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Solving System of Ordinary Differential Equations By Aboodh Transform

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Abstract: Aboodh transform, whose fundamental properties are presented in this paper, is still not widely known, nor used, Aboodh transform may be used to solve problems without resorting to a new frequency domain. Farther, we use Aboodh transform to solve systems of ordinary differential equations

Key words: Aboodh transform-system of differential equations

INTRODUCTION

In order to solve differential equations, several integral transforms were extensively used and applied in theory and application such as the Laplace, Fourier, Mellin, Hankel and Sumudu transforms [11-13], Elzaki Transform [5-10], to name but a few. In the sequence of these transforms. The integral transform methds is also an efficient method to solve the system of ordinary differential equations. Recently, Khalid Aboodh in 2013 introduced a new transform and named as Aboodhtransform [1-4], to facilitate the process of solving ordinary and partial differential equations in the time domain. This transformation has deeper connection with the Laplace and Elzaki Transform .In this study, our purpose is to show the applicability of this interesting new transform and its efficiency in solving the linear system of ordinary differential equations.

Aboodh Transform

Definition: A new transform called the Aboodh transform defined for function of exponential order we consider functions in the set A, defined by:

$$A = \{f(t): \exists\ M, k_1, k_2 > 0, |f(t)| < Me^{-vt}$$

For a given function in the set M must be finite number, k_1, k_2 may be finite or infinite. Aboodh transform which is defined by the integral equation.

$$A[f(t) = K(v) = \frac{1}{v}\int_0^\infty f(t)e^{-vt}\,dt\ \ t \geq 0, k_1 \leq v \leq k_2 \tag{1}$$

Aboodh Transformfor Basic Function:

- let $f(t) = 1$, then $A(1) = \frac{1}{v}\int_0^\infty e^{-vt}dt = \frac{1}{v}\left[\frac{1}{v}e^{-vt}\right]_0^\infty = \frac{1}{v^2}$
- let $f,(t)$ then $A(t) = \frac{1}{v}\int_0^\infty t\ e^{-vt}dt$.

Integratingby parts to find that $A(t) = \frac{1}{v^3}$.

- Let $f(t) = te^{at}$, then

$$A(te^{at}) = \frac{1}{v}\int_0^\infty te^{at}e^{-vt}dt,$$

integrating by parts or using properties of Gamma function to find that.

$$A(te^{at}) = \frac{1}{v(v-a)^2} .$$

Aboodh transform of derivatives:

$$A[f'(t)] = vk(v) - \frac{f(0)}{v}, A[f''(t)] = v^2K(v) - \frac{f'(0)}{v} - f(0)$$

$$A[f^{(n)}(t)] = v^nK(v) - \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{v^{2-n+k}},$$

System of Ordinary Differential Equations: In this section, the effectiveness and the usefulness of Aboodh transform is demonstrated by finding exact solutions System of Ordinary Differential Equations.

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Example (1): (System of First Order Ordinary Differential Equations): We consider a first –order linear system with constant coefficients

$$\left.\begin{matrix} \frac{dy_1}{dt} = a_{11}y_1 + a_{12}y_2 + g_1(t) \\ \frac{dy_2}{dt} = a_{21}y_1 + a_{22}y_2 + g_2(t) \end{matrix}\right\} \quad (2)$$

with initial condition $y_1(0) = y_{10}$ and $y_2(0) = y_{20}$ where a_{11}, a_{12}, a_{21}, a_{22} are constants.

Solution:

Let $\bar{y}_1 = A(y_1), \bar{\bar{y}}_2 = A(y_2), G_1 = A(g_1)$ and $G_2 = A(g_2)$

Introducing the matrices;

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, y_0 = \begin{pmatrix} y_{10} \\ y_{20} \end{pmatrix}, \quad g(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}$$

$$A = \begin{pmatrix} a_{11} \, a_{12} \\ a_{21} \, a_{22} \end{pmatrix}, \quad \frac{dy}{dt} = \begin{pmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \end{pmatrix} \quad (3)$$

We can write the above system in matrix differential system as;

$$\frac{dy}{dt} = Ay + g(t), y_0 = y_0 \quad (4)$$

We take Aboodh transform of the system with the initial conditions to get.

$$\begin{matrix} \{ v(v - a_{11})\overline{y_1}(v) - va_{12}\overline{y_2}(v) = v\overline{g_1}(v) + y_{10} \} \\ \{ v(v - a_{21})\overline{y_1}(v) - va_{22}\overline{y_2}(v) = v\overline{g_2}(v) + y_{20} \} \end{matrix} \quad (5)$$

The solution of this algebraic system is,

$$\overline{y_1}(v) = \frac{\begin{bmatrix} \overline{g_1}(v) + y_{10} & a_{12}(v+1) \\ \overline{g_2}(v) + y_{20} & a_{22}(v+1) \end{bmatrix}}{\begin{bmatrix} v - a_{11} & va_{12} \\ v - a_{21} & va_{22} \end{bmatrix}}$$

$$\overline{y_2}(v) = \frac{\begin{bmatrix} (v-a_{11})(v+1) & \overline{g_1}(v)+y_{10} \\ (v-a_{21})(v+1) & \overline{g_2}(v)+y_{20} \end{bmatrix}}{\begin{bmatrix} v-a_{11} & -va_{12} \\ v-a_{21} & -va_{22} \end{bmatrix}} \quad (6)$$

Example (2): Solve the system of ordinary differential equation.

$$\frac{dy}{dt} = -z \quad , \quad \frac{dz}{dt} = y \quad (7)$$

With initial condition

$$z(0) = 0, y(0) = 1 \quad (8)$$

Solution: Applying Aboodh transform for the equation (7) we have.

$$A\left(\frac{dy}{dt}\right) = A(-z), \qquad A\left(\frac{dz}{dt}\right) = A(y)$$

$$Vy(v) - \frac{y(0)}{v} = -z(v), Vz(v) - \frac{z(0)}{v} = y(v)$$

with the initial condition (8) we get

$v\,y(v) + z(v) = \frac{1}{v}$, $v\,z(v) = y(v)$ so

$$\begin{cases} v\,y(v) - v^2 z(v) = 0 \\ v\,y(v) + z(v) = \frac{1}{v} \end{cases} \quad (9)$$

There fore $z(v) = \frac{1}{v(v^2+1)}$

Then $z(t) = sint, y(t),$ (10)

Example (3):

Consider the system of ordinary differential equation.

$$\begin{cases} \frac{dx}{dt} = 2x - 3y \\ \frac{dy}{dt} = y - 2x \end{cases} t > 0 \quad (11)$$

With the initial condition

$$x(0) = 8, y(0) = 3 \quad (12)$$

Take Aboodh transform of the system with the initial conditions (12) to get.

$$\begin{matrix} A(\frac{dx}{dt}) = A(2x) - A(3y) \\ A(\frac{dy}{dt}) = A(y) - A(2x) \end{matrix} \quad (13),$$

we get,

$$\begin{cases} Vx(v) - \frac{x(0)}{v} = 2x(v) - 3y(v) \\ Vy(v) - \frac{y(0)}{v} = y(v) - 2x(v) \end{cases} \quad (14),$$

with the initial condition we get,

$$(V^2 - 2v)x(v) + 3vy(v) = 8 \quad (15)$$

$$(V^2 - v)y(v) + 2vx(v) = 3 \quad (16)$$

Solve these equations for x and y we find:

$$y(v) = \frac{3v^2-22v}{(v^2-4v)(v^2+v)} = \frac{-2}{(v^2-4v)} + \frac{5}{(v^2+v)} \quad (17)$$

By taking inverse Aboodh transform we have:

$$y(t) = 5e^{-t} - 2e^{4t} \quad (18)$$

Now solve equation (15) and (16) simultaneously from $x(v)$ we have.

$$x(v) = \frac{8v^2-17v}{(v^2-4v)(v^2+v)} = \frac{3}{(v^2-4v)} + \frac{5}{(v^2+v)} \quad (19)$$

By taking inverse Aboodh transform we have:

$$x(t) = 3e^{4t} + 5e^{-t} \quad (20)$$

Example (4):

Solve the second order couplet differential system.

$$\begin{cases} \frac{d^2x}{dt^2} - 3x - 4y = 0 \\ \frac{d^2y}{dt^2} + x + y = 0 \end{cases} \quad t > 0 \quad (21)$$

With the initial condition.

$$x(0) = y(0) = 0, \ \frac{dx}{dt}(0) = 2, \frac{dy}{dt}(0) = 0 \quad (22)$$

Applying Aboodh transform for the equation (21) we have l

$$\begin{aligned} &A\left(\frac{d^2x}{dt^2}\right) - 3A(x) - 4A(y) = A(0) \\ &A\left(\frac{d^2y}{dt^2}\right) + A(x) + A(y) = A(0) \end{aligned} \quad (23)$$

we get,

$$\begin{cases} v^2 x(v) - \frac{x`(0)}{v} - x(0) - 3x(v) - 4y(v) = 0 \\ v^2y(v) + x(v) + y(v) = 0 \end{cases}$$

with the initial condition (22) we get,

$$(v^2 - 3)x(v) - 4y(v) = \frac{2}{v} \quad (24)$$

$$(v^2 + 1)y(v) + x(v) = 0$$

Solve these equations for x and y we find:

$$x(v) = \frac{2(v^2+1)}{v(v^2-1)^2}, \quad (25)$$

By taking inverse Aboodh transform we have

$$x(t) = t(e^t + e^{-t}), so\ y = \frac{1}{4}\left(\frac{d^2x}{dt^2} - 3x\right)$$

$$y(t) = \frac{1}{2}(e^t - e^{-t} - te^t - te^{-t}) \quad (26)$$

Example (5):

Solve the second order couplet differential system

$$\begin{cases} y' + z' + y + z = 1 \\ y' + z = e^t \end{cases} \quad (27)$$

With the initial condition,

$$y(0) = -1, z(0) = 2 \quad (28)$$

Solution: Applying Aboodh transform for the equation (27) we have,

$$\begin{cases} A(y') + A(z') + A(y) + A(z) = A(1) \\ A(y') + A(z) = A(e^t) \end{cases} \quad (29)$$

we get

$$\begin{cases} v\, y(v) - \frac{y(0)}{v} + vz(v) - \frac{z(0)}{v} + y(v) + z(v) = \frac{1}{v^2} \\ v\, y(v) - \frac{y(0)}{v} + z(v) = \frac{1}{v^2 v} \end{cases} \quad (30)$$

with the initial condition (28) we get,

$$\begin{cases} v\, y(v) + \frac{1}{v} + vz(v) - \frac{2}{v} + y(v) + z(v) = \frac{1}{v^2} \\ v\, y(v) + \frac{1}{v} + z(v) = \frac{1}{v^2 v} \end{cases} \quad (31)$$

Solve these equations for x and y we find:

$$y(v) = \frac{1}{v(v-1)^2} - \frac{1}{v-1} - \frac{1}{v^2(v-1)}, \quad (32)$$

By taking inverse Aboodh transform we have,

$$\begin{aligned} &y(t) = 1 - 2e^t + te^t \\ &y`(t) = te^t + e^t - 2e^t = te^t - e^t \\ &z = e^t - y` = e^t - (te^t - e^t) = 2e^t - te^t \end{aligned} \quad (33)$$

Examples 6: Solve the system of second order differential equations.

$$\begin{cases} z'' + y' = \cos x \\ y'' - z = \sin x \end{cases} \quad (34)$$

With the initial condition;

$z(0) = -1, z'(0) = -1, y(0) = 1, y'(0) = 0$ (35)

Solution: Applying Aboodh transform for the equation (34) we have

$$\begin{cases} A(z'') + A(y') = A(\cos x) \\ A(y'') - A(z) = A(\sin x) \end{cases} \quad (36)$$

we get,

$$\begin{cases} v^2 z(v) - \frac{z'(0)}{v} + z(0) + v y(v) + \frac{y(0)}{v} = \frac{1}{v^2+1} \\ v^2 y(v) - \frac{y'(0)}{v} - y(0) + z(v) = \frac{1}{v(v^2+1)} \end{cases} \quad (37)$$

with the initial condition (35) we get,

$$\begin{cases} v\, y(v) + v^2\, z(v) = \frac{1}{v^2+1} - 1 \\ v^2 y(v) - z(v) = \frac{1}{v(v^2+1)} + 1 \end{cases} \quad (38)$$

Solve these equations for x and y we find:

$$y(v) = \frac{v+1}{(v^4+v)(v^2+1)} - \frac{v^2-1}{v^4+v}, \quad (39)$$

By taking inverse Aboodh transform we have,

$y(x) = \cos x, z(x) = y`` - \sin x$

$z(x) = -(\cos x + \sin x)$ (40)

CONCLUSION

In this paper, we use Aboodh transform to solve systems of ordinary differential equations, Aboodh transform provides powerful method for analyzing derivatives. It is heavily used to solve ordinary differential equations, Partial differential equations and system of ordinary differential equations.

Appendix

Aboodh Transform of some Functions

$f(t)$	$A[f(t)] = k(v)$
1	$\frac{1}{v^2}$
$\sin(at$	$\frac{a}{v(v^2+a^2)}$
$\cos(at$	$\frac{1}{(v^2+a^2)}$
te^{-at}	$\frac{1}{v(v+a)^2}$
$t\sin at$	$\frac{2a}{(v^2+a^2)^2}$
$t\cos at$	$\frac{v^2-a^2}{v(v^2+a^2)^2}$
$t\sinh at$	$\frac{2a}{(v^2-a^2)^2}$
$t\cosh at$	$\frac{v^2+a^2}{v(v^2-a^2)^2}$
$e^{at}\sin bt$	$\frac{b}{v((v-a)^2+b^2)}$
$e^{at}\cos bt$	$\frac{v-a}{v((v-a)^2+b^2)}$
$e^{at}\sinh bt$	$\frac{b}{v((v-a)^2-b^2)}$

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