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Some Representations and Specifications of the Generalized Negative Binomial Distribution

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Abstract: In a sequence of dependent Bernoulli trials, the distribution of the number of trials required to obtain r successes, V_r, is called a Generalized Negative Binomial (GNB) distribution. We present a simple representation of this distribution based on moments and consider the conditions under which a GNB distribution follows negative binomial distribution. Also we study the properties of this distribution in Markov chains.

Key words: Negative binomial distribution . Dependent Bernoulli sequences . Generalized negative binomial distribution . Markov chains . Characteristic function

INTRODUCTION

Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of Bernoulli trials and the statistic V_r be the number of trials required to obtain r successes (or failures). When the trials are independent and having the same success probability p, the distribution of V_r is the classical negative binomial (NB) distribution with parameters r, p, NB (r, p). This statistic V_r is used in many different statistical models such as inverse sampling model and crash data and also appears in the context of statistical quality control and Markov chains.

Although, for many situations the independence of trials have been assumed without any questions, but practically this assumption is not very realistic and the trials are dependent. This dependence case has been studied by many authors such as Drezner and Farnum [1], Vellaisamy [2], Bebbington and Lai [3], Hsiau and Yang [4], Shishebor and Towhidi [5]. When the trials are dependent or non-identical, the distribution of V_r is called a generalized negative binomial (GNB) distribution, but it hasn't been introduced a simple form for the probability mass function of this distribution.

A REPRESENTATION OF GNB DISTRIBUTION

In this section, we follow up a way similar to vellaisamy and Punnen [6] and present a simple and useful representation of GNB distribution based on moments.

Theorem 2.1: Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of Bernoulli variables and V_r be the number of trials required to obtain r successes, then for any integer k≥r,

$$P(V_r = k) = \sum_{j=r-1}^{k-1} (-1)^{j-(r-1)} \binom{j}{r-1} T_{jk} \tag{2.1}$$

where, T_{jk} is defined as:

$$T_{jk} = \sum_{1 \le i_1 < i_2 < \ldots < i_j \le k-1} E\left(X_{i_1} X_{i_2} \ldots X_{i_j} X_k\right); \forall k \ge 2,\ 1 \le j \le k-1$$

and

$$T_{0k} = E(X_k) \qquad \forall k \ge r$$

Proof: The probability mass function (pmf) of V_r is given by:

$$P(V_r = k) = \sum_{1 \le i_1 < \ldots < i_{r-1} \le k-1} P\left[\left(X_{i_1} = 1, \ldots, X_{i_{r-1}} = 1, X_k = 1\right) \cap \left(X_j = 0, \quad \text{for all } 1 \le j \le k-1; \quad j \ne i_1, \ldots, i_{r-1}\right)\right]$$

$$= \sum_{1 \le i_1 < \ldots < i_{r-1} \le k-1} P\left(X_{i_1} = 1, \ldots, X_{i_{r-1}} = 1, X_k = 1\right) - \sum_{1 \le i_1 < \ldots < i_{r-1} \le k-1} P\left(\bigcup_{j \ne i_1, \ldots, i_{r-1}} \left(X_{i_1} = 1, \ldots, X_{i_{r-1}} = 1, X_k = 1, X_j = 1\right)\right)$$

$$= T_{(r-1)k} - \binom{r}{r-1} T_{rk} + \binom{r+1}{r-1} T_{(r+1)k} + \ldots + (-1)^{k-r} \binom{k-1}{r-1} T_{(k-1)k} = \sum_{j=r-1}^{k-1} (-1)^{j-(r-1)} \binom{j}{r-1} T_{jk}$$

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Remark 2.1: In the special case k = 1, the pmf of V_r can be written as a simple form:

$$\begin{aligned} P(V_1 = k) &= \sum_{j=0}^{k-1}(-1)^j T_{jk} \\ &= E(X_k) - \sum_{i=1}^{k-1} E(X_i X_k) \\ &\quad + \sum_{1\le i<j\le k-1} E(X_i X_j X_k) + \cdots + (-1)^{k-1} E(X_1 \ldots X_k) \end{aligned}$$

and is called a Generalized Geometirc (GG) distribution.

If the observations are exchangeable, then for any $1\le j\le k-1$, T_{jk}, will be equal to:

$$T_{jk} = \binom{k-1}{j} E(X_1 \ldots X_j X_{j+1})$$

and hence the pmf of V_r has a simple form as:

$$\begin{aligned} P(V_r = k) &= \sum_{j=r-1}^{k-1} (-1)^{j-(r-1)} \binom{j}{r-1}\binom{k-1}{j} E(X_1 \ldots X_{j+1}) \\ &= \binom{k-1}{r-1}\sum_{j=0}^{k-r}\binom{k-r}{j}(-1)^j E(X_1 \ldots X_{r+j}) \end{aligned}$$

In the following theorem, we present a necessary and sufficient condition for the distribution of V_r to be a negative binomial.

Theorem 2.2: Under the assumptions of theorem 2.1, $Vs \sim NB(s, p)$ $\forall s \ge r$
if and only if

$$T_{jk} = \binom{k-1}{j} p^{j+1} \quad \forall k \ge r; \quad r-1 \le j \le k-1 \qquad (2.2)$$

Proof: If the relation (2.2) holds for any $k\ge r$; $r-1\le j \le k-1$, then for each $s\ge r$ the pmf of V_S will be as follows:

$$\begin{aligned} P(V_s = k) &= \sum_{j=s-1}^{k-1} (-1)^{j-(s-1)} \binom{j}{s-1}\binom{k-1}{j} p^{j+1} \\ &= \binom{k-1}{s-1}\sum_{j=s-1}^{k-1}(-1)^{j-(s-1)}\binom{k-r}{j-(s-1)} p^{j+1} \quad \forall k \ge s \end{aligned}$$

By taking $t = j-(s-1)$, we have

$$\begin{aligned} P(V_s = k) &= \binom{k-1}{s-1}\sum_{t=0}^{k-s}(-1)^t\binom{k-s}{t} p^{t+s} \\ &= \binom{k-1}{s-1} p^s q^{k-s} \end{aligned}$$

Which proves that the distribution of V_S is NB (s, p). Conversely, if $Vs \sim NB(s, p)$ $\forall s \ge r$, then the relation (2.2) can be derived by theorem 2.1 and tracing the following steps:

$$P(V_s = s) = T_{(s-1)s} = p^s \qquad \forall s \ge r \qquad (1)$$

$$\begin{aligned} P(V_s = s+1) &= T_{(s-1)(s+1)} - sT_{s(s+1)} \quad \forall s \ge r \\ &= s p^s q \end{aligned} \qquad (2)$$

and hence:

$$T_{(s-1)(s+1)} = sp^s \qquad \forall s \ge r$$

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(t) In step t, by using the steps 1 to (t-1) and the relation (2.1) we have:

$$T_{(s-1)(s+t-1)} = \binom{s+t-2}{s-1} p^s \qquad \forall s \ge r, \ \forall t \ge 1$$

and entirely, we conclude that:

$$T_{jk} = \binom{k-1}{j} p^{j+1} \qquad \forall k \ge r, \ r-1 \le j \le k-1$$

THE DISTRIBUTION OF V_r IN MARKOV CHAINS

In this section we justify the distribution of V_r in Markov chains and interpret some properties of them. In a serially-dependent production process, the probability of finding a conforming or nonconforming unit depends on the status of the immediately preceding unit produced. This process can be represented by a two-state Markov chain with transition matrix

$$P = \begin{array}{c c} & \begin{array}{cc} F & S \end{array} \\ \begin{array}{c} F \\ S \end{array} & \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} \end{array} \qquad (3.1)$$

where

$$a = P(X_{n+1} = S \mid X_n = F)$$

and

$$b = P(X_{n+1} = F \mid X_n = S)$$

In considering such a stochastic process, the distribution of V_r (the number of trials required to obtain r occurrences of the event S) can be stated by the following recursive formulas:

$$\begin{cases} P_1(V_r = k) = (1-b)P_1(V_{r-1} = k-1) + bP_0(V_r = k-1) \\ P_0(V_r = k) = aP_1(V_{r-1} = k-1) + (1-a)P_0(V_r = k-1) \end{cases} \quad (3.2)$$

Where

$$P_0(.) = P(. \mid X_0 = 0) \text{ and } P_1(.) = P(. \mid X_0 = 1)$$

Remark 3.1: Let V_1 be the number of trials required to the first success and P (V_0 = 0) = 1, then the pmf of V_1 follows from the relations (3.2),

$$P_0(V_1 = k) = a(1-a)^{k-1} \qquad \forall k \geq 1$$

and

$$P_1(V_1 = k) = \begin{cases} 1-b & \text{if } k = 1 \\ ab(1-a)^{k-2} & \text{if } k \geq 2 \end{cases}$$

and it can be shown that the pmf of V_2 is of the form :

$$P_0(V_2 = k) = \begin{cases} a(1-b) & \text{if } k = 2 \\ (k-2)a^2b(1-a)^{k-3} + a(1-b)(1-a)^{k-2} & \text{if } k \geq 3 \end{cases}$$

$$P_1(V_2 = k) = \begin{cases} (1-b)^2 & \text{if } k = 2 \\ (k-3)a^2b^2(1-a)^{k-4} + 2ab(1-b)(1-a)^{k-3} & \text{if } k \geq 3 \end{cases}$$

Also, the distribution function of V_1 is obtained by:

$$G_1^0(\nu) = 1 - (1-a)^{[\nu]} \quad ; \quad \nu \geq 1$$

and

$$G_1^1(\nu) = 1 - b(1-a)^{[\nu]-1} \quad ; \quad \nu \geq 1$$

and the distribution function of V_2 has the following form:

$$G_2^0(\nu) = 1 - (1-a)^{[\nu]-1} - ([\nu]-1)ab(1-a)^{[\nu]-2} \; ; \; \nu \geq 2$$

and

$$G_2^1(\nu) = 1 + (([\nu]-1)b - 2)b(1-a)^{[\nu]-2} - ([\nu]-2)b^2(1-a)^{[\nu]-3} \quad ; \nu \geq 2$$

where $\nu \in \Re$, $\nu \geq r$ and [.] denotes the integer part.

The following theorem shows that in a markov chain, GNB distribution of V_r arising out of r independent random variables with generalized geometric (GG) distributins.

Theorem 3.1: In a two-state Markov chain with transition matrix (3.1), the characteristic function of V_r takes the form:

$$\phi_r^0(t) = \left[\frac{ae^{it}}{1-(1-a)e^{it}}\right]\left[(1-b)e^{it} + \frac{abe^{2it}}{1-(1-a)e^{it}}\right]^{r-1}$$

& (3.5)

$$\phi_r^1(t) = \left[(1-b)e^{it} + \frac{abe^{2it}}{1-(1-a)e^{it}}\right]^r$$

where $\phi_r^i(\cdot)$ is the characteristic function of V_r with $X_0 = i$.

Proof: First by using the formulas (3.2), we conclude that:

$$\begin{aligned} \phi_r^0(t) &= \sum_{k=r}^{+\infty} e^{itk} P_0(V_r = k) \\ &= a\sum_{k=r}^{+\infty} e^{itk} P_1(V_{r-1} = k-1) + (1-a)\sum_{k=r}^{+\infty} e^{itk} P_0(V_r = k-1) \\ &= ae^{it}\phi_{r-1}^1(t) + (1-a)e^{it}\phi_r^0(t) \end{aligned}$$

and also

$$\begin{aligned} \phi_r^1(t) &= (1-b)\sum_{k=r}^{+\infty} e^{itk} P_1(V_{r-1} = k-1) + b\sum_{k=r}^{+\infty} e^{itk} P_0(V_r = k-1) \\ &= (1-b)e^{it}\phi_{r-1}^1(t) + be^{it}\phi_r^0(t) \end{aligned}$$

And hence we can find a recursive formulas for $\phi_r^i(\cdot)$ as:

$$\phi_r^0(t) = \left[\frac{ae^{it}}{1-(1-a)e^{it}}\right]\phi_{r-1}^1(t)$$

and

$$\phi_r^1(t) = \left[(1-b)e^{it} + \frac{abe^{2it}}{1-(1-a)e^{it}}\right]\phi_{r-1}^1(t)$$

According to Remark 3.1, the characteristic function of V_1 is equal to:

$$\phi_1^0(t) = ae^{it}\sum_{k=1}^{+\infty} e^{it(k-1)}(1-a)^{k-1} = \frac{ae^{it}}{1-(1-a)e^{it}}$$

and

$$\phi_1^1(t) = (1-b)e^{it} + \sum_{k=2}^{+\infty} e^{itk}[ab(1-a)^{k-2}]$$
$$=(1-b)e^{it}\frac{abe^{2it}}{1-(1-a)e^{it}}$$

Therefore we can specify the characteristic function of V_r as the formulas (3.5).

Remark 3.2: By using theorem 3.1 and the fact that the distribution of V_r is the same as sum of r independent random variables with GG distributions, we can find the expectation of V_r.
Define

$$E_0(V_r) = E(V_r|X_0 = 0) \text{ and } E_1(V_r) = E(V_r|X_0 = 1)$$

since $$\frac{\partial^k}{\partial t^k}\phi_1^0(t)|_{t=0} = i^k E_0(V_1^k)$$

and $$\frac{\partial^k}{\partial t^k}\phi_1^1(t)|_{t=0} = i^k E_1(V_1^k)$$

Hence $$E_0(V_r) = rE_0(V_1) = \frac{r}{a}$$

and $$E_1(V_r) = rE_1(V_1) = \frac{r(a+b)}{a}$$

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