Conversely, if R is I-finite we can write $1 = e_1 + e_2 + ... + e_n$ where the e_i are orthogonal, primitive idempotents. since R is strongly 1-clean,each $e_i Re_i$ is local by proposition 2.11 and hence is strongly 1-clean. Therefore R is semiprefect.

Strongly n-clean endomorphism rings: Recently In [3] camillo proved that the endomorphism ring of continous module is clean.He also proved that the result is not true for any free module over a clean ring. In fact he showed that the endomorphism ring of a free module over a semiperfect ring may not be clean. Here we claim that the result still is true for strongly n-clean rings by corollary 2.9, we know that if M is a free R-module with finite rank and if R is strongly n-clean then so also is M, because in this case $M \cong R^n$ for some n and so

$$\mathrm{End}(M) \cong \mathrm{End}(R^n) \cong M_n(R).$$

We shall show that if R is strongly n-clean then a free R-module M with any rank even uncountable infinite rank also is strongly n-clean,then any free R-module is also strongly n-clean, for seeing this claim we apply the meehan,s method.

Definition 3.1: Let $M = \bigoplus_{i \in I} Re_i$ be a free R-module with rank |I| then we define

(i) The support of element $m = \sum_{i \in I} r_i e_i$ of M by $[m] = \{i \in I | r_i \neq 0\}$ and the support of $0 \in M$ is the empty set ϕ. Note that the support depends on the choice of bassis and that [m] is finite for any $m \in M$
(ii) let X be an arbitrary subset of M. we define the support of X as $X = U_{m \in X}[m]$

Proposition 3.2: Let $M = \oplus_{i<\omega} Re_i$ be a free R-module of countably inifite rank and let ϕ be an endomorphism of M.

Then there exist a strictly increasing sequenece of natural numbers.

$$0 = r_0 < ... < r_s < ...(S < \omega)$$

such that, if $i < \omega$ and $r_s \le i < r_s + 1$ Then

$$[\phi(e_i)] \subseteq \{0,1,...,r_{s+2} - 1\}.$$

Moreover, for any fixed positive integer m, r_s may be chosen so that $r_s - 1$ is a multiple of m.

Proof: See [4, proposition 2.3].

We make some further definitions based on the usual concept of odd and even parts of a function. if θ is any endomorphism of $M = \oplus_{S<\omega} M_S$ we define θ^{odd} and θ^{even} as follows: s

$$\text{We note that } \theta = \theta^{odd} + \theta^{even}$$

Lemma 3.3: Let m be a nonzero positive integer and suppose that a free R-module of finite rank m be strongly n-clean.Then a free R-module of finite rank divisible by m is strongly n-clean too.

Proof: For each positive integer n, let M_n be a free R-module of rank mn. Any endomorphism of M_2 can be expressed as a 2m×2n matrix witch may consider as a 2×2 block matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$. But since each of the two diagonal blocks is strongly 2-clean so we have:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} E_1 + U_1 + U_2 & B \\ C & E_2 + V_1 + V_2 \end{pmatrix}$$
$$= \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} + \begin{pmatrix} U_1 & B \\ 0 & V_1 \end{pmatrix} + \begin{pmatrix} U_2 & 0 \\ C & V_2 \end{pmatrix}$$

Since the matrices U_1, V_1, U_2 and V_2 are all invertible,it is very easy to check that the two later matrices are invertible.Now with a simple induction argument we can see that $\mathrm{End}_R(M_n)$ is strongly n-clean for each $n \in N$.

Theorem 3.4: Let m be a positive integer and suppose that a free R-module of finite rank m be strongly n-clean. if $M = \oplus_{i<\omega} Re_i$ is a free R-module of countably infinite rank, then every endomorphism of M is strongly n-clean.

Proof: Let ϕ be any endomorphism of M. then by proposition 10 there is a sequence $r_0 < r_1 < ... < r_s < ...(s \in \omega)$ such that for

$$r_s \le i < r_{s+1}, [e_i \phi] \subseteq \{0,1,...,r_{s+2} - 1\}$$

and furthermore, r_s - 1 is a multiple of m for all 0, $s \in \omega$. Now we define $M_K = \bigoplus_{r_k \le i < r_{k+1}} Re_i$ and $N_l = \bigoplus_{0 \le k \le j} M_i$, $k < \omega$

Write $\phi = \bigoplus_{S \in \omega} \phi^s$,this being the usual matrix decomposition where ϕ_s denotes the restriction of ϕ to M_s we can now write

$$\phi^s = \phi^s_* + \phi^s_{-1} + \phi^s_0 + \phi^s_1$$

Where

$$\phi^s = \phi^s_*, \phi^s_{-1}, \phi^s_0, \phi^s_1$$

are the composition of ϕ^s with projections onto the four summands N_{s-2}, M_{s-1}, M_s and M_{s+1} respectively.Let $\phi_* = \bigoplus_{s<\omega} \phi^s_*$, $\phi_{-1} = \bigoplus_{s<\omega} \phi^s_{-1}$, $\phi_0 = \bigoplus_{s<\omega} \phi^s_0$ and $\phi_1 = \oplus \phi^s_1$.

By assumption a free R-module of finite rank m, is strongly n-clean and since for each $s \in \omega$, the rank of M_s is a multiple of m, then by lemma 3.3, M_s is strongly n-clean, so we can write $\phi^s_0 = e^s_0 + \alpha^s_0 + \beta^s_0$ where e^s_0 is a nonzero idempotent and α^s_0 and β^s_0 are automorphisms of M_s,for each $s \in \omega$. let $e = \bigoplus_{s<\omega} e^s_0, \alpha = \bigoplus_{s<\omega} \alpha^s_0$ and $\beta = \bigoplus_{s<\omega} \beta^s_0$

Where e is a nonzero idempotetnt of M and α and β are automorphisms of M.

Now decompose ϕ as follows: $\phi = e + \psi_1 + \psi_2$ where

$$\psi_1 = (\alpha + \phi^{odd}_1 + \phi^{odd}_{-1} + \phi_*)$$

and

$$\psi_2 = (\beta + \phi^{even}_1 + \phi^{even}_{-1})$$

As meehan has proved ([4, theorem2.5]) we can see that ψ_1 and ψ_2 are automorphism. therefore M is strongly n-clean.

Now by theorem 16 and applying the method of the proof of theorem 2.7 from [4] immediately we have the following theorem.

Theorem 3.5: Let R be an strongly 2-clean ring then any free R-module with uncountble rank also is strongly n-clean.

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