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Feasible Partial Minimax Estimation of Linear Regression Model

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Abstract: Estimation of the coefficient vector in a linear regression model, when part of it is constrained by ellipsoidal restrictions has been considered. The theory of partial quasi minimax estimator and of partial mock minimax estimator is extended to provide feasible estimators. The large sample approximations of their statistical properties namely the bias, the mean squared error, the quadratic risk and the minimax risk have been derived. A comparative study of the behavior of these proposed estimators based on the bias, the mean squared error and the minimax risk criterion has been done and the superiority conditions have been derived.

Key words: Minimax estimation · Ellipsoidal prior information · Feasibility · Loss function

INTRODUCTION

The classical least square estimator for the coefficient vector in linear regression model is well known in the literature. Howsoever in many applications the coefficient vector of the conceptualized model is constrained by a set of restrictions on the coefficients. It is imperative to incorporate the information on the coefficients in the estimation process so as to get intrinsically more precise estimates. The subject matter of the paper is the prior information on part of the parameter vector β of the regression model constrained to lie in the convex set $B = \{\beta : \beta_1' H_1 \beta_1 \leqq 1\}$ where H_1 a given positive definite matrix. This prior information may be combined with the data set to obtain a partial minimax estimator of the parameter vector.

The theory was initiated by Kuks and Olman [1] who developed an iterative minimization procedure to indicate a point of maximum of the risk function. Lauter [2] gave an explicit presentation of Minimax Linear estimator in the case of regular loss matrix. Later Bunke [3-5], Hoffman [6], Toutenberg [7], Trenkler & Stahlecker [8], Srivastava & Shukla [9], Herring, Trenkler & Stahlecker [9]and Pant [10] developed this theory. Out of all this two distinct approaches emerge. One developed by Trenkler & Stahlacker [8] the Quasi Minimax Estimation theory and Mock Minimax Estimation theory proposed by Srivastava And Shukla [9]. In both the approaches the basic assumption is that the information about the complete parameter vector β is available, how-so-ever quite often there are situations where only a part of the parameter vector is constrained. The approaches where extended to these situations by Herring, Trenkler & Stahlecker [8] and Pant [10] providing Partial Quasi Minimax Estimation theory and Partial Mock Minimax Estimation theory. In both these extensions the unconstrained part of the coefficient vector and the variance component of disturbance term is assumed to be known. In this paper we relax both these assumptions and derived Feasible Partial Quasi Minimax Estimatior, Feasible Partial Mock Minimax Estimatior and their large sample approximations of some statistical properties.

The plan of the paper is as follows. In section 2 of the paper, we describe the model and estimators, while in section 3 we present the large sample approximation of various statistical properties of both the Feasible Partial Estimators. In section 4, we investigate the relative dominance of these estimators over each other and derive the dominance condition. Finally in section 5 we have derived the result.

The Model and the Estimators: Consider the linear regression model

$$y = x\beta + u \tag{2.1}$$

where y is a $T \times 1$ vector of observations on the variable to be explained, x is a $T \times p$ full column rank matrix of observations on explanatory variables, β is a $p \times 1$ vector

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of unknown regression coefficients being estimated and u is a $T \times 1$ vector of disturbances which are assumed to be distributed normally with

$$E(u) = 0, \quad E(uu') = \sigma^2 I_T \tag{2.2}$$

where σ^2 is the scaling factor of disturbances which is usually not known

Let us consider portioning of the coefficient vector β as $\beta=(\beta'_1,\beta'_2)'$ such that β_1 is a $p_1 \times 1$ sub vector of β and β_2 is the remaining $(p - p_1) \times 1$ sub vector of β. The apriori information about the sub vector β_1 is available in the form that the sub vector β_1 is contained in the ellipsoid which is expressible as

$$B = (\beta : \beta'_1 H_1 \beta_1 \leq 1) \tag{2.3}$$

where H_1 is a $p_1 \times p_1$ positive definite and symmetric matrix. The ellipsoidal restrictions can alternatively be written as

$$B = (\beta : \beta' H \beta = \beta_1' H_1 \beta_1 + \beta_2' \beta_2 < 1 + \beta_2' \beta_2 = \eta) \tag{2.4}$$

where $\mathrm{H} = \begin{bmatrix} \mathrm{H}_1 & 0 \\ 0 & I_{p-p_1} \end{bmatrix}$ and $\eta = 1 + \beta_2' \beta_2$.

The best linear unbiased estimator of β which ignores the restrictions is the least square estimator b given by

$$b = (x'x)^{-1} x' y \tag{2.5}$$

Which is normally distributed with mean vector β and dispersion matrix $\sigma^2(x'x)^{-1}$. Assuming σ^2 and η to be known the restrictions can be incorporated in the estimation procedure provides the two partial minimax estimators (Pant [10]). The partial quasi minimax estimator $\hat{\beta}_1$ derived by Herring, Trenkler and Stahlecker [8] given by

$$\hat{\beta}_1 = \left[\mathrm{I} + \frac{\sigma^2}{\eta} (x'x)^{-1} \mathrm{H} \right]^{-1} b \tag{2.6}$$

And the partial mock minimax estimator $\hat{\beta}_2$ derived by Pant (2017) given by

$$\hat{\beta}_2 = \left[\mathrm{I} - \frac{\sigma^2 trA(x'x)^{-1}}{\sigma^2 trA(x'x)^{-1} + \eta \lambda_{\max}(AH^{-1})} \right] b \tag{2.7}$$

where A is the loss matrix associated with the estimation procedure which is positive definite and symmetric and λ_{max} (.) is the largest characteristic root of the matrix function inside the brackets. The problem with these estimators is that they become in operational when either σ^2 or β_2 is unknown. In order to make them operational we replace these unknown terms by their least square estimators as suggested by Herring, Trenkler and Stahlecker [8].

An unbiased estimator of σ^2 and β_2 based on the least squares theory is given by

$$\hat{\sigma}^2 = \frac{1}{T-p} (y - xb)'(y - xb) \tag{2.8}$$

$$b_0 = (x_2' \bar{P}_{x_1} x_2)^{-1} x_2' \bar{P}_{x_1} y \tag{2.9}$$

where x_2 is a $T \times (p - p_1)$ sub matrix of x matrix of T observations on the $(p - p_1)$ explanatory variables corresponding to sub vector β_2 of regression coefficients and $\bar{P}_{x_1} = I_T - x_1(x'_1 x_1)^{-1} x'_1$ is the projection matrix of p_1 explanatory variables contained in x_1 in sub matrix of x. Thus the least squares estimator of η is given by

$$\hat{\eta} = 1 + b'_0 b_0 \tag{2.10}$$

This can be utilized to replace η in the two minimax estimators. Employing the least square estimators of η and σ^2 we get the feasible partial quasi minimax estimator as

$$\hat{b}_1 = \left[\mathrm{I} + \frac{\hat{\sigma}^2}{\hat{\eta}} (x'x)^{-1} \mathrm{H} \right]^{-1} b \tag{2.11}$$

And the feasible partial mock minimax estimator as

$$\hat{b}_2 = \left[1 - \frac{\hat{\sigma}^2 trA(x'x)^{-1}}{\hat{\sigma}^2 trA(x'x)^{-1} + \hat{\eta} \lambda_{max}(AH^{-1})} \right] b \tag{2.12}$$

Large Sample Properties of Feasible Estimators: The large sample approximation for the bias vectors, the dispersion and the mean square matrix and the quadratic and the minimax risks of feasible estimators are derived and presented in following theorem.

Theorem 3.1: The large sample approximations for the bias vector, the dispersion and mean squared matrices and the quadratic and minimax risks of the feasible partial quasi minimax estimators $\hat{b}_1$ upto the order $O_p(T^{-1})$ are respectively given by

$$B(\hat{b}_1) = -\frac{\sigma^2}{\eta}(x'x)^{-1}H\beta + \frac{2\sigma^4}{\eta^2}(x'x)^{-1}H\begin{bmatrix} -(x'_1x_1)^{-1}x'_1x_2 \\ I_{(p-p_1)} \end{bmatrix}(x_2'\bar{P}_{x_1}x_2)^{-1}\beta_2$$

$$-\frac{4\sigma^4}{\eta^3}\left[\beta'_2(x_2'\bar{P}_{x_1}x_2)^{-1}\beta_2\right](x'x)^{-1}H\beta + \frac{\sigma^4}{\eta^2}\left[tr(x_2'\bar{P}_{x_1}x_2)^{-1}\right](x'x)^{-1}H\beta \qquad (3.1)$$

$$+\frac{\sigma^4}{\eta^2}(x'x)^{-1}H(x'x)^{-1}H\beta$$

$$V(\hat{b}_1) = \sigma^2(x'x)^{-1} - \frac{2\sigma^4}{\eta}(x'x)^{-1}H(x'x)^{-1} \qquad (3.2)$$

$$M(\hat{b}_1) = \sigma^2(x'x)^{-1} - \frac{2\sigma^4}{\eta}(x'x)^{-1}H(x'x)^{-1} + \frac{\sigma^4}{\eta^2}L(x'x)^{-1}H\beta\beta'H(x'x)^{-1} \qquad (3.3)$$

$$R(\hat{b}_1) = \sigma^2 trA(x'x)^{-1} - \frac{2\sigma^4}{\eta}trA(x'x)^{-1}H(x'x)^{-1} + \frac{\sigma^4}{\eta^2}\beta'H(x'x)^{-1}AL(x'x)^{-1}H\beta \qquad (3.4)$$

$$\rho(\hat{b}_1) = \sigma^2 trA(x'x)^{-1} - \frac{2\sigma^4}{\eta}trA(x'x)^{-1}H(x'x)^{-1} + \frac{\sigma^4}{\eta}\lambda_{max}[AL(x'x)^{-1}H(x'x)^{-1}] \qquad (3.5)$$

where the matrix L involved in mean squared error matrix and quadratic and minimax risks expressions is given by

$$L = \left[4\left\{0_{p\times p_1} \quad (x'x)^{-1}\begin{pmatrix} 0_{p_1\times p-p_1} \\ I_{p-p_1} \end{pmatrix}\right\} + I_p\right]$$

Theorem 3.2: The large sample approximations for the bias vector, the dispersion and mean squared error matrix and the weighted quadratic and minimax risks of the feasible mock minimax estimator $\hat{b}_2$ upto order $O_p(T^{-1})$ are respectively given by

$$B(\hat{b}_2) = -\frac{\sigma^2}{\eta}\frac{trA(x'x)^{-1}}{\lambda_{max}[AH^{-1}]}\beta + \frac{2\sigma^4}{\eta^2}\frac{trA(x'x)^{-1}}{\lambda_{max}[AH^{-1}]}\begin{bmatrix} -(x'_1x_1)^{-1}x'_1x_2 \\ I_{(p-p_1)} \end{bmatrix}(x_2'\bar{P}_{x_1}x_2)^{-1}\beta_2$$

$$-\frac{4\sigma^4}{\eta^3}\frac{trA(x'x)^{-1}}{\lambda_{max}[AH^{-1}]}\left[\beta'_2(x_2'\bar{P}_{x_1}x_2)^{-1}\beta_2\right]\beta + \frac{\sigma^4}{\eta^2}\frac{trA(x'x)^{-1}}{\lambda_{max}[AH^{-1}]}\left[tr(x_2'\bar{P}_{x_1}x_2)^{-1}\right]\beta \qquad (3.6)$$

$$+\frac{\sigma^4}{\eta^2}\left(\frac{trA(x'x)^{-1}}{\lambda_{max}[AH^{-1}]}\right)^2\beta$$

$$V(\hat{b}_2) = \sigma^2(x'x)^{-1} - \frac{2\sigma^4}{\eta}\frac{trA(x'x)^{-1}}{\lambda_{max}[AH^{-1}]}(x'x)^{-1} \qquad (3.7)$$

$$M(\hat{b}_2) = \sigma^2(x'x)^{-1} - \frac{2\sigma^4}{\eta}\frac{trA(x'x)^{-1}}{\lambda_{max}[AH^{-1}]}(x'x)^{-1} + \frac{\sigma^4}{\eta^2}\frac{trA(x'x)^{-1}}{\lambda_{max}[AH^{-1}]}L\beta\beta' \qquad (3.8)$$

$$R(\hat{b}_2) = \sigma^2 trA(x'x)^{-1} - \frac{2\sigma^4}{\eta}\frac{trA(x'x)^{-1}}{\lambda_{max}[AH^{-1}]}trA(x'x)^{-1} + \frac{\sigma^4}{\eta^2}\frac{trA(x'x)^{-1}}{\lambda_{max}[AH^{-1}]}\beta'AL\beta \qquad (3.9)$$

$$\rho(\hat{b}_2) = \sigma^2 trA(x'x)^{-1} - \frac{2\sigma^4}{\eta}\frac{trA(x'x)^{-1}}{\lambda_{max}[AH^{-1}]}trA(x'x)^{-1} + \frac{\sigma^4}{\eta}\frac{trA(x'x)^{-1}}{\lambda_{max}[AH^{-1}]}\lambda_{max}[ALH^{-1}] \quad (3.10)$$

The results of the theorem 3.1 and theorem 3.2 are derived in appendix.

Performance of Feasible Estimators: In order to study the performance of these feasible partial minimax estimators, we analyze their large sample properties and compare their performance with respect to three criterion of the quadratic loss setup of the decision theory, viz. the bias, the weighted quadratic risk and the minimax risk. There from the optimality of various estimators is considered and dominance conditions are derived.

The Bias Criterion: The classical least squares estimator b is exactly unbiased. The partial quasi minimax estimator $\hat{b}_1$ and partial mock minimax estimator $\hat{b}_2$ both are biased. To compare the two estimators the difference in Euclidean norms of bias vectors of $\hat{b}_1$ and $\hat{b}_2$, to order $O_p(T^{-1})$, is given by

$$\|\hat{b}_1\| - \|\hat{b}_2\| = \frac{\sigma^4}{\eta^2}\left[\beta' \mathrm{H}(x'x)^{-2}\mathrm{H}\beta - \left(\frac{trA(x'x)^{-1}}{\lambda_{max}[AH^{-1}]}\right)^2 \beta'\beta\right] \quad (4.1)$$

Now this is positive as long as we have

$$\lambda_{max}[(x'x)^2 H^{-2}] < \left(\frac{\lambda_{max}[AH^{-1}]}{trA(x'x)^{-1}}\right)^2 \quad (4.2)$$

Thus the feasible partial mock minimax estimator $\hat{b}_2$ will have lesser magnitude of bias than that of the feasible partial quasi minimax estimator $\hat{b}_1$ to the order of our approximations at least as long as the above condition is satisfied in practice. The reverse will hold true, that is the feasible partial quasi minimax estimator $\hat{b}_1$ will have lesser magnitude of bias than that of the feasible partial mock minimax estimator $\hat{b}_2$ to order $O_p(T^{-1})$, at least as long as the following condition is satisfied in a given application.

$$\lambda_{min}[(x'x)^2 H^{-2}] < \left(\frac{\lambda_{max}[AH^{-1}]}{trA(x'x)^{-1}}\right)^2 \quad (4.3)$$

The Quadratic Risk Criterion: The quadratic risk associated with the classical least squares estimator b is

$$R(b) = \sigma^2\, trA(x'x)^{-1} \quad (4.4)$$

Let us compare the least square estimator b with the feasible partial quasi minimax estimator $\hat{b}_1$. The difference in the risk approximations of the two estimators upto order $O_p(T^{-1})$ is given by

$$R(b) - R(\hat{b}_1) = \frac{\sigma^4}{\eta}\left[2trA(x'x)^{-1}H(x'x)^{-1} - \frac{\beta'\mathrm{H}(x'x)^{-1}AL(x'x)^{-1}\mathrm{H}\beta}{\eta}\right] \quad (4.5)$$

Which will be positive if and only if

$$2trA(x'x)^{-1}H(x'x)^{-1} > \frac{\beta'\mathrm{H}(x'x)^{-1}AL(x'x)^{-1}\mathrm{H}\beta}{\eta} \quad (4.6)$$

A sufficient condition to hold this true is

$$2trA(x'x)^{-1}H(x'x)^{-1} > \lambda_{max}\left[AL(x'x)^{-1}\mathrm{H}(x'x)^{-1}\right] \quad (4.7)$$

This is the sufficient condition for the superiority of the feasible partial quasi minimax estimator $\hat{b}_1$ over the classical least square estimator up to the order $O_p(T^{-2})$.

Similarly, comparing the classical least squares estimator $\hat{b}_2$ with the feasible partial mock minimax estimator b, the difference in their risk approximations up to the order $O_p(T^{-2})$ is given by

$$R(b) - R(\hat{b}_2) = \frac{\sigma^4}{\eta}\left(\frac{trA(x'x)^{-1}}{\lambda_{max}\left[A\mathrm{H}^{-1}\right]}\right)\left[2trA(x'x)^{-1} - \frac{\beta' AL\beta}{\eta}\right] \tag{4.8}$$

Which is positive if and only if

$$2trA(x'x)^{-1} > \frac{\beta' AL\beta}{\eta} \tag{4.9}$$

A sufficient condition to hold this true is

$$2trA(x'x)^{-1} > \lambda_{max}\left[AL\mathrm{H}^{-1}\right] \tag{4.10}$$

This is the sufficient condition for the quadratic risk superiority of the feasible partial mock minimax estimator b over the classical least square estimator $\hat{b}_2$ to the order of our approximations.

Finally, comparing the quadratic risk performance of the two feasible partial minimax estimators, the difference in their risk approximations up to the order $O_p(T^{-2})$ is given by

$$\begin{aligned} R(\hat{b}_1) - R(\hat{b}_2) &= \frac{2\sigma^4}{\eta}\, trA(x'x)^{-1}\left[\frac{trA(x'x)^{-1}}{\lambda_{max}\left[A\mathrm{H}^{-1}\right]} I - H(x'x)^{-1}\right] \\ &\quad - \frac{\sigma^4}{\eta^2}\beta'\left[\frac{trA(x'x)^{-1}}{\lambda_{max}\left[A\mathrm{H}^{-1}\right]} AL - H(x'x)^{-1}AL(x'x)^{-1}\mathrm{H}\right]\beta \\ &= \frac{\sigma^4}{\eta}\left[\Delta_1 - \frac{\beta' Q^*\beta}{\eta}\right] \end{aligned} \tag{4.11}$$

where $\Delta_1 = 2\, trA(x'x)^{-1}\left[\frac{trA(x'x)^{-1}}{\lambda_{max}\left[A\mathrm{H}^{-1}\right]} I - H(x'x)^{-1}\right]$

$$Q^* = \left[\frac{trA(x'x)^{-1}}{\lambda_{max}\left[A\mathrm{H}^{-1}\right]} AL - H(x'x)^{-1}AL(x'x)^{-1}\mathrm{H}\right]$$

Now this difference of risk approximations of the two estimators will be positive if and only if, we have

$$\Delta_1 > \frac{\beta' Q^*\beta}{\eta} \tag{4.12}$$

Which will hold true at least as long as we have

$$\Delta_1 > \lambda_{max}\left[Q^*\mathrm{H}^{-1}\right]$$

Thus the feasible partial mock minimax estimator $\hat{b}_2$ will have quadratic risk superiority over the feasible partial quasi minimax estimator $\hat{b}_1$ at least up to order $O_p(T^{-2})$ as long as above condition is satisfied in practice.

The Minimax Risk Criterion : The minimax risk associated with the least square estimator b is the same as its quadratic risk and is given by

$$\rho(b) = \min_{b} \sup_{\beta'\mathrm{H}\beta<\eta} R(b) = \sigma^2 trA(x'x)^{-1} \tag{4.13}$$

The difference in minimax risk approximations of the least squares estimator b and the feasible partial quasi minimax estimator $\hat{b}_1$ up to order $O_p(T^{-2})$ is given by

$$\rho(b) - \rho(\hat{b}_1) = \frac{\sigma^4}{\eta}\left[2trA(x'x)^{-1}\mathrm{H}(x'x)^{-1} - \lambda_{max}[AL(x'x)^{-1}\mathrm{H}(x'x)^{-1}]\right] \tag{4.14}$$

Which is positive if and only if we have

$$2trA(x'x)^{-1}\mathrm{H}(x'x)^{-1} > \lambda_{max}[AL(x'x)^{-1}\mathrm{H}(x'x)^{-1}] \tag{4.15}$$

Which can easily be checked in any given application. Thus the feasible partial quasi minimax estimator $\hat{b}_1$ will have the minimax risk superiority over the classical least square estimator b up to the order $O_p(T^{-2})$, if and only if above condition is satisfied in practice.

Similarly the difference in minimax risk approximations of the estimator $\hat{b}_2$ and b turns out to be

$$\rho(b) - \rho(\hat{b}_2) = \frac{\sigma^4}{\eta}\left(\frac{trA(x'x)^{-1}}{\lambda_{max}[A\mathrm{H}^{-1}]}\right)\left[2trA(x'x)^{-1} - \lambda_{max}[AL\mathrm{H}^{-1}]\right] \tag{4.16}$$

Which is positive if and only if we have

$$2trA(x'x)^{-1} > \lambda_{max}[AL\mathrm{H}^{-1}] \tag{4.17}$$

Thus the feasible partial mock minimax estimator $\hat{b}_2$ will dominate the classical least square estimator b with respect to their minimax risk approximations up to order $O_p(T^{-2})$, if and only if above condition is satisfied.

Finally, the difference in minimax risk approximations of the two feasible partial minimax estimators $\hat{b}_1$ and $\hat{b}_2$ up to the order $O_p(T^{-2})$, is given by

$$\rho(\hat{b}_1) - \rho(\hat{b}_2) = \frac{\sigma^4}{\eta}\left[2trA(x'x)^{-1}\left\{\frac{trA(x'x)^{-1}}{\lambda_{max}[A\mathrm{H}^{-1}]}I - \mathrm{H}(x'x)^{-1}\right\} - \left\{\frac{trA(x'x)^{-1}}{\lambda_{max}[A\mathrm{H}^{-1}]}\lambda_{max}[\mathrm{AL}\mathrm{H}^{-1}] - \lambda_{max}[\mathrm{AL}(x'x)^{-1}\mathrm{H}(x'x)^{-1}]\right\}\right]$$

$$= \frac{\sigma^4}{\eta}(\Delta_1 - \Lambda) \tag{4.18}$$

Where Δ_1 is defined previously and Λ is given by

$$\Lambda = \frac{trA(x'x)^{-1}}{\lambda_{max}[A\mathrm{H}^{-1}]}\lambda_{max}[\mathrm{AL}\mathrm{H}^{-1}] - \lambda_{max}[\mathrm{AL}(x'x)^{-1}\mathrm{H}(x'x)^{-1}] \tag{4.19}$$

This difference of minimax risk approximations is positive if and only if we have

$$\Delta_1 > \Lambda \tag{4.20}$$

Which is necessary and sufficient condition for the minimax risk dominance of the feasible partial mock minimax estimator $\hat{b}_2$ over the feasible partial quasi minimax estimator $\hat{b}_1$, up to the order of our approximations.

Derivation of Results : The feasible partial quasi minimax estimator $\hat{b}_1$ can be written as

$$\hat{b}_1 = \left[I + \frac{\hat{\sigma}^2}{\hat{\eta}}(x'x)^{-1}\mathrm{H}\right]^{-1}\left(\beta + (x'x)^{-1}x'u\right) \tag{5.1}$$

where

$$\hat{\sigma}^2 = \frac{1}{T-p}(y - xb)'(y - xb) \tag{5.2}$$

Now observing that

$$\hat{\sigma}^2 - \sigma^2 = \lambda_{-1/2} + \lambda_{-1} \tag{5.3}$$

$$\lambda_{-1/2} = w \quad (say)$$

$$\lambda_{-1} = \frac{1}{(T-p)}u'x(x'x)^{-1}x'u$$

with $\quad E(w) = 0 \qquad E(w^2) = \frac{2\sigma^4}{T}$

and

$$\hat{\eta} = 1 + b'_0 b_0 \tag{5.4}$$

where b_0 is the least square estimate of β_2

$$b_0 = \beta_2 + (x_2{}'\bar{P}_{x_1}x_2)^{-1}x_2{}'\bar{P}_{x_1}u \tag{5.5}$$

where x_2 is a sub matrix of $x \equiv (x_1, x_2)$ consisting of explanatory variables corresponding to coefficients of β_2 sub vector and $\bar{P}_{x_1} = I_T - x_1(x'_1x_1)^{-1}x'_1$.
Therefore, the estimate of η is given by

$$\hat{\eta} = 1 + b'_0 b_0$$

$$= 1 + \beta'_2\beta_2 + 2\beta'_2 w + w'w \tag{5.6}$$

where $w = (x_2{}'\bar{P}_{x_1}x_2)^{-1}x_2{}'\bar{P}_{x_1}u$. Thus we can write

$$\frac{1}{\hat{\eta}} = \frac{1}{\eta}\left[1 + v_{-1/2} + v_{-1} + o_p(T^{-\frac{3}{2}-j})\right] \qquad j \geq 0 \tag{5.7}$$

where

$$v_{-1/2} = -\frac{2\beta'_2 w}{\eta}$$

$$v_{-1} = \frac{4(\beta'_2 w)^2}{\eta^2} - \frac{w'w}{\eta}$$

Thus the estimation error associated with this estimator can be written as

$$\hat{b}_1 - \beta = e_{-1/2} + e_{-1} + e_{-3/2} + e_{-2} + o_p(T^{-2-j}) \qquad j \geq 0 \tag{5.8}$$

where

$$e_{-1/2} = (x'x)^{-1}x'u$$

$$e_{-1} = -\frac{\sigma^2}{\eta}(x'x)^{-1}\mathrm{H}\beta$$

$$e_{-3/2} = -\frac{\sigma^2}{\eta}(x'x)^{-1}\mathrm{H}(x'x)^{-1}x'u - \frac{\left(\varepsilon - \frac{2\sigma^2}{\eta}\beta'_2 w\right)}{\eta}(x'x)^{-1}\mathrm{H}\beta$$

$$e_{-2} = -\frac{\left(\sigma^2 v_{-1/2} + \lambda_{-1/2}\right)}{\eta}\left(x'x\right)^{-1}\mathrm{H}\left(x'x\right)^{-1}x'u - \frac{\left(\sigma^2 v_{-1} + \lambda_{-1} + \lambda_{-1/2}v_{-1/2}\right)}{\eta}\left(x'x\right)^{-1}\mathrm{H}\beta$$

$$+\frac{\sigma^4}{\eta^2}(x'x)^{-1}\mathrm{H}(x'x)^{-1}\mathrm{H}\beta$$

These results can be employed to obtain the expression no. (3.1),(3.2),(3.3),(3.4) and (3.5) for the Feasible Partial Quasi Minimax estimator.

The partial mock minimax estimator can be written as

$$\hat{b}_2 = \left[1 - \frac{\hat{\sigma}^2 trA(x'x)^{-1}}{\hat{\sigma}^2 trA(x'x)^{-1} + \hat{\eta}\lambda_{max}(A\mathrm{H}^{-1})}\right](\beta + (x'x)^{-1}x'u) \tag{5.9}$$

Whose estimation error upto the order $O_p(T^{-2})$ can be expressed as

$$\hat{b}_2 - \beta = e^*_{-1/2} + e^*_{-1} + e^*_{-3/2} + e^*_{-2} + o_p\left(T^{-2-j}\right) \qquad j \geq 0 \tag{5.10}$$

where

$$e^*_{-1/2} = (x'x)^{-1}x'u$$

$$e^*_{-1} = -\frac{\sigma^2 trA(x'x)^{-1}}{\eta\lambda_{max}(A\mathrm{H}^{-1})}$$

$$e^*_{-3/2} = -\frac{\sigma^2 trA(x'x)^{-1}}{\eta\lambda_{max}(A\mathrm{H}^{-1})}\left[(x'x)^{-1}x'u + \frac{\lambda_{-1/2}}{\sigma^2}\beta - v_{-1/2}\beta\right]$$

$$e^*_{-2} = \lambda_{-1/2}\frac{trA(x'x)^{-1}}{\lambda_{max}(A\mathrm{H}^{-1})}(x'x)^{-1}x'u + \lambda_{-1}\frac{trA(x'x)^{-1}}{\lambda_{max}(A\mathrm{H}^{-1})}\beta$$

Defining the statistical properties for the estimator as follows

1. Bias vector : $Bias\left(\tilde{\beta}\right) = E\left(\tilde{\beta} - \beta\right)$

2. Dispersion matrix : $V\left(\tilde{\beta}\right)=E\left[\tilde{\beta}-E\left(\tilde{\beta}\right)\right]\left[\tilde{\beta}-E\left(\tilde{\beta}\right)\right]'$

3. Mean squared error matrix : $M\left(\tilde{\beta}\right)=E\left[\tilde{\beta}-\beta\right]\left[\tilde{\beta}-\beta\right]'$

4. Weighted quadratic risk : $R\left(\tilde{\beta},A\right)=E\left[\tilde{\beta}-\beta\right]'A\left[\tilde{\beta}-\beta\right]$

5. Minimax risk : $\rho\left(\tilde{\beta},A\right)=\underset{\beta}{Min}\ \underset{\beta' H\beta\leq\eta}{Sup}\ E\left[\beta'-\beta\right]'A\left[\beta'-\beta\right]$

where *A* is the weighing matrix of losses which is assumed to be positive definite and symmetric. On substituting the values of $\hat{b}_1-\beta$ for estimator $\hat{b}_1$ we can evaluate the expression no. (3.1),(3.2),(3.3),(3.4)and (3.5) for the Feasible Partial Quasi Minimax estimator. Similarly for Feasible Mock Minimax Estimator $\hat{b}_2$ substituting the values of $\hat{b}_2-\beta$ we can find the expression no. (3.6),(3.7),(3.8),(3.9)and (3.10).

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