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New Double Kharrat-Toma Transform and its Application in Partials Differential Equations

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Abstract: The aim of this paper is to introduce a new Double integral transform called Double Kharrat-Toma transform which is a generalization of the original Kharrat-Toma transform. The proposed integral transform is applied to solve some illustrative examples. The solutions obtained by application the suggested transform show the accuracy and the efficiency of this techniques.

Key words: Double Integral Transform Kharrat-Toma Transform, Kharrat-Toma transform, Partial differential equations.

INTRODUCTION

Partial differential equation plays vital role in engineering, applied mathematics, mechanics and general physics sciences.

To solve this type of equations many numerical methods were known, such as the finite element method and the finite difference method, as well as semi analytical methods such as: Variational iteration method, Homotopy perturbation method, and Adomian decomposition method.

ln the literature researchers have used many integral transforms combined with some semi analytical methods to solve linear and nonlinear partial differential equations, namely Laplace Transform, Sumudu Transform, Natural Transform, Elzaki Transform, Aboodh Transform, Kamal transform, Mahgoub (Laplace-Carson) transform, Mohand transform, Sawi transform, ARA Transform and Kharrat-Toma Transform [1-12].

In 2020 Kharrat and Toma [11] introduced a new transform called Kharrat-Toma Transform to solve the ordinary differential equations with initial conditions.

In this study, we propose a new double integral transform which called Double Kharrat-Toma transform.

The suggested transform is tested through some initial- boundary value problems. Therefore, the Double Kharrat –Toma transform technique is very convenient and effective.

The rest of the paper is organized as follows. In section 2, Double kharrat-Toma transforms are introduced, in section 3, Double Kharrat-Toma transform of Partial Derivatives, in section 4, Double Kharrat-Toma Transform of Some Functions, in section 5, the application for solving initial- boundary value problems is shown and conclusion in 6.

Double Kharrat-Toma transform:

Let $f(x,t)$ be a function of two variables *x* and *t*, where *x, t > 0*. The double Kharrat-Toma transform of $f(x,t)$ is defined as

$$B_t B_x\left[f(x,t)\right] = F(p,s) = p^3 s^3 \int_0^{\infty}\int_0^{\infty} e^{-\left(\frac{x}{p^2}+\frac{t}{s^2}\right)} f(x,t)\,dx\,dt \tag{1}$$

whenever the improper integral converges. Here *p, s* are complex numbers.

The double Kharrat-Toma integral transform and inversion is defined by

$$f(x,t) = B_t^{-1}B_x^{-1}\left[F(p,s)\right] = B_t^{-1}B_x^{-1}\left[p^3 s^3 \int_0^{\infty}\int_0^{\infty} e^{-\left(\frac{x}{p^2}+\frac{t}{s^2}\right)} f(x,t)\,dx\,dt\right] \tag{2}$$

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Double Kharrat-Toma transform of Partial Derivatives:

Double Kharrat-Toma Transform for first partial derivatives with respect to x is defined asfollows:

$$B_t B_x\left[f_x(x,t)\right] = \frac{1}{p^2} F(p,s) - p^3 F(0,s) \tag{3}$$

Similarly, Double Kharrat-Toma Transform for first partial derivatives with respect to t is given by

$$B_t B_x\left[f_t(x,t)\right] = \frac{1}{s^2} F(p,s) - s^3 F(p,0) \tag{4}$$

Double Kharrat-Toma Transform for second partial derivatives with respect to x is defined by

$$B_t B_x\left[f_{xx}(x,t)\right] = \frac{1}{p^4} F(p,s) - p\,F(0,s) - p^3 F_x(0,s) \tag{5}$$

In a similar manner, Double Kharrat-Toma Transform for second partial derivatives with respect to t can be deduced from a single Kharrat-Toma Transform

$$B_t B_x\left[f_{tt}(x,t)\right] = \frac{1}{s^4} F(p,s) - s\,F(p,0) - s^3 F_t(p,0) \tag{6}$$

Double Kharrat-Toma Transform of Some Functions:
In this section we give Double Kharrat-Toma transform of some functions

$f(x,t)$	$B_t B_x\left[f(x,t)\right] = G(p,s)$
1	$p^5 s^5$
$x\,t$	$p^7 s^7$
$\sin(ax)\sin(bt)$	$\frac{ap^7}{1+a^2p^4}\frac{bs^7}{1+b^2s^4}$
$\cos(at)\cos(bt)$	$\frac{p^5}{1+a^2p^4}\frac{s^5}{1+b^2s^4}$
$\sinh(ax)\sinh(bt)$	$\frac{ap^7}{1-a^2p^4}\frac{bs^7}{1-b^2s^4}$
$\cosh(ax)\cosh(bt)$	$\frac{p^5}{1-a^2p^4}\frac{s^5}{1-b^2s^4}$
$e^x e^t$	$\frac{p^5}{1-p^2}\frac{s^5}{1-s^2}$
$e^{ax} e^{bt}$	$\frac{p^5}{1-ap^2}\frac{s^5}{1-bs^2}$

Some Examples
Example 1 Double Kharrat-Toma Transform & First order Partial Differential Equation
Find the bounded solution of [13]

$$\begin{cases} u_x = 2u_t + u \\ u(x,0) = e^{-3x} \quad x > 0, t > 0 \end{cases} \tag{7}$$

Solution: Taking the Double Kharrat-Toma Transform, we obtain

$$B_t B_x [u_x] = B_t B_x [2u_t + u]$$

$$\frac{1}{p^2} U(p,s) - p^3 U(0,s) = 2\left[\frac{1}{s^2} U(p,s) - s^3 U(p,0)\right] + U(p,s)$$

$$\left[\frac{1}{p^2} - \frac{2}{s^2} - 1\right] U(p,s) = p^3 U(0,s) - 2s^3 U(p,0)$$

$$u(x,0) = e^{-3x} \quad \Rightarrow \quad U(p,0) = \frac{p^5}{1+3p^2}$$

$$U(p,s) = \frac{p^3}{\frac{1}{p^2} - \frac{2}{s^2} - 1} U(0,s) - \frac{2s^3 p^5}{\left(\frac{1}{p^2} - \frac{2}{s^2} - 1\right)\left(1+3p^2\right)}$$

$$U(p,s) = \frac{p^5}{1 - \left(\frac{2}{s^2} + 1\right)p^2} U(0,s) - \frac{s^5}{1+2s^2}\left[\frac{p^5}{1 - \left(\frac{2}{s^2} + 1\right)p^2} - \frac{p^5}{1+3p^2}\right]$$

Using B_x^{-1}, we get

$$U(x,s) = e^{\left(\frac{2}{s^2}+1\right)x} U(0,s) - \frac{s^5}{1+2s^2}\left[e^{\left(\frac{2}{s^2}+1\right)x} - e^{-3x}\right]$$

$$U(x,s) = e^{\left(\frac{2}{s^2}+1\right)x}\left[U(0,s) - \frac{s^5}{1+2s^2}\right] + \frac{s^5}{1+2s^2} e^{-3x}$$

Now $u(x,t)$ is bounded as $x \to \infty$ hence $U(x,s)$ is bounded as $x \to \infty$

Hence,

$$U(0,s) - \frac{s^5}{1+2s^2} = 0$$

$$U(0,s) = \frac{s^5}{1+2s^2}$$

Therefore,

$$U(x,s) = \frac{s^5}{1+2s^2} e^{-3x}$$

By Inverse Kharrat-Toma Transform B_t^{-1}, we get bounded solution

$$u(x,t) = e^{-3x} e^{-2t} = e^{-3x-2t} \tag{8}$$

Example 2 Double Kharrat-Toma Transform & One Dimensional Heat Equation

Solve [13]

$$\begin{cases} u_t = ku_{xx} \\ u(x,0) = \sin \pi x \\ u(0,t) = 0 \\ u(1,t) = 0 \end{cases} \quad , \quad 0 < x < 1 \quad , \quad t > 0 \tag{9}$$

Solution: Taking the Double Kharrat-Toma Transform, we obtain

$$B_t B_x \left[u_t \right] = B_t B_x \left[ku_{xx} \right]$$

$$\frac{1}{s^2} U(p,s) - s^3 U(p,0) = k \left[\frac{1}{p^4} U(p,s) - pU(0,s) - p^3 U_x(0,s) \right]$$

$$u(0,t) = 0 \quad \Rightarrow \quad U(0,s) = 0$$

$$u(x,0) = \sin \pi x \quad \Rightarrow \quad U(p,0) = \frac{\pi p^7}{1 + \pi^2 p^4}$$

$$\frac{1}{s^2} U(p,s) - s^3 \frac{\pi p^7}{1 + \pi^2 p^4} = \frac{k}{p^4} U(p,s) - kp^3 U_x(0,s)$$

$$\left[\frac{k}{p^4} - \frac{1}{s^2} \right] U(p,s) = kp^3 U_x(0,s) - \frac{\pi p^7 s^3}{1 + \pi^2 p^4}$$

$$U(p,s) = \frac{ks^2 p^7}{ks^2 - p^4} U_x(0,s) - \frac{\pi p^{11} s^5}{\left(ks^2 - p^4\right)\left(1 + \pi^2 p^4\right)}$$

$$U(p,s) = \frac{p^7}{1 - \frac{1}{ks^2} p^4} U_x(0,s) - \frac{\pi s^5}{1 + k\pi^2 s^2} \left[\frac{p^7}{1 - \frac{1}{ks^2} p^4} - \frac{p^7}{1 + \pi^2 p^4} \right]$$

$$U(p,s) = \frac{p^7}{1 - \frac{1}{ks^2} p^4} \left[U_x(0,s) - \frac{\pi s^5}{1 + k\pi^2 s^2} \right] + \frac{s^5}{1 + k\pi^2 s^2} \frac{\pi p^7}{1 + \pi^2 p^4}$$

Applying B_x^{-1}, we get

$$U(x,s) = \sqrt{k}\, s \sinh\left(\frac{1}{\sqrt{k}\, s} x \right) \left[U_x(0,s) - \frac{\pi s^5}{1 + k\pi^2 s^2} \right] + \frac{s^5}{1 + k\pi^2 s^2} \sin \pi x$$

Taking limit as $x \to 1$

$$U(1,s) = \sqrt{k}\, s \sinh\left(\frac{1}{\sqrt{k}\, s} \right) \left[U_x(0,s) - \frac{\pi s^5}{1 + k\pi^2 s^2} \right] + \frac{s^5}{1 + k\pi^2 s^2} \sin \pi$$

But $u(1,t) = 0 \quad \Rightarrow \quad U(1,s) = 0$

$$0 = \sqrt{k}\, s \sinh\left(\frac{1}{\sqrt{k}\, s} \right) \left[U_x(0,s) - \frac{\pi s^5}{1 + k\pi^2 s^2} \right] + \frac{s^5}{1 + k\pi^2 s^2} \sin \pi$$

$$U_x(0,s) - \frac{\pi s^5}{1 + k\pi^2 s^2} = 0$$

$$U_x(0,s) = \frac{\pi s^5}{1+k\pi^2 s^2}$$

Therefore,

$$U(x,s) = \frac{s^5}{1+k\pi^2 s^2}\sin \pi x$$

By Inverse Kharrat-Toma Transform B_t^{-1}, we get bounded solution

$$u(x,t) = e^{-k\pi^2 t}\sin \pi x \tag{10}$$

The problem can be interpreted physically. The given equation is the heat equation,where $u(x,t)$ gives the temperature at a point x at time t. Consider a section bounded byplanes $x=0$ & $x=1$& the boundary conditions $u(0, t)=0=u(1, t)$ give the temperature zeroat the planes. The condition $u(x, 0) = \sin\pi x$ indicates the initial temperature in $0<x<1$.The u represents the temperature at time $t>0$ [13].

Example 3 Telegraph Equation

Consider the telegraph equation [14]:

$$u_{xx} = u_{tt} + u_t - u \tag{11}$$

With the boundary conditions:

$$u(0,t) = e^{-2t} \quad , \quad u_x(0,t) = e^{-2t} \tag{12}$$

And the initial conditions:

$$u(x,0) = e^x \quad , \quad u_t(x,0) = -2e^x \tag{13}$$

The exact solution is $u(x,t) = e^{x-2t}$

Solution:

Take the double Kharrat-Toma transform of eq (11), and single Kharrat-Toma transform of conditions (12), (13), and

$$B_t B_x[u_{xx}] = B_t B_x[u_{tt} + u_t - u]$$

$$\frac{1}{p^4}U(p,s) - pU(0,s) - p^3 U_x(0,s) = \frac{1}{s^4}U(p,s) - sU(p,0) - s^3 U_t(p,0) +$$

$$+\frac{1}{s^2}U(p,s) - s^3 U(p,0) - U(p,s)$$

$$s^4 U(p,s) - p^5 s^4 U(0,s) - p^7 s^4 U_x(0,s) = p^4 U(p,s) - p^4 s^5 U(p,0) - p^4 s^7 U_t(p,0) +$$

$$+ p^4 s^2 U(p,s) - p^4 s^7 U(p,0) - p^4 s^4 U(p,s)$$

And the transform of conditions are,

$$U(0,s) = \frac{s^5}{1+2s^2} \quad , \quad U_x(0,s) = \frac{s^5}{1+2s^2}$$

$$U(p,0) = \frac{p^5}{1-p^2} \quad , \quad U_t(p,0) = -2\frac{p^5}{1-p^2}$$

we obtain:

$$U(p,s)=\frac{1}{s^4-p^4-p^4s^2+p^4s^4}\Big[p^5s^4U(0,s)+p^7s^4U_x(0,s)-$$

$$-p^4s^5U(p,0)-p^4s^7U_t(p,0)-p^4s^7U(p,0)\Big]$$

$$U(p,s)=\frac{1}{s^4-p^4-p^4s^2+p^4s^4}\left[p^5s^4\frac{s^5}{1+2s^2}+p^7s^4\frac{s^5}{1+2s^2}-\right.$$

$$\left.-p^4s^5\frac{p^5}{1-p^2}+2p^4s^7\frac{p^5}{1-p^2}-p^4s^7\frac{p^5}{1-p^2}\right]$$

$$U(p,s)=\frac{1}{s^4-p^4-p^4s^2+p^4s^4}\left[\frac{\left(p^5+p^7\right)s^9}{1+2s^2}+\frac{\left(-s^5+s^7\right)p^9}{1-p^2}\right]$$

$$=\frac{1}{s^4-p^4-p^4s^2+p^4s^4}\left[\frac{\left(p^5+p^7\right)s^9}{1+2s^2}+\frac{\left(-s^5+s^7\right)p^9}{1-p^2}\right]$$

$$=\frac{1}{s^4-p^4-p^4s^2+p^4s^4}\left[\frac{\left(p^5+p^7\right)s^9}{1+2s^2}+\frac{\left(-s^5+s^7\right)p^9}{1-p^2}\right]$$

$$U(p,s)=\frac{\left(s^4-p^4-p^4s^2+p^4s^4\right)s^5p^5}{\left(s^4-p^4-p^4s^2+p^4s^4\right)\left(1+2s^2\right)\left(1-p^2\right)}$$

$$=\frac{s^5p^5}{\left(1+2s^2\right)\left(1-p^2\right)}=\frac{s^5}{1+2s^2}\frac{p^5}{1-p^2}$$

By Inverse Kharrat-Toma Transform $B_t^{-1}B_x^{-1}$, we get bounded solution

$$u(x,t)=e^{-3x}e^{-2t}=e^{-3x-2t} \tag{14}$$

CONCLUSION

In this work, double Kharrat-Toma transform is applied to obtain the solution of linear telegraph [15] and One Dimensional Heat Equation [17-19]. It may be concluded that double Kharrat-Toma transform is very powerful and efficient in finding the analytical solution for awide class of partial differential equations.

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