

A Comparison Between the Variational Iteration Method and Trapezoidal Rule for Solving Linear Integro-Differential Equations

^{1,2}*R. Saadati, ³B. Raftari, ²H. Adibi, ²S.M. Vaezpour and ⁴S. Shakeri*

²Faculty of Sciences, University of Shomal, Amol P.O.Box 731, Iran

²Department of Mathematics and Computer Science,

Amirkabir University of Technology, No. 424, Hafez Ave., Tehran, Iran

³Islamic Azad University-Kermanshah Branch, Kermanshah, Iran

⁴Islamic Azad University-Ayatollah Amoli Branch, Amol P.O. Box 678, Iran

Abstract: Two numerical methods for solving the linear integro-differential equations is presented and compared. First method, based upon Trapezoidal rule and numerical differentiation, transforms the integro-differential equation into a system of linear algebraic equations. The simplicity of the method makes it perfect for many applications. Second method, is variational iteration method. Finally, some numerical examples are presented to compare the accuracy of the methods. Comparisons with the Trapezoidal rule reveal that the VIM is very effective and convenient.

Key words: Fredholm • Volterra integro • Differential equations • Trapezoidal rule • Finite differentiation • Variational iteration method

INTRODUCTION

Several numerical methods for approximating Fredholm or Volterra integro-differential equations are known. In this paper, we introduce two numerical method to solve the following linear Volterra and Fredholm integro-differential equations

$$\begin{cases} y'(x) = f(x) + \lambda \int_a^x k(x,t)y(t)dt, & a \leq x \leq b, \\ y(a) = y_a \end{cases} \quad (1)$$

$$\begin{cases} y'(x) = f(x) + \lambda \int_a^b k(x,t)y(t)dt, & a \leq x \leq b, \\ y(a) = y_a \end{cases} \quad (2)$$

where the function $f(x)$ and the kernel $k(x,t)$ are known and $y(x)$ is the solution to be determined.

First method for solving linear Volterra integro-differential equations: In this section, we consider Volterra integro-differential equation (1.1) and subdivide

the interval of integration $[a,x]$ into N equal subintervals of width $h = \frac{x_N - a}{N}$, where x_N is the end point we

choose for x . Since we will be using either t or x as the independent variable, therefore let $x_i = a + ih (= t_i)$ and $f(x_i) = f_i$. Clearly $k(x_i, t_j)$ vanishes for $t_j > x_i$.

So, if we use the Trapezoidal rule with N subintervals, then the Volterra integro-differential equation (1.1) is approximated by

$$\int_a^x k(x,t)y(t)dt \approx h \left[\frac{1}{2}k(x,t_0)y(t_0) + k(x,t_1)y(t_1) + \dots + k(x,t_{N-1})y(t_{N-1}) + \frac{1}{2}k(x,t_N)y(t_N) \right], \quad (2.1)$$

$$h = \frac{t_j - a}{j} = \frac{x - a}{N}, \quad t_j \leq x, j \geq 1, x = x_N (= t_N)$$

Also the integro-differential equation (1.2) is approximated by

$$y'(x) = f(x) + h \begin{bmatrix} \frac{1}{2}k(x, t_0)y(t_0) + k(x, t_1)y(t_1) + \dots \\ + k(x, t_{N-4})y(t_{N-4}) + \frac{1}{2}k(x, t_N)y(t_N) \end{bmatrix},$$

$t_j \leq x, j \geq 1, x = x_N(t_N)$ (2.2)

$$y'(x_i) = f(x_i) + h \begin{bmatrix} \frac{1}{2}k(x_i, t_0)y(t_0) + k(x_i, t_1)y(t_1) + \dots \\ + k(x_i, t_{j-1})y(t_{j-1}) + \frac{1}{2}k(x_i, t_j)y(t_j) \end{bmatrix},$$

$i = 1, 2, \dots, n, \quad t_j \leq x_i,$ (2.3)

and accordingly the system of equations (2.3) can be written in the following more compact form

$$y'_i = f_i + h \left[\frac{1}{2}k_{i0}y_0 + k_{i1}y_1 + \dots + k_{i,j-1}y_{j-1} + \frac{1}{2}k_{ij}y_j \right], \quad i = 1, 2, \dots, N, j \leq i.$$

(2.4)

Now we take advantage of finite differentiation to get

$$\frac{y_{i+1} - y_{i-1}}{2h} = f_i + h \left[\frac{1}{2}k_{i0}y_0 + k_{i1}y_1 + \dots + k_{i,j-1}y_{j-1} + \frac{1}{2}k_{ij}y_j \right],$$

$i = 1, 2, \dots, N-1, \quad j \leq i,$ (2.5)

and

$$\frac{3y_N - 4y_{N-1} + y_{N-2}}{2h} = f_N + h \begin{bmatrix} \frac{1}{2}k_{N0}y_0 + k_{N1}y_1 + \dots \\ + k_{N,N-1}y_{N-1} + \frac{1}{2}k_{NN}y_N \end{bmatrix},$$

as a system consisting N equations which by virtue of (2.4) and (2.5) can be written in the following matrix form

$$KY = F$$

where

$$K = \begin{pmatrix} -h^2 k_{11} & 1 & 0 & 0 & \dots & 0 & 0 \\ (2h^2 k_{21} + 1) & -h^2 k_{22} & 1 & 0 & 0 & \dots & 0 \\ -h^2 k_{31} & -h^2 k_{32} & 1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ -h^2 k_{N-1,1} & -h^2 k_{N-1,2} & -h^2 k_{N-1,3} & (2h^2 k_{N-1,N-2} + 1) & -h^2 k_{N-1,N-1} & 1 & 0 \\ -h^2 k_{N,1} & -h^2 k_{N,2} & -h^2 k_{N,3} & -h^2 k_{N,4} & -h^2 k_{N,5} & -h^2 k_{N,6} & -h^2 k_{N,7} \end{pmatrix}$$

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix}$$

and

$$F = \begin{bmatrix} 2hf_1 + (h^2 k_{10} + 1)y_0 \\ 2hf_2 + h^2 k_{20}y_0 \\ \vdots \\ 2hf_{N-1} + h^2 k_{N-1,0}y_0 \\ 2hf_N + h^2 k_{N,0}y_0 \end{bmatrix}.$$

First method for solving linear Fredholm integro-differential equations: Consider the linear Fredholm integro-differential equation (1.2). Adopting as the preceding section with $k(x_i, t_j) = k_{ij}$,

$y(x_i) = y_i, (= y(t_i)), i = 1, 2, \dots, N$ and utilizing the trapezoidal rule, we get

$$y'(x) \approx f(x) + h \begin{bmatrix} \frac{1}{2}k(x, t_0)y(t_0) + k(x, t_1)y(t_1) + \dots \\ + k(x, t_{N-1})y(t_{N-1}) + \frac{1}{2}k(x, t_N)y(t_N) \end{bmatrix},$$

(3.1)

Accordingly, for $y'_i = y'(x_i), (= y'(t_i)), i = 1, 2, \dots, N$, the relation (3.1) gets replaced by

$$y'_i = f_i + h \left[\frac{1}{2}k_{i0}y_0 + k_{i1}y_1 + \dots + k_{i,N-1}y_{N-1} + \frac{1}{2}k_{iN}y_N \right],$$

$i = 1, 2, \dots, N.$

If so, we have

$$\frac{y_{i+1} - y_{i-1}}{2h} = f_i + h \left[\frac{1}{2}k_{i0}y_0 + k_{i1}y_1 + \dots + k_{i,N-1}y_{N-1} + \frac{1}{2}k_{iN}y_N \right],$$

$i = 1, 2, \dots, N-1,$

and

$$\frac{3y_N - 4y_{N-1} + y_{N-2}}{2h} = f_N + h \begin{bmatrix} \frac{1}{2}k_{N0}y_0 + k_{N1}y_1 + \dots \\ + k_{N,N-1}y_{N-1} + \frac{1}{2}k_{NN}y_N \end{bmatrix},$$

which are N equations in y_i 's

Now from (3.3) and (3.4), we get a system of equations for y_1, y_2, \dots, y_N , which can be written in the matrix form

$$KY = F \quad (3.5)$$

where

$$K = \begin{pmatrix} -2h^2k_{11} & -(2h^2k_{12}-1) & -2h^2k_{13} & \dots & -2h^2k_{1,N-1} & -h^2k_{1N} \\ -(2h^2k_{21}+1) & -2h^2k_{22} & -(2h^2k_{23}-1) & -2h^2k_{24} & \dots & -2h^2k_{2,N-1} & -2h^2k_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -2h^2k_{N-1,1} & \dots & -2h^2k_{N-1,N-3} & -(2h^2k_{N-1,N-2}+1) & -2h^2k_{N-1,N-1} & -(h^2k_{N-1,N}-1) \\ -2h^2k_{NN} & \dots & -2h^2k_{N,N-3} & -(2h^2k_{N,N-2}-1) & -(2h^2k_{N,N-1}+4) & -(2h^2k_{N,N}-3) \end{pmatrix} \quad (3.6)$$

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix}$$

and

$$F = \begin{pmatrix} 2hf_1 + (h^2k_{10}+1)y_0 \\ 2hf_2 + h^2k_{20}y_0 \\ \vdots \\ 2hf_N + h^2k_{N0}y_0 \end{pmatrix}.$$

Variational iteration method: Various kinds of analytical methods and numerical methods [1,2] were used to solve integral equations. In this section, we apply He's variational iteration method [3-9] to solve integral equations. The method can solve various different nonlinear equations [10-17]. To illustrate the basic idea of the method, we consider following general nonlinear system:

$$L[u(t)] + N[u(t)] = g(t), \quad (4.1)$$

Table 1: Numerical results for Example 1

x	$ U_{Exact} - U_{Tr} $	$ U_{Exact} - U_{VIM} $
0.04167	0.000021431634307012	0.0000033362
0.08333	0.000023157760403958	0.0000033447
0.12500	0.000043694619093926	0.0000000001
0.16667	0.000044586972802962	0.0000033772
0.20833	0.000064351267891061	0.0000034007
0.25000	0.000064532251764948	0.0000000039
0.29167	0.000083643867841987	0.0000034623
0.33333	0.000083233963874996	0.0000034996
0.37500	0.000101815005884021	0.0000000001
0.41667	0.000100933531027025	0.0000035867
0.45833	0.000119104036649942	0.0000036360
0.50000	0.000117874040123034	0.0000000002
0.54167	0.000135756711352020	0.0000037468
0.58333	0.000134300444016966	0.0000038051
0.62500	0.000152021404810021	0.0000000026
0.66667	0.000150465780536013	0.0000039418
0.70833	0.000168151643201997	0.0000039994
0.75000	0.000166627623651994	0.0000000143
0.79167	0.000184412109674986	0.0000041825
0.83333	0.000183053825299304	0.0000041965
0.87500	0.000201073204312500	0.0000000667
0.91667	0.000200022086380400	0.0000045108
0.95833	0.000218421708768596	0.0000043309
1.00000	0.000217824322773008	0.0000002505

Where L is a linear operator, N is a nonlinear operator and $g(t)$ is a given continuous function. The basic character of:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(s)[Lu_n(s) + N\bar{u}_n(s) + N\tilde{u}_n(s) - g(s)]ds,$$

Illustrative examples

Example 1: Consider the Volterra integro-differential equation

$$\begin{cases} y'(x) = 1 + \sin(x) + \int_0^x y(t)dt, \\ y(0) = -1 \end{cases}$$

which has the exact solution

$$y(x) = \frac{1}{4}e^x - \frac{3}{4}e^{-x} - \frac{1}{2}\cos(x).$$

Table 2: Numerical results for Example 2

x	$ U_{Exact} - U_{Tr} $	$ U_{Exact} - U_{VIM} $
0.04167	0.0002289030726834	0.39099128892
0.08333	0.0000771370953401	0.00000392492
0.12500	0.000311048867308011	0.000000000000
0.16667	0.0001645231435910	0.00000459420
0.20833	0.0004039179333090	0.00000496060
0.25000	0.0002631314102510	0.000000000000
0.29167	0.0005085337923760	0.00000576390
0.33333	0.0003740370896630	0.00000621070
0.37500	0.0006260262536459	0.000000000000
0.41667	0.0004984272679199	0.00000716340
0.45833	0.0007576422290870	0.00000768770
0.50000	0.0006376133268120	0.000000000000
0.54167	0.0009047588216130	0.00000883320
0.58333	0.0007930406858201	0.00000945800
0.62500	0.0010688958353300	0.00001800000
0.66667	0.0009663073926098	0.00001082100
0.70833	0.0012517312517402	0.00001156400
0.75000	0.1606743483941800	0.15951517500
0.79167	0.0014551137767900	0.00001318100
0.83333	0.0013735805565001	0.00001406100
0.87500	0.0016810821831803	0.000000000000
0.91667	0.0016116754479501	0.00001597800
0.95833	0.0019318895486600	0.00001702000
1.00000	0.0018758160369199	0.000000000000

The numerical results with $N=24$ for first method and $N=5$ for second method are represented in Table 1 and are compared with the exact solution.

Example 2: As the second example consider the Fredholm integro-differential equation:

$$\begin{cases} y'(x) = xe^x + e^x - x + \int_0^1 xy(t)dt, \\ y(0) = 0 \end{cases}$$

with the exact solution $y(x) = xe^x$. The numerical results with $N=24$ are represented in Table 2 and are compared with the exact solution.

CONCLUSION

Integro-differential equations are usually difficult to solve analytically. In many cases, it is required to obtain the approximate solutions. In this study we have solved linear integro-differential equations by using Trapezoidal rule and Numerical differentiation and VIM. Comparisons with the Trapezoidal rule reveal that the VIM is very effective and convenient.

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