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Combine Kharrat-Toma Transform and Variational Iteration Method to Solve Nonlinear Boundary Value Problems

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Abstract: In the present work, variational iteration method is combined with Kharrat-Toma transform method to solve nonlinear problems. Some illustrative numerical examples are given. The solutions obtained by this method show the accuracy and the efficiency of the suggested combined method.

Key words: Kharrat-Toma transform · Variational iteration method and nonlinear partial differential equation

INTRODUCTION

Linear and non-linear differential equations can model many phenomena in different fields of science and engineering in order to present their behaviors and effects by mathematical concepts. Most of the equations do not have exact solution which can be handled by semi-analytical or numerical methods.

In order to obtain analytical solution of nonlinear differential equations, semi-analytical methods such as the variational Iteration method (VIM) and homotopy perturbation method (HPM) are considered.

Variational iteration method was first proposed by the Chinese mathematician Ji-Huan He in 1997 [1]. It does not require a small parameter which has a significant advantage to provide an analytical solution for a wide range of linear and nonlinear problems in applied sciences. Later, Abdou and Soliman [2] showed the significant results by using variational iteration method for solving burger's and coupled burger's equations. Nadeem *et al*. [3] presented the application of variational iteration method for solving non-homogeneous Cauchy Euler differential equations.

Currently, there are many methods used to solve nonlinear equations such as Elzaki transform decomposition algorithm [4], natural transform Adomian decomposition method [5], variational iteration method coupled with Laplace transform method [6], Elzaki transform variational iteration method [7], homotopy perturbation Sumudu transform method [8, 9], homotopy perturbation Elzaki transform method [10, 11]. In 2020 Kharrat and Toma [12, 13, 14] introduced a new transform called Kharrat-Toma Transform to solve the ordinary differential equations with initial conditions. Several methods were been modified using Kharrat-Toma transform, such as Adomian Decomposition Method is combined with Kharrat-Toma Transform in 2021 [15].

Preliminaries and Notations: In this section, we give some basic notions about Variational iteration method and Kharrat-Toma transform which are used further in this paper.

Basic Idea Variational Iteration Method: To clarify the VIM, we begin by considering a differential equation in the formal form [1, 16]:

$$Lu + Nu = g(x, t),$$

where L and N are linear and nonlinear operators respectively and $g(x, t)$ is a known analytical function. According to VIM, we can write down a correction functional as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^x \lambda(s)\left[L u_n(s) + N\tilde{u}_n(s) - g(s)\right] ds$$

where λ is a general Lagrange multiplier [17], which can be identified optimally via variational theory. The subscript n indicates the nth approximation and $\tilde{u}_n$ is considered as a restricted variation, i.e. $\delta\tilde{u}_n = 0$.

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Kharrat-Toma Transform "KHTM"

Definition 1: [12]. The function $f(x)$ is said to have exponential order on every finite interval in $[0,+\infty]$ If there exist a positive number M that satisfying:

$$|f(x)| \leq Me^{\alpha x}, M > 0, \alpha > 0, \forall x \geq 0$$

Definition 2: [12]. The Kharrat-Toma integral transform and inversion is defined by;

$$B\left[f(x)\right] = G\left(S\right) = s^3 \int_0^\infty f(x)\, e^{\frac{-x}{s^2}}\, dx \quad , \quad x \geq 0$$

$$f(x) = B^{-1}\left[G\left(S\right)\right] = B^{-1}\left[s^3 \int_0^\infty f(x)\, e^{\frac{-x}{s^2}}\, dx \right]$$

The B integral transform states that, if $f(x)$ is piecewise continuous on $[0, +\infty)$ and has exponential order. The B^{-1} will be the inverse of the B integral transform.

Kharrat-Toma Transform of Some Functions: In this section we give Kharrat-Toma transform of some functions:

$f(t)$	$B[f(t)] = G(s)$
1	s^5
t	s^7
t^n	$n!.s^{2n+5}$, $n = 0,1,2,...$
$\sin(kt)$	$\frac{ks^7}{1+k^2s^4}$
$\cos(kt)$	$\frac{s^5}{1+k^2s^4}$
$\sinh(kt)$	$\frac{ks^7}{1-k^2s^4}$
$\cosh(kt)$	$\frac{s^5}{1-k^2s^4}$

The VIM-KHTM Technique: To illustrate the basic idea of this method, we consider a general nonlinear non-homogenous partial differential equation with the initial conditions of the form:

$$L[u(x,t)] + N[u(x,t)] = g(x, t) \tag{1}$$

where L is the linear differential operator, N represents the general nonlinear differential operator and g is the source term.

According to the Variational iteration method, we can construct a correction functional as following:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(x,\xi)[Lu_n(x,\xi) + N\tilde{u}_n(x,\xi) - g(x,\xi)]d\xi \tag{2}$$

where $N\tilde{u}_n(x,\xi)$ is considered as a restricted variation reads $\delta N\tilde{u}_n(x,\xi) = 0$ and λ is a general Lagrange multiplier [17].

Take Kharrat-Toma transform; then, the correction functional will be constructed in the form

$$B\left[u_{n+1}(x,t)\right] = B\left[u_n(x,t)\right] + B[\int_0^t \lambda(x,\xi)[Lu_n(x,\xi) + N\tilde{u}_n(x,\xi) - g(x,\xi)]d\xi] \tag{3}$$

$$\begin{aligned} B\left[u_{n+1}(x,t)\right] &= B\left[u_n(x,t)\right] + \\ &+ B\left[\lambda(x,t) * [Lu_n(x,t) + N\tilde{u}_n(x,t) - g(x,t)]\right] \\ &= B\left[u_n(x,t)\right] + \frac{1}{s^3} B\left[\lambda(x,t)\right].B\left[Lu_n(x,t) + N\tilde{u}_n(x,t) - g(x,t)\right] \end{aligned} \tag{4}$$

where $*$is a single convolution with respect to t.

To find the optimal value of λ, we first take the variation δ with respect to $u_n(x,t)$. Thus;

$$B\left[\delta\, u_{n+1}(x,t)\right] = B\left[\delta\, u_n(x,t)\right] + \frac{1}{s^3}\delta\, B\left[\lambda(x,t)\right]. .B\left[Lu_n(x,t) + N\tilde{u}_n(x,t) - g(x,t)\right] \tag{5}$$

To obtain the solution u we using the differentiation property of Kharrat-Toma transform, applying the initial conditions and take the inverse Kharrat-Toma transform. Finally we obtain the solution u by;

$$u = \lim_{n\to\infty}{}^{u_n} \tag{6}$$

Now we apply the proposed technique to solve some non-linear examples.

Numerical Example: Consider the nonlinear wave equation given by [18]:

$$\begin{cases} u_t + {u_x}^2 = 0 \\ u(x,0) = -x^2 \end{cases} \tag{7}$$

By using the above technique, we obtain;

$$u_{n+1}(x)=u_n(x)+\int_0^t \lambda[(u_n)_\xi+(\tilde{u}_n)_x^2]d\xi,\ n\geq 0 \tag{8}$$

Take Kharrat-Toma transform on (8), we obtain

$$B[u_{n+1}(x)]=B[u_n(x)]+B[\int_0^t \lambda[(u_n)_\xi+(\tilde{u}_n)_x^2]d\xi]$$

$$B[u_{n+1}]=B[u_n(x)]+B\left[\lambda*[(u_n)_t+(\tilde{u}_n)_x^2]\right]$$

$$B[u_{n+1}]=B[u_n]+\frac{1}{s^3}B[\lambda].B\left[(u_n)_t+(\tilde{u}_n)_x^2\right] \tag{9}$$

$$B[u_{n+1}]=B[u_n]+\frac{1}{s^3}B[\lambda].\left\{\frac{1}{s^2}B[u_n]-s^3u_n(x,0)+B\left[(\tilde{u}_n)_x^2\right]\right\}$$

Taking the variation δ with respect to u_n. Thus, That gives;

$$B[\delta\ u_{n+1}]=B[\delta u_n]+\frac{1}{s^3}B[\lambda].\left\{.\frac{1}{s^2}B[\delta u_n]\right\} \tag{10}$$

The restricted variations condition of $u_n(x, t)$ requires that $\delta u_{n+1}(x, t) = 0$. Hence, we have;

$$\begin{aligned}&0=B[\delta\ u_n]\left\{1+\frac{1}{s^5}B[\lambda]\right\}\\&1+\frac{1}{s^5}B[\lambda]=0\\&B[\lambda]=-s^5\end{aligned} \tag{11}$$

Applying the inverse Kharrat-Tomatrans form, we get

$$\lambda=-1 \tag{12}$$

Substituting into (9), we obtain;

$$B[u_{n+1}]=B[u_n]-s^2B\left[(u_n)_t+(u_n)_x^2\right] \tag{13}$$

Considering the initial solution:

$$u_0=(x,0)=-x^2 \tag{14}$$

Substituting into (13), we obtain;

$$B[u_1]=B[u_0]-s^2B\left[(u_0)_t+(u_0)_x^2\right] \tag{15}$$

$$B[u_1(x,t)]=-x^2-4x^2s^7 \tag{16}$$

Appling the inverse Kharrat-Toma transform;

$$u_1(x,t)=-x^2-4x^2t \tag{17}$$

Substituting (17) into (13), we obtain;

$$B[u_2]=B[u_1]-s^2B\left[(u_1)_t+(u_1)_x^2\right] \tag{18}$$

$$B[u_2(x,t)]=-x^2-4x^2s^7\text{-}32x^2s^9\text{-}128x^2s^{11} \tag{19}$$

The inverse of Kharrat-Toma transform implies that;

$$u_2(x,t)=-x^2-4x^2t-16x^2t^2-\frac{128}{6}x^2t^3 \tag{20}$$

Continue this process, we obtain;

$$\begin{aligned}u_3(x,t)=&-x^2-4x^2t-16x^2t^2-64x^2t^3+\frac{512}{3}x^2t^4-\\&-\frac{1024}{3}x^2t^5-\frac{4096}{9}x^2t^6-\frac{16384}{63}x^2t^7\end{aligned} \tag{21}$$

Then, the exact solution of (7) is;

$$u(x,t)=\lim_{n\to\infty}u_n(x,t)=\frac{x^2}{4t-1} \tag{22}$$

Consider the non-homogenous, nonlinear gas dynamic equation [19]:

$$\begin{cases}v_t(x,t)+v(x,t)v_x(x,t)-v(x,t)+v^2(x,t)=-e^{t-x}\\v(x,0)=1-e^{-x}\end{cases} \tag{23}$$

Where the exact solution is given by;

$$v(x,t)=1-e^{t-x}$$

In a similar fashion, its correction functional can be written as following:

$$\begin{aligned}v_{n+1}(x,t)=v_n(x,t)+\int_0^t \lambda[(v_n)_\xi(x,\xi)-\tilde{v}_n(x,\xi)(\tilde{v}_n)_x(x,\xi)+\\-\tilde{v}_n(x,\xi)+\tilde{v}_n^2(x,\xi)+e^{\xi-x}]d\xi\ \ ,\ n\geq 0\end{aligned} \tag{24}$$

Using Kharrat-Toma transform in (24), we have;

$$B\left[v_{n+1}(x,t)\right]=B\left[v_n(x,t)\right]+B\left[\int_0^t \lambda[(v_n)_\xi(x,\xi)-\right.$$
$$\left.-\tilde{v}_n(x,\xi).(\tilde{v}_n)_x(x,\xi)-\tilde{v}_n(x,\xi)+\tilde{v}_n^2(x,\xi)+e^{\xi-x}]d\xi\right.$$

$$\begin{aligned}
&B\left[v_{n+1}(x,t)\right]=B\left[v_n(x,t)\right]+\\
&+B\left[\lambda(x,t)*[(v_n)_t(x,t)-\tilde{v}_n(x,t).(\tilde{v}_n)_x(x,t)-\tilde{v}_n(x,t)+\tilde{v}_n^2(x,t)+e^{t-x}\right]\\
&=B\left[v_n(x,t)\right]+\frac{1}{s^3}B\left[\lambda(x,t)\right].\\
&\quad .B\left[(v_n)_t(x,t)-\tilde{v}_n(x,t).(\tilde{v}_n)_x(x,t)-\tilde{v}_n(x,t)+\tilde{v}_n^2(x,t)+e^{t-x}\right]
\end{aligned} \tag{25}$$

$$B\left[v_{n+1}(x,t)\right]=B\left[v_n(x,t)\right]+\frac{1}{s^3}B\left[\lambda(x,t)\right].\left\{\frac{1}{s^2}B\left[v_n(x,t)\right]-s^3v_n(x,0)-\right.$$
$$\left.+B\left[-\tilde{v}_n(x,t).(\tilde{v}_n)_x(x,t)-\tilde{v}_n(x,t)+\tilde{v}_n^2(x,t)+e^{t-x}\right]\right\}$$

Taking the variation δ with respect to v_n. Thus, That gives;

$$B\left[\delta v_{n+1}(x,t)\right]=B\left[\delta v_n(x,t)\right]+\frac{1}{s^3}B\left[\lambda(x,t)\right].\left\{\frac{1}{s^2}B\left[\delta v_n(x,t)\right]\right\} \tag{26}$$

The restricted variations condition of $v_n(x,t)$ means that $\delta\, v_{n+1}(x,t) = 0$. Hence, we have;

$$\begin{aligned}
&0=B\left[\delta\ u_n\right]\left\{1+\frac{1}{s^5}B[\lambda]\right\}\\
&1+\frac{1}{s^5}B[\lambda]=0\\
&B[\lambda]=-s^5
\end{aligned} \tag{27}$$

Taking the inverse Kharrat-Toma transform get;

$$\lambda = -\ 1 \tag{28}$$

Substituting the value of Lagrange multiplier into (25), we obtain;

$$\begin{aligned}
&B\left[v_{n+1}(x,t)\right]=B\left[v_n(x,t)\right]-\\
&-s^2.B\left[(v_n)_t(x,t)-v_n(x,t).(v_n)_x(x,t)-v_n(x,t)+v_n^2(x,t)+e^{t-x}\right]
\end{aligned} \tag{29}$$

Considering the initial solution as:

$$v_0(x,t)=1-e^{-x} \tag{30}$$

Then, from (29), we have;

$$B\left[v_1(x,t)\right] = B\left[v_0(x,t)\right] - \\ -s^2.B\left[(v_0)_t(x,t) - v_0(x,t).(v_0)_x(x,t) - v_0(x,t) + v_0^2(x,t) + e^{t-x}\right] \quad (31)$$

$$B\left[v_1(x,t)\right] = B\left[1-e^{-x}\right] - s^2 B\left[e^{t-x}\right] \\ = 1 - e^{-x} - e^{-x}\frac{s^7}{1-s^2} \quad (32)$$

Then, the first approximate solution of (29) by applying the inverse Kharrat-Toma transform is;

$$v_1(x,t) = 1 - e^{t-x}$$

Continue this process, we obtain;

$$v_n(x,t) = 0, n > 1$$

Finally, the exact solution is;

$$v(x,t) = 1 - e^{t-x} \quad (33)$$

We note the application of VIM-KHTM permit to obtained the exact solution by a few iterates.

CONCLUSION

In this paper, the new integral transform KHARRAT-Toma transform combined with Variational iteration method, have been applied to find the solution of nonlinear problems. We demonstrated the accuracy and efficiency of the proposed method by solving some examples. It suggests that the Variational Iteration method combined with KHTT. Is accurate, reliable and easy touse.

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