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On The Convergence of the Ishikawa Iterates to a Common Fixed Point in Probabilistic Metric Spaces

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Abstract: In this paper, it is shown that if the sequence of Ishikawa iterations associated with G and H converges, then its limit point is the common fixed point of G and H in probabilistic metric spaces.

Key words: Ishikawa iterations method . Probabilistic metric space . Common fixed point theorem

INTRODUCTION

K. Menger introduced the notion of a probabilistic metric space in 1942 and since then the theory of probabilistic metric spaces has developed in many directions [1]. The idea of K. Menger was to use distribution functions instead of nonnegative real numbers as values of the metric. The notion of a probabilistic metric space corresponds to situations when we do not know exactly the distance between two points, but we know probabilities of possible values of this distance.

A probabilistic generalization of metric spaces appears to be interest in the investigation of physical quantities and physiological thresholds. It is also of fundamental importance in probabilistic functional analysis. In the recent years several authors [2-6] have studied the convergence of the sequence of the Mann iterates [7] of a mapping H to a fixed point of H, under various contractive conditions. The Ishikawa iteration scheme [4] was first used to establish the strong convergence for a pseudo contractive self-mapping of a convex compact subset of a Hilbert space.

Very soon both iterative processes were used to establish the strong convergence of the respective iterates for some contractive type mappings in Hilbert spaces and then in more general normed linear spaces. In this paper, we used Ishikawa iterations scheme for finding a common fixed point for two mappings in probabilistic metric spaces. In the sequel, we shall adopt the usual terminology, notation and conventions of probabilistic metric spaces introduced by Schweizer and Sklar [1].

Throughout this paper, the space of all probability distribution functions (briefly, d.f.'s) is denoted by

$\Delta^+ = \{F : R \bigcup \{-\infty, +\infty\} \to [0,1] : F$ is left-continuous and non-decreasing on R, $F(0) = 0$ and $F(+\infty) = 1\}$ and the subset $D^+ \subseteq \Delta^+$ is the set $D^+ = \{F \in \Delta^+ : l^- F(+\infty) = 1\}$.

Here, $l^- f(x) = \lim_{t \to x^-} f(t)$.

Definition 1.1: ([1]) A mapping $T : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if T satisfies the following conditions :

- T is commutative and associative;
- T is continuous;
- $T(a,1) = a$ for all $a \in [0,1]$;
- $T(a,b) \le T(c,d)$ whenever $a \le c$ and $b \le d$ for $a,b,c,d \in [0,1]$.

Two typical examples of continuous t-norm are $T(a,b) = ab$ and $T(a,b) = \min(a,b)$.

Now t-norms are recursively defined by $T^1 = T$ and

$$T^n(x_1,\dots,x_{n+1}) = T(T^{n-1}(x_1,\dots,x_n), x_{n+1})$$

for $n \ge 2$ and $x_i \in [0,1]$, for all $i \in \{1,2,\dots,n+1\}$. A t-norm is *Hadzic type* if $T(a,b) \ge \min(a,b)$ for every $a,b \in [0,1]$ (see [8]).

Definition 1.2: A Menger Probabilistic Metric space (briefly, Menger PM-space) is a triple (X,F,T), where X is a nonempty set, T is a continuous t-norm and F is a mapping from $X \times X$ into D^+ such that, if $F_{x,y}$ denotes the value of F at the pair (x,y), the following conditions hold: for all x,y,z in X,

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(PM1) $F_{x,y}(t)=1$ for all $t>0$ if and only if x = y;

(PM2) $F_{x,y}(t)=F_{y,x}(t)$;

(PM3) $F_{x,y}(t+s)\geq T(F_{x,y}(t),F_{y,z}(s))$ for all $x,y,z\in X$ and $t,s\geq 0$.

If, in the above definition, the triangular inequality (PM3) is replaced by

(NA) $F_{x,y}(\max\{t,s\})\geq T(F_{x,y}(t),F_{y,z}(s))$

for all $x,y,z\in X$ and $t,s\geq 0$.

Or, equivalently,

$$F_{x,y}(t)\geq T(F_{x,y}(t),F_{y,z}(t))$$

for all $x,y,z\in X$ and $t\geq 0$, then the triple (X,F,T) is called a non-Archimedean Menger PM space [9,10].

Definition 1.3: Let (X,F,T) be a Menger PM-space.

(1) A sequence $\{x_n\}$ in X is said to be *convergent* to x in X if, for every $\varepsilon>0$ and $\lambda>0$, there exists positive integer N such that $F_{x_n,x}(\varepsilon)>1-\mu$ whenever $n\geq N$.

(2) A sequence $\{x_n\}$ in X is called *Cauchy* sequence if, for $\varepsilon>0$ and $\mu>0$, there exists positive integer N such that $F_{x_n,x_m}(\varepsilon)>1-\mu$ whenever $n,m\geq N$.

(3) A Menger PM-space (X,F,T) is said to be *complete* if and only if every Cauchy sequence in X is convergent to a point in X.

Definition 1.4: Let (X,F,T) be a Menger PM space. For each p in X and $\mu>0$ the strong μ neighborhood of p is the set

$$N_p(\mu)=\{q\in X : F_{p,q}(\mu)>1-\mu\}$$

and the strong neighborhood system for X is the union $\bigcup_{p\in v}N_P$ where

$$N_p=\{N_p(\mu)\ ;\mu>0\}$$

The strong neighborhood system for X determines a Hausdorff topology for X.

Theorem 1.5: [1,11] *If* (X,F,T) *is a* PM*-space and* $\{p_n\}$ *and* $\{q_n\}$ *are sequences such that* $p_n\to p$ *and* $q_n\to q$ *then* $\lim n\to\infty\ F_{p_n,q_n}(t)=F_{p,q}(t)$

Lemma 1.6: [12] *Let* (X,F,T) *be a Menger PM-space and define* $E_{\lambda,F}: X^2\to R^+\cup\{0\}$ *by*

$$E_{\lambda,F}(x,y)=\inf\{\ t>0: F_{x,y}(t)>1-\lambda\}$$

for each $\lambda\in]0,1[$ *and* $x,y\in X$. *Then we have*

- *For any* $\mu\in]0,1[$ *there exists* $\lambda\in]0,1[$ *such that*

$$E_{\mu,F}(x_1,x_n)\leq E_{\lambda,F}(x_1,x_2)+\cdots+E_{\lambda,F}(x_{n-1},x_n)$$

for any $x_1,\ldots,x_n\in X$;

- *The sequence* $\{x_n\}$ *is convergent with respect to Menger PM* F *if and only if* $E_{\lambda,F}(x_n,x)\to 0$. *Also the sequence* $\{x_n\}$ *is a Cauchy sequence with respect to Menger PM* F *if and only if it is a Cauchy sequence with* $E_{\lambda,F}$.

RESULTS AND DISCUSSION

Definition 2.1: Let (X,F,T) be a Menger PM-space and $I=[0,1]$ the closed unit interval. A continuous mapping $W: X^2\times I\to X$ is said to be a convex structure on X if for all $x,y\in X$ and $k\in I$

(2.1)

$$E_{F,\lambda}[u,W(x,y,k)]\leq kE_{F,\lambda}(u,x)+(1-k)E_{F,\lambda}(u,y)$$

for all u in X. A Menger PM-space (X,F,T) together with a convex structure is called a convex Menger PM-space [4, 13].

Theorem 2.2. *Let C be a nonempty closed convex subset of a convex non-Archimedean Menger PM-space* (X,F,T) *in which T is Hadzic type. Let* $G,H: X \to X$ *be a self-mappings satisfying*

(2.2) $$F_{G(x),H(y)}(t) \geq T^2\left(F_{x,y}(t/h), F_{x,H(y)}(t/h), F_{G(x),y}(t/h)\right)$$

for all $x, y \in C$ *and t > 0 in which* $\lambda \in]0,1[$. *Suppose that* $\{x_n\}$ *is Ishikawa type iterative scheme with G and H, defined by*

(1) $x_n \in c$;

(2) $y_n = W(Gx_n, x_n, \beta_n), n \geq o$

(3) $x_{n+1} = W(Hy_n, x_n, \alpha_n), n \geq o$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *satisfy* $0 \leq \alpha_n, \beta_n \leq 1$ *and* $\{\alpha_n\}$ *is away from zero. If* $\{x_n\}$ *converges to some* $p \in C$ *,*

then p is common fixed point of G and H.

Proof: From (3) it follows that

$$E_{\lambda,F}(x_n, x_{n+1}) = E_{\lambda,F}\left[x_n, W(H(y_n), x_n, \alpha_n)\right] = \alpha_n E_{\lambda,F}(x_n, H(x_n)).$$

Since $x_n \to p, F_{x_n,x_{n+1}}(t) \to 1$ and by Lemma 1.6 (ii) $E_{\lambda,F}(x_n, x_{n+1}) \to 0$

Since $\{\alpha_n\}$ is away from zero, it follows that

(2.3) $$\lim_{n\to\infty} E_{\lambda,F}(x_n, H(y_n)) = 0$$

Using (2.2) we get:

(2.4) $$E_{G(x_n)H(y_n)}(t) \geq T^2\left(F_{x_n,y_n}\left(\frac{t}{h}\right), F_{x_n,H(y_n)}\left(\frac{t}{h}\right), F_{G(x_n),y_n}\left(\frac{t}{h}\right)\right).$$

By a property of E we have that

$$E_{\lambda,F}(G(x_n), H(y_n))$$

$$= \inf\{\ t > 0 : F_{G(x_n)H(y_n)}(t) > 1-\lambda\ \}$$

$$\leq \inf\left\{t > 0:\ T^2\left(F_{x_n,y_n}\left(\frac{t}{h}\right), F_{x_n,H(y_n)}\left(\frac{t}{h}\right), F_{G(x_n),y_n}\left(\frac{t}{h}\right)\right) > 1-\lambda\ \right\}$$

$$\leq h \inf\left\{t > 0:\ T^2\left(F_{x_n,y_n}(t), F_{x_n,H(y_n)}(t), F_{G(x_n),y_n}(t)\right) > 1-\lambda\ \right\}$$

$$\leq h\ \inf\ \left\{t > 0:\ \min\left(F_{x_n,y_n}(t), F_{x_n,H(y_n)}(t), F_{G(x_n),y_n}(t)\right) > 1-\lambda\ \right\}$$

$$\leq h\left[\ E_{\lambda,F}(x_n, y_n) + E_{\lambda,F}(x_n, H(y_n)) + E_{\lambda,F}(G(x_n), y_n)\ \right].$$

From (2) and (3) we have that

$$E_{\lambda,F}(x_n, y_n) = E_{\lambda,F}\left[x_n, W\left(G(x_n), x_n, \beta_n\right)\right] = \beta_n E_{\lambda,F}\left(x_n, G(x_n)\right),$$

$$E_{\lambda,F}\left(G(x_n), y_n\right) = E_{\lambda F}\left[G(x_n), W\left(G(x_n), x_n, \beta_n\right)\right] = (1-\beta_n) E_{\lambda,F}\left(x_n, G(x_n)\right).$$

Thus we have

$$(2.5) \qquad E_{\lambda,F}\left(G(x_n), H(y_n)\right) \le h\left[E_{\lambda,F}\left(x_n, G(x_n)\right) + E_{\lambda,F}\left(x_n, H(y_n)\right)\right].$$

By triangular inequality (NA), we have

$$(2.6) \qquad T\left(F_{H(y_n)G(x_n)}(t), F_{x_n, H(y_n)}(t)\right) \le F_{x_n, G(x_n)}(t).$$

By a property of E and since the t–norm is Hadzic type, we have that

$$E_{\lambda,F}\left(x_n, G(x_n)\right) = \inf\left\{t > 0 : F_{x_n, G(x_n)}(t) > 1-\lambda\right\}$$

$$\le \inf\left\{t > 0 : T\left(F_{H(y_n)G(x_n)}(t), F_{x_n, H(y_n)}(t)\right) > 1-\lambda\right\}$$

$$\le \inf\left\{t > 0 : \min\left(F_{H(y_n)G(x_n)}(t), F_{x_n, H(y_n)}(t)\right) > 1-\lambda\right\}$$

$$\le E_{\lambda,F}\left(H(y_n), G(x_n)\right) + E_{\lambda,F}\left(H(y_n), x_n\right).$$

Hence, from (2.5) and last inequality we have

$$E_{\lambda,F}\left(H(y_n), G(x_n)\right) \le \frac{2h}{1-h} E_{\lambda,F}\left(H(y_n), x_n\right).$$

Taking the limit as $n \to \infty$, by (2.3), we obtain

$$\lim_{n\to\infty} E_{\lambda,F}\left(H(y_n), G(x_n)\right) = 0.$$

Since $H(y_n) \to p$, it follows that $G(x_n) \to p$. Since $E_{\lambda,F}(x_n, y_n) = \beta_{nE_{\lambda,F}}\left(x_n, G(x_n)\right)$, it follows that $y_n \to p$.

From (2.2) and by a property of E we have

$$E_{\lambda,F}\left(G(x_n), H(p)\right) \le h\left[E_{\lambda,F}(x_n, p) + E_{\lambda,F}\left(x_n, H(p)\right) + E_{\lambda,F}\left(p, G(x_n)\right)\right].$$

Taking the limit as $n \to 1$, we obtain

$$E_{\lambda,F}\left(p, H(p)\right) \le h E_{\lambda,F}\left(p, H(p)\right).$$

Since $h \in (0,1)$ then $E_{\lambda,F}\left(p, H(p)\right) = 0$. Hence H(p) = p. Similarly, From (2.2) and by a property of E we have

$$E_{\lambda,F}\left(G(p), H(xn)\right) \le h\left[E_{\lambda,F}(xn, p) + E_{\lambda,F}\left(p, H(xn)\right) + E_{\lambda,F}\left(xn, G(p)\right)\right].$$

Taking the limit as $n \to 1$, we obtain

$$E_{\lambda,F}\left(p, G(p)\right) \le h E_{\lambda,F}\left(p, G(p)\right).$$

Hence $G(P) = P$. Therefore, $G(P) = H(P) = P$ and the proof is complete.

CONCLUSIONS

In this paper, it is shown that if the sequence of Ishikawa iterations associated with G and H converges, then its limit point is the common fixed point of G and H in Menger probabilistic metric spaces.

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