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# Numerical Solution of Van Der Pol Oscillator Problem Using a New Hybrid Method

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**Abstract:** This paper introduces an efficient proposed technique combining the homotopy perturbation method and natural transform to obtain a numerical solution of an important initial value problem arising in applied dynamics, called Van Der Pol Oscillator problem. The objective of this work is to investigate the efficiency of this proposed hybrid method. The results showed that the new method is simple, effective and accurate.

**Key words:** Homotopy Perturbation Method · Natural Transform · Van Der Pol Oscillator · Initial Value Problem · Approximate Solution

## INTRODUCTION

There are many nonlinear differential equations with strong nonlinearity which is very difficult to solve either analytically or numerically in science and engineering. Van der Pol equation represents a classical nonlinear problem. This kind of nonlinear oscillator represents a good model of the first vacuum tube circuits, i.e., part of the early radios was first extensively studied by the Dutch electrical engineer Van der Pol in the early 1920s and other many mathematicians and scientists. The Van der Pol equation has been studied extensively by many authors using various methods. M. Kumar and P. Varshney [1] presented numerical simulation of Van der Pol equation using multiple scales modified lindstedt-poincare method. J. H. M. Darbyshire [2] used collocation method for numerical solution of Van der pol equation, A. S. Soomro *et al*. [3] compared between improved Heun's (IH) method against the classical Runge-Kutta (RK4) and Mid-point (MP) methods for Van der Pol equation. A. Rasedee *et. al* [4] used the backward difference formulation to solve Duffing- Van der Pol equation. A. R. Vahidi *et al*. [5] using restarted Adomain decomposition method. H. Vazquez-Leal *et al*. [6] applied rational biparameter homotopy perturbation method and Laplace-Pade` coupled version as a novel tool with the potential to find approximate solutions for nonlinear differential equations. Kharrat *et al*. [7-12] interested in the improvement and hybridization of the homotopy perturbation method to solve many boundary-initial value problems represented by nonlinear, ordinary, partial, integral differential equations provided by boundary-initial conditions.

The homotopy perturbation method (HPM) was proposed first by J. Juan. He [13-15] to solve linear and nonlinear boundary value problems and initial value problems. The HPM method is based on the use of a power series, which transforms the original nonlinear differential equation into a series of linear differential equations. In Addition, the natural transform was proposed by Khan *et al*. in 2008 [16]. The natural transform operator was denoted by $N[.]$ and it has been defined by the integral equation as follows:

$$N[v(t)] = R(s,u) = \int_0^\infty e^{-st} v(ut)\, dt, \quad \mathrm{Re}(s) > 0, \quad v \in (-\tau_1, \tau_2)$$

Provided the function $v(t) \in R^2$ is defined in the set

$$A = \{v(t) | \exists M, \tau_1, \tau_2 > 0, |v(t)| < M e^{\frac{|t|}{\tau_j}}, if\ t \in (-1)^j \times [0,\infty)\}$$

The integral transformations played an essential role in many fields of science especially, engineering mathematics, mathematical physics [17-20].

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In this work, we propose a new hybrid method by using HPM and natural transform (NT) to find an approximate solution of Van Der Pol Oscillator problem represented by nonlinear second order differential equation and initial conditions.

This paper is organised as follows. In section 2 illustrate the methodology of proposed hybrid method (NTHPM). Numerical solution by (NTHPM) for Van Der Pol Oscillator equation are presented in section 3. Finally, in section 4 the conclusion is shown.

**Proposed Hybrid Method (NTHPM):** Consider the following nonlinear equation

$$v^{(n)}(x) = \sum_{m=0}^{n-1} \varphi_m(x) v^{(m)}(x) + N\big(v(x)\big) + f(x) \tag{1}$$

With the initial conditions

$$v^{(k)}(0) = \alpha_k = const\ ,\ \ k = \overline{0, n-1}$$

where $N(v(x))$ is the nonlinear term and $f(x)$ is an analytical function and $x \in [0, a]$.

Taking the natural transform on equation (1), yields

$$\frac{s^n}{u^n} N[v] - \sum_{k=0}^{n-1} \frac{s^{n-k-1}}{u^{n-k}} v^{(k)}(0) = N\left[\sum_{m=0}^{n-1} \varphi_m(x) v^{(m)}(x) + N\big(v(x)\big) + f(x)\right]$$

Then

$$N[v] = \frac{u^n}{s^n} \sum_{k=0}^{n-1} \frac{s^{n-k-1}}{u^{n-k}} \alpha_k + \frac{u^n}{s^n} N\left[\sum_{m=0}^{n-1} \varphi_m(x) v^{(m)}(x) + N\big(v(x)\big) + f(x)\right] \tag{2}$$

The homotopy of equation (2) can be written as follows

$$N[v] = \frac{u^n}{s^n} \sum_{k=0}^{n-1} \frac{s^{n-k-1}}{u^{n-k}} \alpha_k + p\frac{u^n}{s^n} N\left[\sum_{m=0}^{n-1} \varphi_m(x) v^{(m)}(x) + N\big(v(x)\big) + f(x)\right] \tag{3}$$

where $p \in [0,1]$ is an embedding parameter.

According to the HPM the solution of equation (3) can be written as a power series in $p$

$$v = \sum_{i=0}^{\infty} p^i v_i \tag{4}$$

Substituting equation (4) into equation (3), yields

$$N\left[\sum_{i=0}^{\infty} p^i v_i\right] = \frac{u^n}{s^n} \sum_{k=0}^{n-1} \frac{s^{n-k-1}}{u^{n-k}} \alpha_k + p\frac{u^n}{s^n} N\left[\begin{array}{l} \displaystyle\sum_{m=0}^{n-1}\sum_{i=0}^{\infty} p^i \varphi_m(x) v_i^{(m)}(x) \\ \displaystyle + N\left(\sum_{i=0}^{\infty} p^i v_i\right) + f(x) \end{array}\right] \tag{5}$$

Comparing coefficients of the terms with identical powers of $p$ in equation (5), leads to

$$p^0 : N[v_0] = \frac{u^n}{s^n} \sum_{k=0}^{n-1} \frac{s^{n-k-1}}{u^{n-k}} \alpha_k \tag{6}$$

$$p^{i+1} : N[v_{i+1}] = \frac{u^n}{s^n} N\left[\begin{array}{l} \displaystyle\sum_{m=0}^{n-1} \varphi_m(x) v_i^{(m)}(x) \\ + N(v_i) + f(x) \end{array}\right] \tag{7}$$

where $i = 0,1,2, \ldots$

Taking the inverse natural transform on equations (6) and (7), gives

$v_i;\ i = 0,1,2, \ldots$

Setting $p = 1$, we have the approximate solution of equations (1)

$$v(x) = \sum_{i=0}^{\infty} v_i(x) \tag{8}$$

The last series is convergent in most cases and the convergence rate of the series depends on the nonlinear operator.

**Van Der Pol Oscillator:** Consider the Van Der Pol Oscillator problem [6]:

$$v'' = -v' - v - v^2 v', \qquad v = v(x) \tag{9}$$

where $x \in [0,0.5]$
With the initial conditions

$$v(0)=0\ ,\ \ v'(0)=1$$
$$f(x)=0, \qquad N(v)=-v^2 v'$$
$$\varphi_0(x)=-1, \qquad \varphi_1(x)=-1$$

Taking the natural transform on equation (9), yields

$$\frac{s^2}{u^2}N(v)-\frac{s}{u^2}v(0)-\frac{1}{u}v'(0)$$
$$=N\left[\sum_{m=0}^{1}\varphi_m(x)v^{(m)}(x)-v^2 v'\right]$$

Then

$$\frac{s^2}{u^2}N(v)=\frac{1}{u}+N\left[-v-v'-v^2 v'\right]$$

Then we have

$$N(v)=\frac{u}{s^2}+\frac{u^2}{s^2}N\left[-v-v'-v^2 v'\right] \tag{10}$$

The homotopy of equation (10) can be written as follows

$$N(v)=\frac{u}{s^2}+p\frac{u^2}{s^2}N\left[-v-v'-v^2 v'\right] \tag{11}$$

Substituting equation (4) into equation (11), yields

$$N\left(\sum_{i=0}^{\infty}p^i v_i\right)=\frac{u}{s^2}+p\frac{u^2}{s^2}N\left[\begin{array}{l}-\sum_{i=0}^{\infty}p^i v_i-\sum_{i=0}^{\infty}p^i v_i' \\ -\left(\sum_{i=0}^{\infty}p^i v_i\right)^2\left(\sum_{i=0}^{\infty}p^i v_i'\right)\end{array}\right] \tag{12}$$

Comparing coefficients of the terms with identical powers of $p$ in equation (12) and taking the inverse natural transform on the result equations, yields

$$v_0=x$$
$$v_1=-\frac{x^2}{2}-\frac{x^3}{6}-\frac{x^4}{12}$$

Table 1: Comparison of absolute errors in the numerical solution by the proposed hybrid method (NTHPM)

| X | Error of NTHPM n=4 | Error of NTHPM n=5 | Error of NTHPM n=6 |
|---|---|---|---|
| 0 | 0.00000 | 0.00000 | 0.00000 |
| 0.0001 | 1.6669 e -13 | 4.1671 e -18 | 8.3342 e -23 |
| 0.0003 | 4.5136 e -12 | 3.3760 e -16 | 2.0256 e -20 |
| 0.0005 | 2.0844 e -11 | 2.6055 e -15 | 2.6055 e -19 |
| 0.0007 | 5.7207 e -11 | 1.0011 e -14 | 1.4016 e -18 |
| 0.0009 | 1.2161 e -10 | 2.7363 e -14 | 4.9254 e -18 |
| 0.001 | 1.6684 e -10 | 4.1710 e -14 | 8.3422 e -18 |
| 0.003 | 4.5140 e -09 | 3.3860 e -12 | 2.0323 e -15 |
| 0.005 | 2.0944 e -08 | 2.6190 e -11 | 2.6215 e -14 |
| 0.007 | 5.7602 e -08 | 1.0088 e -10 | 1.4150 e -13 |
| 0.009 | 1.2272 e -07 | 2.7647 e -10 | 4.9918 e -13 |
| 0.01 | 1.6854 e -07 | 4.2203 e -10 | 8.4726 e -13 |
| 0.03 | 4.6564 e -06 | 3.5651 e -08 | 2.2109 e -10 |
| 0.05 | 2.2544 e -05 | 2.9282 e -07 | 3.1924 e -09 |
| 0.07 | 6.4812 e -05 | 1.2192 e -06 | 1.9945 e -08 |
| 0.09 | 1.4540 e -04 | 3.6657 e -06 | 8.3277 e -08 |
| 0.1 | 2.0543 e -04 | 5.8880 e -06 | 1.5459 e -07 |
| 0.2 | 2.3611 e -03 | 1.7433 e -04 | 1.2833 e -05 |
| 0.3 | 1.2138 e -02 | 1.6803 e -03 | 2.3107 e -04 |
| 0.4 | 4.3974 e -02 | 9.6843 e -03 | 2.0796 e -03 |
| 0.5 | 1.2884 e -01 | 7.0980 e -02 | 1.2531 e -02 |

$$v_2=\frac{x^3}{6}+\frac{x^4}{12}+\frac{x^5}{8}+\frac{11x^6}{360}+\frac{x^7}{84}$$

$$v_3=-\frac{x^4}{24}-\frac{x^5}{40}-\frac{67x^6}{720}-\frac{31x^7}{720}-\frac{53x^8}{1680}-\frac{67x^9}{10080}-\frac{19x^{10}}{10080}$$

$$v_4=\frac{x^5}{120}+\frac{x^6}{180}+\frac{25x^7}{504}+\frac{11x^8}{336}+\frac{13921x^9}{362880}+\frac{14039x^{10}}{907200}+\frac{8347x^{11}}{1108800}+\frac{1889x^{12}}{1330560}+\frac{737x^{13}}{2358720}$$

$$v_5=-\frac{x^6}{720}-\frac{x^7}{1008}-\frac{107x^8}{5040}-\frac{3221x^9}{181440}-\frac{113293x^{10}}{3628800}-\frac{246359x^{11}}{13305600}-\frac{3169829x^{12}}{239500800}-\frac{243187x^{13}}{518918400}-\frac{629383x^{14}}{363242880}-\frac{323179x^{15}}{1089728640}-\frac{14051x^{16}}{26176640}$$

It leads to the solution of equation (9)

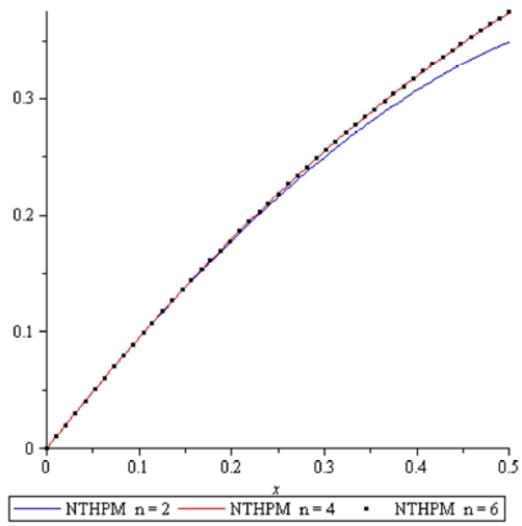


Fig. 1: The approximate solution using NTHPM

$$v(x) = \sum_{i=0}^{\infty} v_i(x)$$

In the following Table 1, we present the comparison of absolute error for the approximate solution by the hybrid method (NTHPM) of Eqn. (9) by taking four terms $v_0 + v_1 + v_2 + v_3$ and five terms $v_0 + v_1 + v_2 + v_3 + v_4$ and six terms $v_0 + v_1 + v_2 + v_3 + v_4 + v_5$ of the solution series. Where (n) is the number of terms of the equation (8)

The comparison between the approximate solutions is shown in Figure 1

## CONCLUSION

In this work, we have proposed a new hybrid method combining the homotopy perturbation method with the natural transform for solving Van Der Pol Oscillator equation represented by nonlinear second order differential equation and initial conditions. The obtained results show that the NTHPM is a powerful and good technique for obtaining numerical solution of nonlinear initial value problem. The computations presented in this work are performed by using the Maple software.

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