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Differential Transform Method For Solving Initial and Boundary Value Problems Represented By Linear and Nonlinear Ordinary Differential Equations of 14th Order

Bachir Nour Kharrat and George Toma

Department of Mathematics, Faculty of Science, Aleppo University, Aleppo-Syria

Abstract: In this paper, differential transform method (DTM) is applied to linear and nonlinear initial and boundary value problems represented by ordinary differential equations of 14th order. So as to show the capability and robustness, some examples are solved as numerical examples that we are compared our results with the Homotopy perturbation method (HPM) and the exact solutions. We show that the (DTM) is very simple and very effective.

Key words: Differential Transform Method · Initial Value Problem · Boundary Value Problem · Ordinary Differential Equation

INTRODUCTION

The Differential Transform Method was proposed first by Zhou [1-7] for solving linear and nonlinear Boundary Value Problems and Initial Value Problems. In this paper, the Differential Transform Method has been utilized for solving initial and boundary value problems represented by ordinary differential equations of 14th order from the form: $u^{(14)}(x) = f\left(x, y, y', y'', \ldots, y^{(13)}\right)$ where *f* is a continuous function and nonlinear in a general case.

The Comparisons of the results of (DTM) on three applications with Homotopy perturbation Method (HPM) [8] and the exact solutions reveal that (DTM) is very effective and convenient.

Differential Transform Method (DTM): The differential transform of the ***kth*** derivative of function $u(x)$ is defined as follows [2-7].

$$U(k) = \frac{1}{k!}\left[\frac{d^k u}{dx^k}(x)\right]_{x=x_0} \tag{1}$$

where $u(x)$ is the original function and $U(k)$ is the transformed function.

And the differential inverse transform of $U(k)$ is defined as

$$u(x) = \sum_{k=0}^{\infty} U(k)(x - x_0)^k \tag{2}$$

The differential transform verified The following properties [2-7].

1) If $u(x) = u_1(x) \pm u_2(x)$, then $U(k) = U_1(k) \pm U_2(k)$
2) If $u(x) = cu_1(x)$, then $U(k) = cU_1(x)$, where c is a constant.
3) If $u(x) = \frac{d^n u_1(x)}{dx^n}$, then $U(k) = \frac{(k+n)!}{k!} U_1(k+n)$
4) If $u(x) = u_1(x)\, u_2(x)$, then $U(k) = \sum_{r=0}^{k} U_1(r) U_2(k-r)$
5) If $u(x) = u_1(x) u_2(x) \ldots u_n(x)$, then

$$U(k) = \sum_{r_{n-1}=0}^{k} \sum_{r_{n-2}=0}^{r_{n-1}} \cdots \sum_{r_2=0}^{r_3} \sum_{r_1=0}^{r_2} U_1(r_1) U_2(r_2 - r_1) \ldots U_{n-1}(r_{n-1} - r_{n-2}) U_n(k - r_{n-1})$$

6) If $u(x) = x^m$, then $U(k) = \delta(k-m) = \begin{cases} 1 & , k = m \\ 0 & , k \neq m \end{cases}$
7) If $u(x) = e^{\alpha x}$, then $U(k) = \frac{\alpha^k}{k!}$, where α is a constant

Numerical Examples: In this section, some examples show the usage of DTM for solving the linear and nonlinear ordinary differential equations of 14th order.

Corresponding Author: Bachir Nour Kharrat, Department of Mathematics, Faculty of Science, Aleppo University, Aleppo-Syria.

Example 1: First let us consider the following linear ordinary differential equation of 14th order

$$u^{(14)}(x) = -u(x) + 2e^x \tag{3}$$

With initial conditions

$$u^{(m)}(0) = 1,\ m = 0,1,2,...,13$$

The exact solution of this problem is

$$u(x) = e^x$$

Applying the DTM By using properties 1, 2, 3 and 7 choosing $x_0 = 0$, equation (3) is transformed in the following form:

$$\frac{(k+14)!}{k!}U(k+14) = -U(k) + \frac{2}{k!}$$

$$u(0) = 1 \rightarrow U(0) = 1$$
$$u'(0) = 1 \rightarrow U(1) = 1$$
$$u^{(n)}(0) = 1 \rightarrow U(n) = \frac{1}{n!} \quad , \quad n = 2,3,...,13$$

$$k = 0 \Rightarrow 14!\,U(14) = -U(0) + 2 \Rightarrow U(14) = \frac{1}{14!}$$

It leads to the solution of equation (3)

$$u(x) = \sum_{k=0}^{\infty} U(k)\,x^k = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^{14}}{14!} + ... = e^x$$

Example. 2: Consider the linear ordinary differential equation

$$u^{(14)}(x) = u^{(4)}(x) + u''(x) + 193\sin x + 32x\cos x - x^2\sin x \quad 0 \le x \le 1 \tag{4}$$

With boundary conditions

$$u(0) = 0,\quad u'(0) = -1,\quad u''(0) = 0,\quad u^{(3)}(0) = 7$$
$$u^{(4)}(0) = 0,\quad u^{(5)}(0) = -21,\quad u^{(6)}(0) = 0,\quad u^{(7)}(0) = 43$$
$$u^{(8)}(0) = 0,\quad u^{(9)}(0) = -73,\quad u^{(10)}(0) = 0$$
$$u(1) = 0,\quad u'(1) = 2\sin(1),\quad u''(1) = 2\sin(1) + 4\cos(1)$$

Table 1: Results numerical for example 2

x	Error of DTM	Error of HPM (N=13)
0	0.0000000000	3.00101297 e -08
0.1	3.4057540 e -20	8.10009510 e -08
0.2	5.5805353 e -17	1.75358592 e -07
0.3	3.7548517 e -15	4.16492591 e -07
0.4	6.6742854 e -14	8.82872394 e -07
0.5	5.5470486 e -13	1.86871407 e -06
0.6	2.7488389 e -12	3.98441791 e -06
0.7	3.0837880 e -11	8.38888894 e -06
0.8	1.9564711 e -11	1.71998727 e -05
0.9	2.3892980 e -11	3.38833966 e -05
1	1.1142988 e -15	6.39618159 e -05

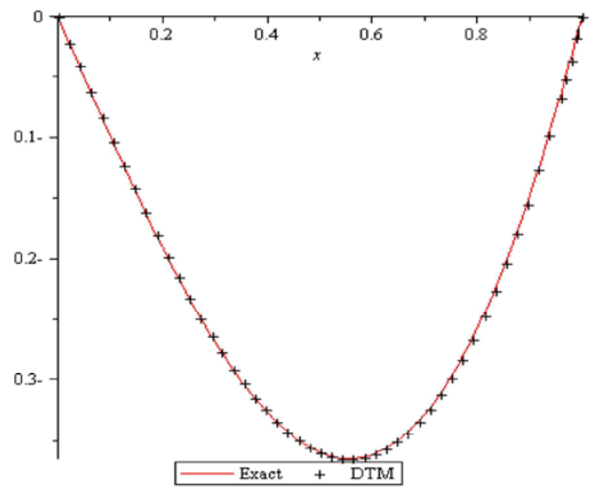

Fig. 1: The approximate solution by using DTM and the exact solution for example 2

The exact solution of this problem is:

$$u(x) = (x^2 - 1)\sin x$$

Developing $(193\sin x + 32x\cos x - x^2\sin x)$ by Maclaurin series, then Equation (4) can be written as follows

$$u^{(14)}(x) = u^{(4)}(x) + u''(x) + 225x - \frac{295}{6}x^3 + \frac{373}{120}x^5 - \frac{51}{560}x^7 \tag{5}$$

Applying the DTM by using properties 1, 2, 3 and 6 choosing $x_0 = 0$, equation (5) is transformed as follows:

$$\frac{(k+14)!}{k!}U(k+14) = \frac{(k+4)!}{k!}U(k+4) + \frac{(k+2)!}{k!}U(k+2) + 225\delta(k-1) - \frac{295}{6}\delta(k-3) + \frac{373}{120}\delta(k-5) - \frac{51}{560}\delta(k-7)$$

$$u(0)=0 \rightarrow U(0)=0$$
$$u'(0)=-1 \rightarrow U(1)=-1$$
$$u''(0)=0 \rightarrow U(2)=0$$
$$u^{(3)}(0)=7 \rightarrow U(3)=\frac{7}{3!}$$
$$u^{(4)}(0)=0 \rightarrow U(4)=0$$
$$u^{(5)}(0)=-21 \rightarrow U(5)=\frac{-21}{5!}$$
$$u^{(6)}(0)=0 \rightarrow U(6)=0$$
$$u^{(7)}(0)=43 \rightarrow U(7)=\frac{43}{7!}$$
$$u^{(8)}(0)=0 \rightarrow U(8)=0$$
$$u^{(9)}(0)=-73 \rightarrow U(9)=\frac{-73}{9!}$$
$$u^{(10)}(0)=0 \rightarrow U(10)=0$$
$$u^{(11)}(0)=\alpha \rightarrow U(11)=\frac{\alpha}{11!}$$
$$u^{(12)}(0)=\beta \rightarrow U(12)=\frac{\beta}{12!}$$
$$u^{(13)}(0)=\gamma \rightarrow U(13)=\frac{\gamma}{13!}$$

Then we obtain the solution of equation (4) as follow

$$u(x)=\sum_{k=0}^{\infty} U(k)\, x^k = -x+\frac{7}{3!}x^3-\frac{21}{5!}x^5+\frac{43}{7!}x^7-\frac{73}{9!}x^9 +\frac{\alpha}{11!}x^{11}+\frac{\beta}{12!}x^{12}+\frac{\gamma}{13!}x^{13}+\frac{211}{15!}x^{15}$$

By using

$$u(1)=0,\quad u'(1)=2\sin(1),\quad u''(1)=2\sin(1)+4\cos(1)$$

We get

$$\alpha = 110.8338553\,,\ \beta = 3.804900618\,,\ \gamma = -180.5498898$$

In the following table we give the approximate solution of equation (4)

Example 3: Consider the nonlinear ordinary differential equation

$$u^{(14)}(x)=e^{-x}u^2(x)-(x^2-x+14)e^x \tag{6}$$

With boundary conditions

$$u^{(m)}(0)=1-m \quad , \quad m=0,1,...,11$$
$$u^{(m)}(1)=-me \quad , \quad m=0,1 \qquad 0 \le x \le 1$$

The exact solution of this problem is

$$u(x)=(1-x)e^x$$

Developing e^x by Maclaurin series, then Equation (6) can be written as follows

$$u^{(14)}(x)=e^{-x}u^2(x)-14-13x-7x^2-\frac{17}{6}x^3-\frac{11}{12}x^4 \tag{7}$$

Applying the DTM by using properties 1, 2, 3, 5 and 6 choosing x_0 = 0, equation (7) is transformed in the following form:

$$\frac{(k+14)!}{k\,!}U(k+14)=\sum_{r_1=0}^{k}\sum_{r=0}^{r_1}\frac{(-1)^r}{r\,!}U(r_1-r)U(k-r_1) -14\delta(k)-13\delta(k-1)+7\delta(k-2) -\frac{17}{6}\delta(k-3)-\frac{11}{12}\delta(k-4)$$

$$u(0)=1 \rightarrow U(0)=1$$
$$u'(0)=0 \rightarrow U(1)=0$$
$$u''(0)=-1 \rightarrow U(2)=\frac{-1}{2!}$$
$$u^{(3)}(0)=-2 \rightarrow U(3)=\frac{-2}{3!}$$
$$u^{(4)}(0)=-3 \rightarrow U(4)=\frac{-3}{4!}$$
$$u^{(5)}(0)=-4 \rightarrow U(5)=\frac{-4}{5!}$$
$$u^{(6)}(0)=-5 \rightarrow U(6)=\frac{-5}{6!}$$
$$u^{(7)}(0)=-6 \rightarrow U(7)=\frac{-6}{7!}$$
$$u^{(8)}(0)=-7 \rightarrow U(8)=\frac{-7}{8!}$$

$$u^{(9)}(0) = -8 \to U(9) = \frac{-8}{9!}$$

$$u^{(10)}(0) = -9 \to U(10) = \frac{-9}{10!}$$

$$u^{(11)}(0) = -10 \to U(11) = \frac{-10}{11!}$$

$$u^{(12)}(0) = \alpha \to U(12) = \frac{\alpha}{12!}$$

$$u^{(13)}(0) = \beta \to U(13) = \frac{\beta}{13!}$$

$$k = 0 \Rightarrow 14!U(14) = 2U(0) - 14 = -13$$

$$\Rightarrow U(14) = \frac{-13}{14!}$$

$$k = 1 \Rightarrow 15!U(15) = 2U(0)U(1) - U^2(0) - 13$$

$$\Rightarrow U(15) = \frac{-14}{15!}$$

Then we obtain the solution of equation (6) as follow

$$u(x) = \sum_{k=0}^{\infty} U(k)\, x^k$$
$$= 1 - \frac{1}{2!}x^2 - \frac{2}{3!}x^3 - \frac{3}{4!}x^4 - \frac{4}{5!}x^5 - \frac{5}{6!}x^6$$
$$- \frac{6}{7!}x^7 - \frac{7}{8!}x^8 - \frac{8}{9!}x^9 - \frac{9}{10!}x^{10} - \frac{10}{11!}x^{11}$$
$$+ \frac{\alpha}{12!}x^{12} + \frac{\beta}{13!}x^{13} - \frac{13}{14!}x^{14} - \frac{14}{15!}x^{15}$$

By using

$$u(1) = 0, \quad u'(1) = -e$$

We get

$$\alpha = -10.99887678\,, \quad \beta = -12.01936377$$

Table 2: Results numerical for example 3

x	Error of DTM	Error of HPM (N=15)
0	0.0000000000	0.00000000
0.1	2.0340276 e -24	1.4732 e -08
0.2	7.0621326 e -21	5.2800 e -10
0.3	7.5355493 e -19	6.9700 e -10
0.4	1.8788628 e -17	1.4150 e -09
0.5	2.0418871 e -16	3.5000 e -10
0.6	1.2531222 e -15	8.4400 e -10
0.7	4.8194247 e -15	7.5900 e -10
0.8	1.1430092 e -14	6.9800 e -10
0.9	1.2615079 e -14	8.8400 e -10
1	8.4118801 e -19	6.2354 e -10

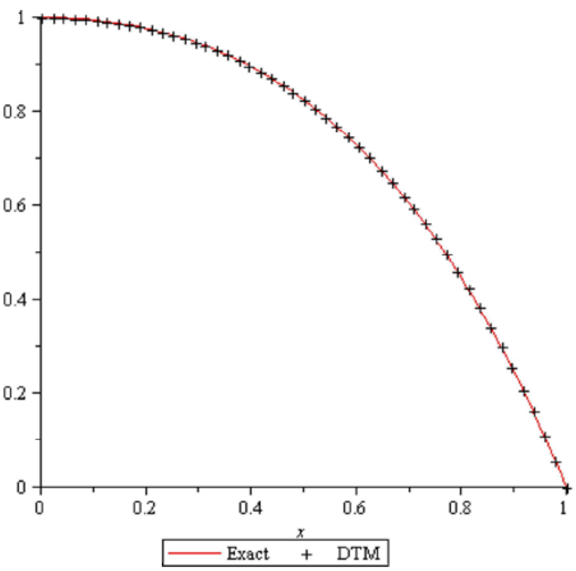

Fig. 2: The approximate solution by using DTM and the exact solution for example 3

In the following table we give the approximate solution of equation (6)

CONCLUSION

In this research, differential transform method (DTM) has been implemented to find the solutions of various kinds of linear and nonlinear ordinary differential equations of 14th order. The comparisons the basic HPM and exact solution with DTM, clearly show the simplicity and accuracy.

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