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A New Integral Transform: Kharrat-Toma Transform and Its Properties

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Abstract: In this paper, we propose a new integral transform, called the Kharrat-Toma Transform which can be considered as a base for a number of potential new integral transforms. Many fundamental properties about this new integral transform which were created in this work, include (for example) the existence theorem, transportation theorem, convolution theorem and inversion equation. The main advantage of this new technique is that it solves ordinary differential equations with variable and constant coefficients. To show the efficiency and use of the presented transform for solving differential equations, some examples are given.

Key words: Integral Transform · Kharrat-Toma Transform · Ordinary differential equations

INTRODUCTION

The differential equation plays an important role in Physics Science, applied mathematics, Chemistry, Physiology applications and Engineering; therefore, the researchers suggest to develop new methods for obtaining solution which approximate the exact solution.

In the literature, there are many integral transforms that have widely used to solve the differential equations and thus there are several works on the integral transform such as the Laplace transform was introduced by P. S. Laplace in 1780s [1], Laplace transform is the oldest integral transform and the most used. The Stieltjes Transform was firstly introduced by T.S. Stieltjes in [2], R.H. Mellin was the first to give a systematic formulation of the Mellin transformation in [3], the Hankel transform was first developed by Hermann Hankel. It is also known as the Fourier–Bessel transform [4], D. Hilbert suggested the Hilbert transform in [5], J. Radon founded the Radon transform in [6], Laguerre transform by Edmond Laguerre in [7].

Furthermore, of recent, G. K. Watugula introduced the Sumudu transform in [8].

The Natural transform was initiated by Khan and Khan in [9]. The Elzaki transform was presented by Tarig M. Elzaki in [10], The Aboodh transform was introduced by Khalid S. Aboodh in [11]. The new integral transform " M-transform" was suggested by Srivastava in [12]. The ZZ transform was devised by Zafar in [13]. A. Kamal and H. Sedeeg proposed the Kamal Transform in [14], the Yang Transform was by Xiao-Jun Yang in [15] The Mohand Transform was introduced by Mohand M. Mahgoub in [16], S. Ahmadi *et al*. proposed a new integral transform to solve higher order linear Laguerre and Hermite differential equations in [17]. and finally R. Saadeh *et al* introduced ARA Transform in [18].

Kharrat *et al*. also interested in integral transform methods, where they applied the Differential Transform to solve boundary value problems represented by differential equations from higher orders and also to solve a system of differential equations [19-21]. In addition, they suggested hybridization the homotopy perturbation method with Sumudu Transform to solve initial value problems for nonlinear partial differential equations [22]. They also introduced the hybridization of the Natural Transform method with the homotopy perturbation method to solve Van Der Pol Oscillator problem [23].

The purpose of this paper is to show the efficiency and applicability of this new integral transform and applied it to solve ordinary differential equations with variable and constant coefficients. The rest of the paper is as follows: We present the basic idea of Kharrat-Toma transform in section 2. In section 3, Kharrat-Toma transform of some functions is introduced and we proof some properties, in section 4, the application for solving ordinary differential is shown and conclusion in 5.

Kharrat-Toma Transform: Definition 1. The function $f(x)$ is said to have exponential order on every finite interval in $[0, +\infty)$ If there exist a positive number M that satisfying:
$\left| f(x) \right| \le M\, e^{\alpha x}, \quad M > 0, \quad \alpha > 0, \quad \forall x \ge 0$

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Definition 2: The Kharrat-Toma integral transform and inversion is defined by.

$$B\left[f(x)\right]=G\left(S\right)=s^3\int_0^\infty f(x)\,e^{\frac{-x}{s^2}}\,dx \quad , \quad x\geq 0$$

$$f(x)=B^{-1}\left[G\left(S\right)\right]=B^{-1}\left[s^3\int_0^\infty f(x)\,e^{\frac{-x}{s^2}}\,dx\right]$$

The B integral transform states that, if $f(x)$ is piecewise continuous on $[0,+\infty)$ and has exponential order. The B^{-1} will be the inverse of the B integral transform.

Theorem 1: [Sufficient Condition for Existence of a Kharrat-Toma Transform]: The Kharrat-Toma transform $B[f(x)]$ exists if it has exponential order and $\int_0^b |f(x)|\,dx$ exists for any $b > 0$.

Proof:

$$s^3\int_0^\infty \left| f(x)\,e^{\frac{-x}{s^2}}\right| dx = s^3\int_0^n \left| f(x)\,e^{\frac{-x}{s^2}}\right| dx + s^3\int_n^\infty \left| f(x)\,e^{\frac{-x}{s^2}}\right| dx$$

$$\leq s^3\int_0^n |f(x)|\,dx + s^3\int_0^\infty |f(x)|\,e^{\frac{-x}{s^2}}\,dx$$

$$\leq s^3\int_0^n |f(x)|\,dx + M\,s^3\int_0^\infty e^{\alpha x}\,e^{\frac{-x}{s^2}}\,dx$$

$$= s^3\int_0^n |f(x)|\,dx + M\,s^3\int_0^\infty e^{-\left(\frac{1}{s^2}-\alpha\right)x}\,dx$$

$$= s^3\int_0^n |f(x)|\,dx + \frac{M\,s^3}{-\left(\frac{1}{s^2}-\alpha\right)}\lim_{B\to\infty}\left[e^{-\left(\frac{1}{s^2}-\alpha\right)x}\right]_0^B \quad ; \quad \frac{1}{s^2}>\alpha$$

$$= s^3\int_0^n |f(x)|\,dx + \frac{M\,s^3}{\frac{1}{s^2}-\alpha}$$

The first integral exists, and the second term is finite for $\frac{1}{s^2}>\alpha$, so the integral $s^3\int_0^\infty f(x)\,e^{\frac{-x}{s^2}}\,dx$ converges absolutely and the Kharrat-Toma $B[f(x)]$ exists.

Kharrat-Toma Transform of Some Functions: In this section we find Kharrat-Toma transform of some functions;

$$f(x)=1 \underset{B^{-1}}{\overset{B}{\rightleftarrows}} G(s)=s^5 \tag{1}$$

$$f(x)=x^n \underset{B^{-1}}{\overset{B}{\rightleftarrows}} G(s)=s^{2n+5}.\,n! \tag{2}$$

$$f(x)=\sin(kx) \underset{B^{-1}}{\overset{B}{\rightleftarrows}} G(s)=\frac{k\,s^7}{1+k^2 s^4} \tag{3}$$

$$f(x)=\cos(kx) \underset{B^{-1}}{\overset{B}{\rightleftarrows}} G(s)=\frac{s^5}{1+k^2 s^4} \tag{4}$$

$$f(x)=sh(kx) \underset{B^{-1}}{\overset{B}{\rightleftarrows}} G(s)=\frac{k\,s^7}{1-k^2 s^4} \tag{5}$$

$$f(x)=\cosh(kx)\ \underset{B^{-1}}{\overset{B}{\rightleftarrows}}\ G(s)=\frac{s^5}{1-k^2s^4} \tag{6}$$

Proof:

$$B[1]=s^3\int_0^\infty e^{\frac{-x}{s^2}}dx=-s^5\lim_{B\to\infty}\left[e^{\frac{-x}{s^2}}\right]_0^B=s^5$$

$$where\quad s^2>0 \tag{1}$$

$$B\left[x^n\right]=s^3\int_0^\infty x^n e^{\frac{-x}{s^2}}dx$$

$$u=x^n\Rightarrow du=n\,x^{n-1}dx$$

$$dv=e^{\frac{-x}{s^2}}dx\Rightarrow v=-s^2e^{\frac{-x}{s^2}} \tag{2}$$

Then we get,

$$B\left[x^n\right]=s^3\left[-s^2x^ne^{\frac{-x}{s^2}}\Big|_0^\infty+n\,s^2\int_0^\infty x^{n-1}e^{\frac{-x}{s^2}}dx\right]$$

$$=n\,s^5\int_0^\infty x^{n-1}e^{\frac{-x}{s^2}}dx$$

$$u=x^{n-1}\Rightarrow du=(n-1)x^{n-2}dx$$

$$dv=e^{\frac{-x}{s^2}}dx\Rightarrow v=-s^2e^{\frac{-x}{s^2}}$$

Yields

$$B\left[x^n\right]=n\,s^5\left[-s^2x^{n-1}e^{\frac{-x}{s^2}}\Big|_0^\infty+(n-1)s^2\int_0^\infty x^{n-2}e^{\frac{-x}{s^2}}dx\right]$$

$$=n(n-1)s^7\int_0^\infty x^{n-2}e^{\frac{-x}{s^2}}dx=.....=s^{2n+5}.n!$$

$$where\quad s^2>0 \tag{3}$$

$$B[\sin(kx)]=s^3\int_0^\infty \sin(kx)e^{\frac{-x}{s^2}}dx$$

$$u=\sin(kx)\Rightarrow du=k\cos(kx)dx$$

$$dv=e^{\frac{-x}{s^2}}dx\Rightarrow v=-s^2e^{\frac{-x}{s^2}}$$

Then we get,

$$B[\sin(kx)]=s^3\left[-s^2\sin(kx)e^{\frac{-x}{s^2}}\Big|_0^\infty+k\,s^2\int_0^\infty\cos(kx)e^{\frac{-x}{s^2}}dx\right]$$

$$=k\,s^5\int_0^\infty\cos(kx)e^{\frac{-x}{s^2}}dx\quad ;\ s^2>0$$

$$u=\cos(kx)\Rightarrow du=-k\sin(kx)dx$$

$$dv=e^{\frac{-x}{s^2}}dx\Rightarrow v=-s^2e^{\frac{-x}{s^2}}$$

Yields

$$B[\sin(kx)]=k\,s^5\left[-s^2\cos(k\,x)e^{\frac{-x}{s^2}}\Big|_0^\infty-k\,s^2\int_0^\infty\sin(k\,x)e^{\frac{-x}{s^2}}dx\right]$$

$$=k\,s^5\left[s^2-\frac{k}{s}B[\sin(k\,x)]\right]$$

Then we get

$$B[\sin(kx)]=\frac{k\,s^7}{1+k^2s^4} \tag{4}$$

Proof in the same way as in (3)

$$B[\sinh(kx)]=s^3\int_0^\infty\sinh(kx)e^{\frac{-x}{s^2}}dx$$

$$u=\sinh(kx)\Rightarrow du=k\cosh(kx)dx \tag{5}$$

$$dv=e^{\frac{-x}{s^2}}dx\Rightarrow v=-s^2e^{\frac{-x}{s^2}}$$

Then we get

$$B[\sinh(kx)]=s^3\left[-s^2\sinh(kx)e^{\frac{-x}{s^2}}\Big|_0^\infty+k\,s^2\int_0^\infty\cosh(kx)e^{\frac{-x}{s^2}}dx\right]$$

$$=k\,s^5\int_0^\infty\cosh(kx)e^{\frac{-x}{s^2}}dx$$

$$u=\cosh(kx)\Rightarrow du=k\sinh(kx)dx$$

$$dv=e^{\frac{-x}{s^2}}dx\Rightarrow v=-s^2e^{\frac{-x}{s^2}}$$

Yields

$$B[\sinh(kx)]=k\,s^5\left[-s^2\cosh(kx)e^{\frac{-x}{s^2}}\Big|_0^\infty+k\,s^2\int_0^\infty\sinh(kx)e^{\frac{-x}{s^2}}dx\right]$$

$$=k\,s^5\left[s^2+\frac{k}{s}B[\sinh(kx)]\right]\quad ;\ s^2>0$$

Then we get

$$B\left[\sinh(kx)\right]=\frac{k\,s^7}{1-k^2 s^4}$$

(6) Proof in the same way as in (5)

Theorem 2:

Let $B\left[f_1(x)\right]=G_1(s)$,..., $B\left[f_n(x)\right]=G_n(s)$ and the constants c_1...,c_n, then

$$B\left[\sum_{i=1}^{n} c_i\, f_i(x)\right]=\sum_{i=1}^{n} c_i\, B\left[f_i(x)\right]$$

Proof:

$$B\left[\sum_{i=1}^{n} c_i\, f_i(x)\right]=s^3\int_0^\infty \sum_{i=1}^{n} c_i\, f_i(x)\, e^{\frac{-x}{s^2}}dx=\sum_{i=1}^{n} c_i\left(S^3\int_0^\infty f_i(x)\, e^{\frac{-x}{s^2}}dx\right)$$

$$=\sum_{i=1}^{n} c_i\, B\left[f_i(x)\right]$$

Theorem 3: [Translation Property]

Let $B[f(x)] = G(s)$ and the constants $\alpha > 0$.

then $B\left[f(\alpha x)\right]=\dfrac{1}{\alpha^2\sqrt{\alpha}}G\left[\sqrt{\alpha}\, s\right]$

Proof:

$$B\left[f(\alpha x)\right]=s^3\int_0^\infty f(\alpha x)\, e^{\frac{-x}{s^2}}dx\;=\frac{s^3}{\alpha}\int_0^\infty f(\alpha x)\, e^{\frac{-\alpha x}{\alpha s^2}}d(\alpha x)$$

$$=\frac{\left(\sqrt{\alpha}\, s\right)^3}{\alpha^2\sqrt{\alpha}}\int_0^\infty f(\alpha x)\, e^{\frac{-\alpha x}{\left(\sqrt{\alpha}\, s\right)^2}}d(\alpha x)$$

When $u = \alpha x$ then $x=\dfrac{u}{\alpha}$

where x:0 → ∞ ⇒ u: 0 → ∞ yields

$$B\left[f(\alpha x)\right]=\frac{1}{\alpha^2\sqrt{\alpha}}\left[\left(\sqrt{\alpha}\, s\right)^3\int_0^\infty f(u)\, e^{\frac{-u}{\left(\sqrt{\alpha}\, s\right)^2}}du\right]$$

$$=\frac{1}{\alpha^2\sqrt{\alpha}}G\left[\sqrt{\alpha}\, s\right]$$

Theorem 4: [First Shifting Property]

Let $B\,[f(x)] = G(s)$ and $\alpha \in \mathbb{R}$, then

$$B\left[e^{\alpha x}f(x)\right]=\left(1-\alpha s^2\right)\sqrt{1-\alpha s^2}\;\; G\left[\frac{s}{\sqrt{1-\alpha s^2}}\right]$$

Proof: From definition of a new integral transform we have:

$$B\left[e^{\alpha x}f(x)\right]=s^3\int_0^\infty e^{\alpha x}f(x)\, e^{\frac{-x}{s^2}}dx\;=s^3\int_0^\infty f(x)\, e^{-\left(\frac{1-\alpha s^2}{s^2}\right)x}dx$$

$$=\left(1-\alpha s^2\right)\sqrt{1-\alpha s^2}\left[\left(\frac{s}{\sqrt{1-\alpha s^2}}\right)^3\int_0^\infty f(x)\, e^{\frac{-x}{\left(\frac{s}{\sqrt{1-\alpha s^2}}\right)^2}}dx\right]$$

$$=\left(1-\alpha s^2\right)\sqrt{1-\alpha s^2}\, G\left[\frac{s}{\sqrt{1-\alpha s^2}}\right]$$

Theorem 5: [Convolution Theorem]

Let $B[f(x)] = M(s)$, $B[g(x)] = N(s)$, then

$$B\left[f(x)*g(x)\right]=\frac{1}{s^3}M(s).N(s)$$

Proof:

We know that

$$f(x)*g(x)=\int_0^x f(T)\,g(x-T)dT$$

then

$$B\left[f(x)*g(x)\right]=s^3\int_0^\infty\left[f(x)*g(x)\right]e^{\frac{-x}{s^2}}dx$$

$$=s^3\int_0^\infty e^{\frac{-x}{s^2}}dx\int_T^x f(T)\,g(x-T)dT;\quad 0\le T\le x<\infty$$

$$=s^3\int_0^\infty f(T)dT\int_T^\infty g(x-T)e^{\frac{-x}{s^2}}dx$$

cWhen $x - T = u$ then $x = T + u$

where x: $T \to \infty \Rightarrow u$: $0 \to \infty$ yields

$$B\left[f(x)*g(x)\right]=s^3\int_0^\infty f(T)dT\int_0^\infty g(u)e^{\frac{-(T+u)}{s^2}}du$$

$$=\frac{1}{s^3}\left(s^3\int_0^\infty f(T)e^{\frac{-T}{s^2}}dT\right).\left(s^3\int_0^\infty g(u)e^{\frac{-u}{s^2}}du\right)$$

$$=\frac{1}{s^3}M(s).N(s)$$

So we have proof the theorem.

Theorem 6: Kharrat-Toma Transform of $x^n f(x)$; $n \geq 1$

If $B[f(x)] = G(s)$, then

$$B\left[x f(x)\right]=\frac{s^3}{2}\frac{dG(s)}{ds}-\frac{3}{2}s^2G(s) \quad (1)$$

$$B\left[x^2 f(x)\right]=\frac{s^6}{4}\frac{d^2G(s)}{ds^2}-\frac{3}{4}s^5\frac{dG(s)}{ds}+\frac{3}{4}s^4G(s) \quad (2)$$

$$B\left[x^3 f(x)\right]=\frac{s^9}{8}\frac{d^3G(s)}{ds^3}-\frac{3}{8}s^7\frac{dG(s)}{ds}+\frac{3}{8}s^6G(s) \quad (3)$$

Proof:

$$\frac{dG(s)}{ds}=\frac{d}{ds}\left(\int_0^\infty s^3f(x)e^{\frac{-x}{s^2}}dx\right)=\int_0^\infty\frac{\partial}{\partial s}\left(s^3f(x)e^{\frac{-x}{s^2}}\right)dx$$

$$=\int_0^\infty 3s^2f(x)e^{\frac{-x}{s^2}}dx+\int_0^\infty 2xf(x)e^{\frac{-x}{s^2}}dx$$

$$=\frac{3}{s}\int_0^\infty s^3f(x)e^{\frac{-x}{s^2}}dx+\frac{2}{s^3}\int_0^\infty s^3\left(xf(x)\right)e^{\frac{-x}{s^2}}dx$$

$$=\frac{3}{s}G(s)+\frac{2}{s^3}B\left[xf(x)\right] \quad (1)$$

Then we have

$$B\left[x f(x)\right]=\frac{s^3}{2}\frac{dG(s)}{ds}-\frac{3}{2}s^2G(s)$$

$$\frac{d^2G(s)}{ds^2}=\frac{6}{s^2}\int_0^\infty s^3f(x)e^{\frac{-x}{s^2}}dx+\frac{6}{s^4}\int_0^\infty s^3\left(xf(x)\right)e^{\frac{-x}{s^2}}dx$$

$$+\frac{4}{s^6}\int_0^\infty s^3\left(x^2f(x)\right)e^{\frac{-x}{s^2}}dx$$

$$=\frac{6}{s^2}G(s)+\frac{6}{s^4}B\left[xf(x)\right]+\frac{4}{s^6}B\left[x^2f(x)\right]$$

$$=\frac{6}{s^2}G(s)+\frac{6}{s^4}\left(\frac{s^3}{2}\frac{dG(s)}{ds}-\frac{3}{2}s^2G(s)\right)+\frac{4}{s^6}B\left[x^2f(x)\right]$$

$$=\frac{6}{s^2}G(s)+\frac{3}{s}\frac{dG(s)}{ds}-\frac{9}{s^2}G(s)+\frac{4}{s^6}B\left[x^2f(x)\right] \quad (2)$$

Then we have,

$$B\left[x^2 f(x)\right]=\frac{s^6}{4}\frac{d^2G(s)}{ds^2}-\frac{3}{4}s^5\frac{dG(s)}{ds}+\frac{3}{4}s^4G(s) \quad (3)$$

$$\frac{d^3G(s)}{ds^3}=\frac{6}{s^3}\int_0^\infty s^3f(x)e^{\frac{-x}{s^2}}dx+\frac{6}{s^5}\int_0^\infty s^3\left(xf(x)\right)e^{\frac{-x}{s^2}}dx$$

$$+\frac{8}{s^9}\int_0^\infty s^3\left(x^3f(x)\right)e^{\frac{-x}{s^2}}dx$$

$$=\frac{6}{s^3}G(s)+\frac{6}{s^5}B\left[xf(x)\right]+\frac{8}{s^9}B\left[x^3f(x)\right]$$

$$=\frac{6}{s^3}G(s)+\frac{6}{s^5}\left(\frac{s^3}{2}\frac{dG(s)}{ds}-\frac{3}{2}s^2G(s)\right)+\frac{8}{s^9}B\left[x^3f(x)\right]$$

$$=\frac{6}{s^3}G(s)+\frac{3}{s^2}\frac{dG(s)}{ds}-\frac{9}{s^3}G(s)+\frac{8}{s^9}B\left[x^3f(x)\right]$$

Then we have,

$$B\left[x^3 f(x)\right]=\frac{s^9}{8}\frac{d^3G(s)}{ds^3}-\frac{3}{8}s^7\frac{dG(s)}{ds}+\frac{3}{8}s^6G(s)$$

Theorem 7: Kharrat-Toma Transform of Derivatives

Let $B[f(x)] = G(s)$, then

$$B\left[f'(x)\right]=\frac{1}{s^2}G(s)-s^3f(0) \quad (1)$$

$$B\left[f''(x)\right]=\frac{1}{s^4}G(s)-sf(0)-s^3f'(0) \quad (2)$$

$$B\left[f^{(n)}(x)\right]=\frac{1}{s^{2n}}G(s)-\sum_{k=0}^{n-1}s^{-2n+2k+5}f^{(k)}(0);\ n\geq 1 \quad (3)$$

$$B\left[x f'(x)\right] = \frac{s}{2}\frac{dG(s)}{ds} - \frac{5}{2}G(s) \quad (4)$$

$$B\left[x f''(x)\right] = \frac{1}{2s}\frac{dG(s)}{ds} - \frac{7}{2s^2}G(s) + s^3 f(0) \quad (5)$$

$$B\left[x^2 f'(x)\right] = \frac{s^4}{4}\frac{d^2G(s)}{ds^2} - \frac{7}{4}s^3\frac{dG(s)}{ds} + \frac{15}{4}s^2 G(s) \quad (6)$$

$$B\left[x^2 f''(x)\right] = \frac{s^2}{4}\frac{d^2G(s)}{ds^2} - \frac{11}{4}s\frac{dG(s)}{ds} + \frac{35}{4}G(s) \quad (7)$$

Proof:

$$B\left[f'(x)\right] = s^3\int_0^\infty f'(x)\, e^{\frac{-x}{s^2}}\, dx \quad (1)$$

Consider

$$u = e^{\frac{-x}{s^2}} \Rightarrow du = -\frac{1}{s^2}e^{\frac{-x}{s^2}}dx$$

$$dv = f'(x)dx \Rightarrow v = f(x)$$

Then we get,

$$B\left[f'(x)\right] = s^3\left[e^{\frac{-x}{s^2}} f(x)\Big|_0^\infty + \frac{1}{s^2}\int_0^\infty f(x)\, e^{\frac{-x}{s^2}}dx\right] \quad (2)$$

$$= s^3\left[-f(0) + \frac{1}{s^5}G(s)\right] = \frac{1}{s^2}G(s) - s^3 f(0)$$

$$B\left[f''(x)\right] = \frac{1}{s^2}B\left[f'(x)\right] - s^3 f'(0)$$

$$= \frac{1}{s^2}\left(\frac{1}{s^2}G(s) - s^3 f(0)\right) - s^3 f'(0)$$

$$= \frac{1}{s^4}G(s) - s\, f(0) - s^3 f'(0) \quad (3)$$

For $n = 1$ we have

$$L = B\left[f'(x)\right] = \frac{1}{s^2}G(s) - \sum_{k=0}^{0} s^{2k+3} f^{(k)}(0) = \frac{1}{s^2}G(s) - s^3 f(0) = R$$

Assuming the formula is true for n

Let us prove for $n + 1$

$$B\left[f^{(n+1)}(x)\right] = \frac{1}{s^{2n+2}}G(s) - \sum_{k=0}^{n} s^{-2n+2k+3} f^{(k)}(0)$$

$$L = B\left[f^{(n+1)}(x)\right] = B\left[\left(f^{(n)}(x)\right)'\right]$$

$$= \frac{1}{s^{2n}}B\left[f'(x)\right] - \sum_{k=0}^{n-1} s^{-2n+2k+5} f^{(k+1)}(0)$$

$$= \frac{1}{s^{2n}}\left(\frac{1}{s^2}G(s) - s^3 f(0)\right) - \sum_{k=0}^{n-1} s^{-2n+2k+5} f^{(k+1)}(0) \ ; \ m = k+1$$

$$= \frac{1}{s^{2n+2}}G(s) - s^{3-2n} f(0) - \sum_{m=1}^{n} s^{-2n+2(m-1)+5} f^{(m)}(0)$$

$$= \frac{1}{s^{2n+2}}G(s) - s^{3-2n} f(0) - \sum_{m=1}^{n} s^{-2n+2m+3} f^{(m)}(0)$$

$$= \frac{1}{s^{2n+2}}G(s) - \sum_{m=0}^{n} s^{-2n+2m+3} f^{(m)}(0)$$

$$= \frac{1}{s^{2n+2}}G(s) - \sum_{k=0}^{n} s^{-2n+2k+3} f^{(k)}(0) = R \quad (4)$$

$$B\left[x f'(x)\right] = \frac{s^3}{2}\frac{d}{ds}\left[\frac{1}{s^2}G(s) - s^3 f(0)\right] - \frac{3}{2}s^2\left[\frac{1}{s^2}G(s) - s^3 f(0)\right]$$

$$= -G(s) - \frac{s}{2}\frac{dG(s)}{ds} - \frac{3}{2}s^5 f(0) - \frac{3}{2}G(s) + \frac{3}{2}s^5 f(0)$$

$$= \frac{s}{2}\frac{dG(s)}{ds} - \frac{5}{2}G(s) \quad (5)$$

$$B\left[x f''(x)\right] = \frac{s^3}{2}\frac{d}{ds}\left[\frac{1}{s^4}G(s) - s\, f(0) - s^3 f'(0)\right]$$

$$-\frac{3}{2}s^2\left[\frac{1}{s^4}G(s) - s\, f(0) - s^3 f'(0)\right]$$

$$= -\frac{2}{s^2}G(s) + \frac{1}{2s}\frac{dG(s)}{ds} - \frac{1}{2}s^3 f(0) - \frac{3}{2}s^5 f'(0) - \frac{3}{2s^2}G(s)$$

$$+\frac{3}{2}s^3 f(0) + \frac{3}{2}s^5 f'(0)$$

$$= \frac{1}{2s}\frac{dG(s)}{ds} - \frac{7}{2s^2}G(s) + s^3 f(0) \quad (6)$$

$$B\left[x^2 f'(x)\right] = \frac{s^6}{4}\frac{d^2}{ds^2}\left[\frac{1}{s^2}G(s) - s^3 f(0)\right] - \frac{3}{4}s^5\frac{d}{ds}\left[\frac{1}{s^2}G(s) - s^3 f(0)\right]$$

$$+\frac{3}{4}s^4\left[\frac{1}{s^2}G(s) - s^3 f(0)\right]$$

$$= \frac{3}{2}s^2 G(s) - \frac{s^3}{2}\frac{dG(s)}{ds} - \frac{s^3}{2}\frac{dG(s)}{ds} + \frac{s^4}{4}\frac{d^2G(s)}{ds^2} - \frac{3}{2}s^7 f(0)$$

$$+\frac{3}{4}s^2 G(s) - \frac{3}{4}s^7 f(0) + \frac{3}{2}s^2 G(s) - \frac{3s^3}{4}\frac{dG(s)}{ds} + \frac{9}{4}s^7 f(0)$$

$$= \frac{s^4}{4}\frac{d^2G(s)}{ds^2} - \frac{7}{4}s^3\frac{dG(s)}{ds} + \frac{15}{4}s^2 G(s) \quad (7)$$

$$B\left[x^2 f''(x)\right] = \frac{s^6}{4}\frac{d^2}{ds^2}\left[\frac{1}{s^4}G(s) - s f(0) - s^3 f'(0)\right]$$
$$-\frac{3}{4}s^5\frac{d}{ds}\left[\frac{1}{s^4}G(s) - s f(0) - s^3 f'(0)\right]$$
$$+\frac{3}{4}s^4\left[\frac{1}{s^4}G(s) - s f(0) - s^3 f'(0)\right]$$
$$= \frac{s^2}{4}\frac{d^2 G(s)}{ds^2} - \frac{11}{4}s\frac{dG(s)}{ds} + \frac{35}{4}G(s)$$

Applications: In this section, we introduce the methodology of application of Kharrat-Toma transform for solving initial value problem. This new integral transform can be used as an effective tool for solving ordinary differential equations with initial conditions. The following examples show the use and efficiency of this integral transform.

Example 1:

Consider the initial value problem

$$\begin{cases} u'(x) + u(x) = 3 \\ u(0) = 1 \end{cases} \tag{1}$$

Applying the kharrat-Toma transform on (1), we get

$$\frac{1}{s^2}G(s) - s^3 u(0) + G(s) = 3s^5$$
$$\Rightarrow G(s)\left[\frac{1}{s^2} + 1\right] = 3s^5 + s^3$$
$$\Rightarrow G(s) = \frac{s^5}{1+s^2} + \frac{3s^7}{1+s^2} = \frac{s^5}{1+s^2} + 3s^5 - \frac{3s^5}{1+s^2} \tag{2}$$
$$\Rightarrow G(s) = \frac{-2s^5}{1+s^2} + 3s^5$$

Appling the inverse Kharrat-Toma transform on (2), then the exact solution for IVP (1) is;

$$u(x) = -2e^{-x} + 3$$

Example 2:

Consider the initial value problem

$$\begin{cases} u''(x) - 2u'(x) - 3u(x) = 0 \\ u(0) = 1 \quad , \quad u'(0) = 2 \end{cases} \tag{3}$$

Applying the kharrat-Toma transform on (3), we get

$$\frac{1}{s^4}G(s) - s u(0) - s^3 u'(0) - 2\left[\frac{1}{s^2}G(s) - s^3 u(0)\right] - 3G(s) = 0$$
$$\Rightarrow G(s)\left[\frac{1}{s^4} - \frac{2}{s^2} - 3\right] = s$$
$$\Rightarrow G(s) = \frac{s^5}{-3s^4 - 2s^2 + 1} = \frac{1}{4}\frac{s^5}{1+s^2} + \frac{3}{4}\frac{s^5}{1-3s^2}$$
$$= B\left[\frac{1}{4}e^{-x}\right] + B\left[\frac{3}{4}e^{3x}\right] \tag{4}$$

Appling the inverse Kharrat-Toma transform on (4), then the exact solution for IVP (3) is;

$$u(x) = \frac{1}{4}e^{-x} + \frac{3}{4}e^{3x}$$

Example 3:

Consider the initial value problem

$$\begin{cases} u''(x) + 2u'(x) + u(x) = 3xe^{-x} \\ u(0) = 4 \quad , \quad u'(0) = 2 \end{cases} \tag{5}$$

Applying the kharrat-Toma transform on (5), we get,

$$\frac{1}{s^4}G(s) - s u(0) - s^3 u'(0) + 2\left[\frac{1}{s^2}G(s) - s^3 u(0)\right] + G(s)$$
$$= \frac{3}{2}\frac{3s^9 + 5s^7}{\left(1+s^2\right)^2} - \frac{9}{2}\frac{s^7}{1+s^2}$$
$$\Rightarrow G(s)\left[\frac{1}{s^4} + \frac{2}{s^2} + 1\right] = 10s^3 + 4s + \frac{3}{2}\frac{3s^9 + 5s^7}{\left(1+s^2\right)^2} - \frac{9}{2}\frac{s^7}{1+s^2}$$
$$\Rightarrow G(s) = \frac{10s^7 + 4s^5}{\left(1+s^2\right)^2} + \frac{3}{2}\frac{3s^{13} + 5s^{11}}{\left(1+s^2\right)^4} - \frac{9}{2}\frac{s^{11}}{\left(1+s^2\right)^3}$$
$$= \frac{4s^5}{1+s^2} + \frac{6s^7}{\left(1+s^2\right)^2} + \frac{1}{2}\frac{6s^{11}}{\left(1+s^2\right)^4}$$
$$= B\left[4e^{-x}\right] + B\left[6xe^{-x}\right] + B\left[\frac{1}{2}x^3 e^{-x}\right] \tag{6}$$

Appling the inverse Kharrat-Toma transform on (6), then the exact solution for IVP (5) is;

$$u(x) = 4e^{-x} + 6xe^{-x} + \frac{1}{2}x^3 e^{-x}$$

CONCLUSION

The main aim of this work is to present some fundamental properties of newly defined integral transform “Kharrat-Toma Transform”. Convolution theorem is also proved for Kharrat-Toma transform. It provides a new mathematical tool to solve ordinary differential equations of variable and constant coefficients with initial condition.

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