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Dynamic Modeling of Robot Manipulators in D-H Frames

Mohammad Mehdi Fateh

Department of Electrical and Robotic Engineering, Shahrood University of Technology, Shahrood, Iran

Abstract: This study presents a novel dynamic modelling of a rigid serial manipulator using a compact formula of inertia tensor to simplify dynamic modelling and mathematical transformations. The matrix transformation is used appropriately as a powerful tool for this purpose. The kinetic energy, the potential energy and the inertia matrix of manipulator all are formulated in Denavit-Hartenberg frames (D-H) for deriving the dynamic model. This modelling approach is derived directly from the kinematic results in D-H frames. The proposed approach is simpler with fewer calculations in comparing with conventional approach which requires additional calculations in the center of mass frames (COM). The extreme calculations in the conventional approach are done to form inertia matrix of manipulator where the terms of manipulator Jacobian must be reformed due to COM frames.

Key words: Denavit-Hartenberg frames . Dynamic modelling . Inertia formulation . Serial manipulator

INTRODUCTION

The dynamic equations of manipulators can be simplified using suitable coordinate frames to analyze the motion of manipulators. The coordinate frames can be chosen based on the configuration of manipulator and the geometry of motion for kinematic and dynamic analysis. The Denavit-Hartenberg convention is commonly used to select the coordinate frames for formulating the kinematic problem of serial manipulator [1]. The obtained presentations and frames from kinematic solutions can be used for formulating the dynamic equations. However, the complexity increases for dynamic modelling in frames which are not attached to the center of mass of links. The complications will arise with computing inertia tensor of a moving rigid body in a fixed frame.

One solution is to apply COM frames for simplifying the dynamic calculations. These frames are attached to the links located on the center of mass of links since the inertia of each link will be fixed into the attached frame. However, COM frames are not used in D-H convention and the forward kinematic results are not provided in them. Thus, the provided calculations are then transformed. Therefore, there will be many calculations to cope with these frames for deriving the dynamic equations.

In this study, a mathematical approach is proposed to solve this problem by transforming inertia tensor. A compact and simple form of inertia tensor will be presented for applying the required transformations. Inertia matrix has been widely used in the modeling of kinematics and dynamics of robotic mechanisms [2]. An operational space inertia matrix was derived using matrix transformations to reflect the dynamic properties of a robot manipulator to its tip [3]. The principles and applications of tensors are considered in the linear algebra [4, 5]. Skew symmetric matrixes, rotation matrixes and homogeneous transformations of frames are well used to derive the kinematic and dynamic equations of a manipulator in rigid motions [6].

Dynamic modeling of manipulators is a very active field of research. Dynamic equations of motions can be used to investigate the system responses and system properties. The system stability is one of the main active fields of research which is based on dynamic equations. Many approaches based on the dynamic equations were presented on dynamic, control, identification and simulation of manipulators [7-9]. Model based control approaches play significant roles to control manipulators.

Euler-Lagrange Equations are frequently used in the field of dynamic modelling of manipulators, especially for modelling of rigid serial manipulators such as industrial robot manipulators. The dynamic model of a gear-driven rigid robot manipulator was derived using the Lagrange formulation [10]. The point and joint coordinates were applied to model a serial robot manipulator [11]. The analytical dynamic model was derived for six-DOF industrial robotic manipulators of containing closed chain [12]. The principle of virtual work and the Lagrange method was

Corresponding Author: Dr. Mohammad Mehdi Fateh, Department of Electrical and Robotic Engineering, Shahrood University of Technology, Shahrood, Iran

used to standard dynamics formulation of 6 degree of freedom fully parallel manipulator with elastic joints [13].

Applications of Euler-Lagrange Equations were extended for parallel manipulators where a recursive matrix method was used to model the manipulator [14]. However, there is a contradiction of conditions to apply the Euler-Lagrange equations for flexible mechanisms. Alternatively, the numerical solutions were used for dynamic modelling of non-rigid manipulators such as a rigid-flexible manipulator [15] and for a flexible manipulator [16]. The Lagrange finite element formulation was used for dynamic modeling of a flexible-link planar parallel platform [17].

SKEW SYMETRIC MATRIX

Some properties of these matrixes are introduced as follows [6].

$$S + S_r = 0 \tag{1}$$

where, S denotes a 3×3 skew symmetric matrix. The skew symmetric matrix is introduced as

$$S(\mathbf{a}) = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \tag{2}$$

where, **a** is a vector of the form $a = [a_x, a_y, a_z]^T$. Skew symmetric matrix is a linear operator such that

$$S(\alpha\mathbf{a} + \beta\mathbf{b}) = \alpha S(\mathbf{a}) + \beta S(\mathbf{b}) \tag{3}$$

where, α and β are real constants, **a** and **b** are vectors. The cross product of two vectors can be computed as

$$\mathbf{a} \times \mathbf{b} = S(\mathbf{a})\mathbf{b} \tag{4}$$

Since the skew symmetric matrix is a linear operator, the derivative and integral of skew symmetric matrix leads to

$$\dot{S}(\mathbf{a}) = S(\dot{\mathbf{a}}) \tag{5}$$

$$\int S(\mathbf{a})dt = S(\int \mathbf{a}dt) \tag{6}$$

A relation between the skew symmetric matrix and a rotation matrix is introduced by

$$S(Ra) = RS(\mathbf{a})R^T \tag{7}$$

where, R is a 3×3 rotation matrix and $\boldsymbol{a}$ is the rotation axes of rotation matrix. R is an orthogonal matrix such that $RR^T = I$ and I is a unit matrix. The time derivative of rotation matrix is computed as:

$$\dot{R} = S(\omega)R^T \tag{8}$$

where, $\omega = \mathbf{a}\dot{\theta}$ is angular velocity vector and $\dot{\theta}$ is the angular velocity of the rotation matrix.

KINETIC ENERGY

Assume a rigid body denoted b is rotated with respect to a fixed reference frame denoted A. Attach a coordinate frame named B to the body b. Position of a point $\boldsymbol{p}$ on the body b in the frame A is calculated as:

$$\mathbf{p_A} = R_A^B\mathbf{p_B} + \mathbf{d_A^B} \tag{9}$$

where, p_B is the position of point p with respect to the frame B and R_A^B is a 3×3 rotation matrix to specify the orientation of the coordinate frame B relative to the frame A.

Velocity of point p_A is computed by time derivative of p_A as follows:

$$\dot{\mathbf{p}}_\mathbf{A} = \dot{R}_A^B\mathbf{p_B} + \dot{\mathbf{d}}_\mathbf{A}^\mathbf{B} \tag{10}$$

The kinetic energy denoted by K can be calculated as:

$$K = \frac{1}{2}\int_b \dot{\mathbf{p}}_\mathbf{A}^\mathbf{T}\dot{\mathbf{p}}_\mathbf{A}dm \tag{11}$$

Substituting (10) into (11) yields

$$K = \frac{1}{2}\int_b (\dot{R}_A^B\mathbf{p_B} + \dot{\mathbf{d}}_\mathbf{A}^\mathbf{B})^T(\dot{R}_A^B\mathbf{p_B} + \dot{\mathbf{d}}_\mathbf{A}^\mathbf{B})dm \tag{12}$$

By expanding (12), we have

$$\begin{aligned} K = &\frac{1}{2}\int_b (\dot{R}_A^B\mathbf{p_B})^T\dot{R}_A^B\mathbf{p_B}dm + \frac{1}{2}\int_b \dot{\mathbf{d}}_\mathbf{A}^{\mathbf{B}T}\dot{R}_A^B\mathbf{p_B}dm \\ &+\frac{1}{2}\int_b (\dot{R}_A^B\mathbf{p_B})^T\dot{\mathbf{d}}_\mathbf{A}^\mathbf{B}dm + \frac{1}{2}\int_b \dot{\mathbf{d}}_\mathbf{A}^{\mathbf{B}T}\dot{\mathbf{d}}_\mathbf{A}^\mathbf{B}dm \end{aligned} \tag{13}$$

The second term is equal to the third term in Equation (13) since both are scalar and are transpose to each other. So

$$\frac{1}{2}\int_b \dot{\mathbf{d}}_\mathbf{A}^{\mathbf{B}T}\dot{R}_A^B\mathbf{p_B}dm = \frac{1}{2}\int_b (\dot{R}_A^B\mathbf{p_B})^T\dot{\mathbf{d}}_\mathbf{A}^\mathbf{B}dm \tag{14}$$

Since $\dot{\mathbf{d}}_\mathbf{A}^\mathbf{B}$ and $\dot{R}_A^B$ are not dependent to m,

$$\int_b \dot{\mathbf{d}}_\mathbf{A}^{\mathbf{B}T}\dot{R}_A^B\mathbf{p_B}dm = \dot{\mathbf{d}}_\mathbf{A}^{\mathbf{B}T}\dot{R}_A^B\int_b \mathbf{p_B}dm \tag{15}$$

The center of mass can be calculated as:

$$\mathbf{r}_B = \frac{1}{m}\int_b \mathbf{p}_\mathbf{B} dm \tag{16}$$

where, r_B denotes the center of mass vector described in frame *B*. Substituting (16) into (15), yields

$$\int_b \dot{\mathbf{d}}_\mathbf{A}^{\mathbf{B}\,T} \dot{R}_A^B \mathbf{p}_\mathbf{B} dm = m\dot{\mathbf{d}}_\mathbf{A}^{B\,T} \dot{R}_A^B \mathbf{r}_B \tag{17}$$

Use of (8) for $\dot{R}_A^B = S(\omega_\mathbf{A}^\mathbf{B})R_A^B$ leads to:

$$m\dot{\mathbf{d}}_\mathbf{A}^{B\,T}\dot{R}_A^B\mathbf{r}_\mathbf{B} = m\dot{\mathbf{d}}_\mathbf{A}^{B\,T}S(\omega_\mathbf{A}^\mathbf{B})R_A^B\mathbf{r}_\mathbf{B} \tag{18}$$

Interchanging for $S(\omega_\mathbf{A}^\mathbf{B})R_A^B\mathbf{r}_B = -\omega_\mathbf{A}^\mathbf{B}S(R_A^B\mathbf{r}_B)$ and use of (7) to have $S(R_A^B\mathbf{r}_\mathbf{B}) = R_A^B S(\mathbf{r}_\mathbf{B})R_A^{B\,T}$ yields

$$m\dot{\mathbf{d}}_\mathbf{A}^{B\,T}\dot{R}_A^B\mathbf{r}_B = -m\dot{\mathbf{d}}_\mathbf{A}^{B\,T}R_A^B S(\mathbf{r}_B)R_A^{B\,T}\omega_\mathbf{A}^\mathbf{B} \tag{19}$$

The forth term in (13) is simplified as:

$$\frac{1}{2}\int_b \dot{\mathbf{d}}_\mathbf{A}^{\mathbf{B}\,T}\dot{\mathbf{d}}_\mathbf{A}^\mathbf{B} dm = \frac{1}{2}m\dot{\mathbf{d}}_\mathbf{A}^{\mathbf{B}\,T}\dot{\mathbf{d}}_\mathbf{A}^\mathbf{B} \tag{20}$$

Substituting $\dot{R}_A^B = S(\omega_\mathbf{A}^\mathbf{B})R_A^B$ into the first term of (13), yields

$$\begin{aligned}&\frac{1}{2}\int_B (\dot{R}_A^B\mathbf{p}_\mathbf{B})^T(\dot{R}_A^B\mathbf{p}_\mathbf{B})dm \\ &= \frac{1}{2}\int_B (S(\omega_\mathbf{A}^\mathbf{B})R_A^B\mathbf{p}_\mathbf{B})^T(S(\omega_\mathbf{A}^\mathbf{B})R_A^B\mathbf{p}_\mathbf{B})dm\end{aligned} \tag{21}$$

By $S(\omega_\mathbf{A}^\mathbf{B})R_A^B\mathbf{p}_\mathbf{B} = -\omega_\mathbf{A}^\mathbf{B}S(R_A^B\mathbf{p}_\mathbf{B})$, we have

$$\begin{aligned}\frac{1}{2}\int_B (\dot{R}_A^B\mathbf{p}_\mathbf{B})^T(\dot{R}_A^B\mathbf{p}_\mathbf{B})dm &= \frac{1}{2}\int_b (S(R_A^B\mathbf{p}_\mathbf{B})\omega_\mathbf{A}^\mathbf{B})^T S(R_A^B\mathbf{p}_\mathbf{B})\omega_\mathbf{A}^\mathbf{B} dm \\ &= \frac{1}{2}\int_b \omega_\mathbf{A}^{\mathbf{B}\,T}S(R_A^B\mathbf{p}_\mathbf{B})^T S(R_A^B\mathbf{p}_\mathbf{B})\omega_\mathbf{A}^\mathbf{B} dm\end{aligned} \tag{22}$$

Since $\omega_\mathbf{A}^\mathbf{B}$ is not dependent to m, so

$$\frac{1}{2}\int_B (\dot{R}_A^B\mathbf{p}_\mathbf{B})^T(\dot{R}_A^B\mathbf{p}_\mathbf{B})dm = \frac{1}{2}\omega_\mathbf{A}^{\mathbf{B}\,T}\left(\int_b S(R_A^B\mathbf{p}_\mathbf{B})^T S(R_A^B\mathbf{p}_\mathbf{B})dm\right)\omega_\mathbf{A}^\mathbf{B} \tag{23}$$

From $S(R_A^B\mathbf{p}_\mathbf{B}) = R_A^B S(\mathbf{p}_\mathbf{B})R_A^{B\,T}$

$$\begin{aligned}&\frac{1}{2}\int_B (\dot{R}_A^B\mathbf{p}_\mathbf{B})^T(\dot{R}_A^B\mathbf{p}_\mathbf{B})dm = \\ &\frac{1}{2}\omega_\mathbf{A}^{\mathbf{B}T}\left(\int_b R_A^B S(\mathbf{p}_\mathbf{B})^T R_A^{B\,T}R_A^B S(\mathbf{p}_\mathbf{B})R_A^{B\,T}dm\right)\omega_\mathbf{A}^\mathbf{B}\end{aligned} \tag{24}$$

R_A^B is an orthogonal matrix, thus $R_A^{BT}R_A^B = I$. This results in

$$\begin{aligned}&\frac{1}{2}\int_B (\dot{R}_A^B\mathbf{p}_\mathbf{B})^T(\dot{R}_A^B\mathbf{p}_\mathbf{B})dm = \\ &\frac{1}{2}\omega_\mathbf{A}^{\mathbf{B}T}\left(\int_b R_A^B S(\mathbf{p}_\mathbf{B})^T S(\mathbf{p}_\mathbf{B})R_A^{B\,T}dm\right)\omega_\mathbf{A}^\mathbf{B}\end{aligned} \tag{25}$$

Since R_A^B is not dependent to m, so

$$\begin{aligned}&\frac{1}{2}\int_B (\dot{R}_A^B\mathbf{p}_\mathbf{B})^T(\dot{R}_A^B\mathbf{p}_\mathbf{B})dm \\ &= \frac{1}{2}\omega_\mathbf{A}^{\mathbf{B}\,T}R_A^B\left(\int_b S(\mathbf{p}_\mathbf{B})^T S(\mathbf{p}_\mathbf{B})dm\right)R_A^{B\,T}\omega_\mathbf{A}^\mathbf{B}\end{aligned} \tag{26}$$

We define

$$I_B = \int_B S(\mathbf{p}_\mathbf{B})^T S(\mathbf{p}_\mathbf{B})dm \tag{27}$$

Substituting I_B into (26), yields

$$\frac{1}{2}\int_B (\dot{R}_A^B\mathbf{p}_\mathbf{B})^T(\dot{R}_A^B\mathbf{p}_\mathbf{B})dm = \frac{1}{2}\omega_\mathbf{A}^{\mathbf{B}\,T}R_A^B I_B R_A^{B\,T}\omega_\mathbf{A}^\mathbf{B} \tag{28}$$

Consequently

$$K = \frac{1}{2}m\dot{\mathbf{d}}_\mathbf{A}^{B\,T}\dot{\mathbf{d}}_\mathbf{A}^B + \frac{1}{2}\omega_A^{B\,T}R_A^B I_B R_A^{B\,T}\omega_A^B - m\dot{\mathbf{d}}_\mathbf{A}^{BT}R_A^B S(\mathbf{r}_B)R_A^{B\,T}\omega_\mathbf{A}^\mathbf{B} \tag{29}$$

INERTIA TENSOR

We explain (27) that is a compact form of inertia tensor of the rigid body *b* as follows. Assume the origin of frame *A* is the center of mass and to be coinciding on origin frame *B*, means in the case of rotational motion about the center of mass, the kinetic energy is calculated as

$$\mathbf{d}_\mathbf{A}^\mathbf{B} = 0,\ \mathbf{p}_\mathbf{A} = R_A^B\mathbf{p}_\mathbf{B} \tag{30}$$

$$K = \frac{1}{2}\omega_A^{B\,T}R_A^B I_B R_A^{B\,T}\omega_A^B \tag{31}$$

Compare the kinetic energy obtained by (31) with the definition of kinetic energy in this case,

$$K = \frac{1}{2}\omega^T I\omega \tag{32}$$

where, ω is the angular velocity in the fixed frame and I is the inertia tensor of body in the fixed frame, it is concluded that

$$\omega_A^B = \omega, \quad R_A^B I_B R_A^{B\,T} = I \tag{33}$$

Since the moment of inertia I is calculated in the fixed reference frame, thus

$$R_A^B I_B R_A^{B\,T} = I_A \tag{34}$$

The moment of inertia of a rigid body in the fixed frame, I_A is calculated by a transformation of the moment of inertia of the rigid body in the attached frame to the body, I_B given by (34). Substituting (27) into (34), results

$$I_A = R_A^B \left(\int_b S(\mathbf{p_B})^T S(\mathbf{p_B}) dm R_A^{B\,T}\right. \tag{35}$$

Since R_A^B is not dependent to m, we put R_A^B into the integral and since $R_A^{BT} R_A^B = I$, we can put it into (35) to obtain

$$I_A = \int_b R_A^B S(\mathbf{p_B})^T R_A^{B\,T} R_A^B S(\mathbf{p_B}) R_A^{B\,T} dm \tag{36}$$

Use of (7) yields

$$I_A = \int_b S(R_A^B \mathbf{p_B})^T S(R_A^B \mathbf{p_B}) dm \tag{37}$$

From (30), we have

$$I_A = \int_b S(\mathbf{p}_A)^T S(\mathbf{p}_A) dm \tag{38}$$

Now, it is concluded that the inertia matrix of the rigid body in a desired frame D, is defined

$$I_D = \int_b S(\mathbf{p}_D)^T S(\mathbf{p}_D) dm \tag{39}$$

where, p_D is the position vector of the particle dm of the body b described in the frame D. It will be useful if it is noted that the frame D is a desired frame. There are some points to note:

- For a case of selecting frame D attached to the rigid body b, the inertia matrix I_D is a constant matrix.
- The inertia matrix I_D is a symmetric matrix. It is verified by taking transpose of (39).

KINETIC ENERGY

We use the standard version of Denavit-Hartenbeg convention (D-H convention) described in [6] to define the D-H frames. For a serial manipulator with n independent joints which has n degrees of freedom, the D-H frames are defined as follows.

Links are numbered from 0 to n such that the first link numbered 0 (base of manipulator) and the last link numbered n. Joints are numbered from 1 to n. The D-H frames are numbered from 0 to n as each frame attached to its corresponding link.

The z axis of frame i is aligned with the joint i, the x axis of frame i is vertical and crossed to the z axis of frame $i - 1$ and the y axis of frame i is determined such that a right-wise coordinate frame to be formed. The x axis of frame 0 and the z axis of frame n are determined arbitrary.

The kinetic energy of the i-th link given by (29) is calculated using the Denavit-Hartenberg representation as

$$K_i = \frac{1}{2} m_i \dot{\mathbf{d}}_0^{i\,T} \dot{\mathbf{d}}_0^i + \frac{1}{2} \omega_0^{i\,T} R_0^i I_i R_0^{i\,T} \omega - m_i \dot{\mathbf{d}}_0^{i\,T} R_0^i S(\mathbf{r}_i) R_0^{i\,T} \omega_0^i \tag{40}$$

where, I_i is the inertia matrix of link i in frame i, m_i is the mass of link i, r_i is the center of mass of link i in frame i, $\dot{\mathbf{d}}_0^i$ is the linear velocity vector of frame i respect to frame 0 and ω_0^i is the angular velocity vector of frame i respect to frame 0 and R_0^i is the rotation of frame i respect to frame 0. The linear velocity vector $\dot{\mathbf{d}}_0^i$ is given [6] as

$$\dot{\mathbf{d}}_0^i = \frac{\partial \mathbf{d_0^i}}{\partial \mathbf{q}} \dot{\mathbf{q}} = J_{vi} \dot{\mathbf{q}} \tag{41}$$

where, $\mathbf{q}$ is the position joint vector, $\dot{\mathbf{q}}$ is the position velocity vector and J_{vi} is defined as

$$J_{vi} = \begin{bmatrix} \frac{\partial \mathbf{d_0^i}}{\partial q_1} & \frac{\partial \mathbf{d_0^i}}{\partial q_2} & \cdots & \frac{\partial \mathbf{d_0^i}}{\partial q_n} \end{bmatrix} \tag{42}$$

The elements of J_{vi} are calculated as For j = 1,..,i

$$\frac{\partial \mathbf{d_0^i}}{\partial q_j} = z_{j-1} \qquad \text{if joint j is revolute.}$$

$$\frac{\partial \mathbf{d_0^i}}{\partial q_j} = z_{j-1} \times [o_n \text{-} o_{j-1}] \qquad \text{if joint j is prismatic.}$$

For j = 1=1,..,n $\frac{\partial \mathbf{d_0^i}}{\partial q_j} = 0$

where, z_{j-1} is the z axis of frame $j-1$, o_n is the origin of frame n and o_{j-1} is the origin of frame $j-1$.

$$\omega_0^i = J_{\omega i}\dot{\mathbf{q}} \tag{43}$$

where, $J_{\omega I}$ is defined as

$$J_{\omega i} = \begin{bmatrix} \rho_1 z_0 & \rho_2 z_1 & \dots & \rho_n z_{n-1} \end{bmatrix} \tag{44}$$

ρ_j is calculated as

For j = 1,…,i

$$\rho_j = 1 \quad \text{if joint j is revolute.}$$
$$\rho_j = 0 \quad \text{if joint j is prismatic.}$$

For j = 1+1,…,n

$$\rho_j = 0$$

The inertia matrix I_i is calculated from (39) to obtain

$$\mathbf{I}_i = \int_{\mathbf{b}} \mathbf{S}(\mathbf{P}_i)^{\mathbf{T}}\mathbf{S}(\mathbf{P}_i)dm \tag{45}$$

where, $P_i = [x\ y\ z]^T$ is position of particle dm in frame i and $S(P_i)$ is formed by (2). I_i is a constant matrix since I_i is calculated in frame i. Substituting $S(P_i)$ into (45), yields

$$\mathbf{S}(\mathbf{P}_i) = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$$

$$\mathbf{I}_C = \begin{bmatrix} \int (y^2+z^2)dm & -\int xydm & -\int xzdm \\ -\int xydm & \int (x^2+z^2)dm & -\int yzdm \\ -\int xzdm & -\int yzdm & \int (x^2+y^2)dm \end{bmatrix} \tag{46}$$

The total kinetic energy for a manipulator is then computed as:

$$K = \frac{1}{2}\sum_{i=1}^{n}(m_i \dot{\mathbf{d}}_0^{i\,T}\dot{\mathbf{d}}_0^i + \omega_0^{i\,T} R_0^i I_i R_0^{i\,T}\omega_0^i - 2m_i \dot{\mathbf{d}}_0^{i\,T} R_0^i S(\mathbf{r}_i) R_0^{i\,T}\omega_0^i) \tag{47}$$

Substituting (41) and (43) into (47), yields

$$K = \frac{1}{2}\dot{\mathbf{q}}^T D\dot{\mathbf{q}} \tag{48}$$

where, D is called the inertia matrix of manipulator. The kinetic energy K is a positive scalar. Therefore, (48) yields that the inertia matrix D is a symmetric positive definite matrix.

$$D = \sum_{i=1}^{n}\left[m_i J_{vi}^T J_{vi} + J_{\omega i}^T R_0^i I_i R_0^{i\,T} J_{\omega i} - 2m_i J_{vi}^T R_0^i S(\mathbf{r}_i) R_0^{i\,T} J_{\omega i}\right] \tag{49}$$

According to (49), the transformations of D-H frames, the link's mass and the center of mass of each link in its D-H frame are required to form the inertia matrix.

POTENTIAL ENERGY

The potential energy of body b is of the form

$$V = \int_b \mathbf{g}^T \mathbf{p}_A dm \tag{50}$$

where, $\boldsymbol{g}$ is the gravity acceleration in frame A. Substituting (9) into (50), yields

$$V = \int \mathbf{g}^T (R_A^B p_B + \mathbf{d}_A^B)dm \tag{51}$$

From Equation (16) and since $\mathbf{d}_A^B$ is independent on m, we have

$$V = m\mathbf{g}^T R_A^B \mathbf{r}_B + m\mathbf{g}^T \mathbf{d}_A^B = m\mathbf{g}^T(R_A^B \mathbf{r}_B + \mathbf{d}_A^B) \tag{52}$$

The total potential energy is summation of the potential energy of all links. It is of the form

$$V = \sum_{i=1}^{n} m_i \mathbf{g}^T (R_0^i \mathbf{r}_i + \mathbf{d}_0^i) \tag{53}$$

Equation (53) shows the potential energy in D-H frames.

DYNAMIC EQUATIONS

The Euler-Lagrange equations of motion are applied to derive the dynamic equations of a serial manipulator, which possess n degrees of freedom. The

dynamic equations of a serial rigid manipulator [6] given as:

$$D(\mathbf{q})\ddot{\mathbf{q}} + C(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + G(\mathbf{q}) = \mathbf{t} \tag{54}$$

where, $D(\mathbf{q})$ is the inertia matrix of manipulator, $C(\mathbf{q},\dot{\mathbf{q}})$ is a n×n matrix related to the centrifugal and Coriolis terms,

$$G(\mathbf{q}) = \frac{\partial V}{\partial q}$$

and τ is the joint generalized force.
The matrix $C(\mathbf{q},\dot{\mathbf{q}})$ is calculated from matrix $D(\mathbf{q})$ as:

$$C_{kj} = \frac{1}{2}\sum_{i=1}^{n}\left(\frac{\partial D_{kj}}{\partial q_i} + \frac{\partial D_{ki}}{\partial q_j} - \frac{\partial D_{ij}}{\partial q_k}\right)\dot{q}_i \quad (55)$$

where, $\frac{\partial V}{\partial \mathbf{q}}$ is obtained from (53) as

$$\frac{\partial V}{\partial \mathbf{q}} = \mathbf{g}^T\left[\sum_{i=1}^{n} m_i\left(\frac{\partial (R_0^i \mathbf{r}_i)}{\partial \mathbf{q}} + \frac{\partial \mathbf{d}_0^i}{\partial \mathbf{q}}\right)\right] \quad (56)$$

Use of $J_{vi} = \frac{\partial \mathbf{d}_0^i}{\partial \mathbf{q}}$, yields

$$\frac{\partial V}{\partial \mathbf{q}} = \mathbf{g}^T\left[\sum_{i=1}^{n} m_i\left(\frac{\partial (R_0^i \mathbf{r}_i)}{\partial \mathbf{q}} + J_{vi}\right)\right] \quad (57)$$

The inertia matrix of manipulator $D(\mathbf{q})$ is a significant base for calculations. Therefore, the dynamic equations can be provided in Denavit-Hartenberg frames if we use the $D(\mathbf{q})$ obtained in Denavit-Hartenberg frames by (49) and $G(\mathbf{q})$ by (57).

CONCLUSION

Matrix calculations on skew symmetric matrixes and rotation matrixes have been used to manipulate the equations for deriving the dynamic modelling of serial manipulators in the D-H frames. The kinetic energy and potential energy were calculated in D-H frames to apply the Euler-Lagrange equations. A novel compact presentation of inertia tensor of a rigid body was analytically derived and used for simplifying dynamic calculations. This modelling approach was derived directly from the kinematic results in D-H frames. The proposed approach is simpler with fewer calculations in comparing with conventional approach which requires additional calculations in the center of mass frames (COM).

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