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Conserved Quantities of Non-Linear Third Order Systems of PDEs

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Abstract: Noether approach is applied to derive the conserved quantities for two non-variational third order systems of partial differential equation (PDEs) i.e. bi-Hamiltonian Boussinesq system and system of dispersive wave equations. To apply Noether approach we use a transformation to convert the considered systems into variational problems and thus standard Lagrangian for each variational system is reported. Further, inverse transformation is applied to get the corresponding conserved quantities for the considered non-variational problems.

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Key words: Lagrangian · Noether operator · Conserved quantities

INTRODUCTION

Conserved quantities play a vital role in the theory of differential equations. Solution and reduction of PDEs are some of the aspects where conserved quantities have its significant importance. On the derivation of conservation laws active research efforts have been made in the last few decades. One of the elegant and systematic approaches is developed by Noether [1]. Direct method [2], characteristics approach [3] and the partial Noether approach [4] are the approaches which have been used frequently in literature. Computing utilities for the calculation of conservation laws are also practiced in past [5-11].

A first systematic method for finding the conservation laws [1] of the variational problems is given by Emmy Noether. She found a link between the symmetries and conservation laws [1]. She concluded that each symmetry corresponds to a conserved quantity. This approach is valid for the variational problems and thus depends on the existence of standard Lagrangian. There is a big class of PDEs and system of PDEs which do not possess standard Lagrangian, thus Noether approach is not applicable. In this article, we apply Noether approach to the non-variational system of PDEs. For this we use the following steps:

- Convert the non-variational system of PDEs into variational problem by using transforma- tion.
- Standard Lagrangian is calculated and Noether approach is applied in new coordinates.
- Noether operators and corresponding conservation laws are computed.
- Use inverse transformation to convert the conserved quantities into the coordinates of original problem.

The pattern of this paper is as follow. In section 2, fundamental operators are discussed. The conserved quantities for the bi-Hamiltonian Boussinesq system and system of dispersive wave equa- tions are discussed in Section 3. Conclusion is given at the end.

Fundamental Operators: Let (t, x) be independent variables and (u, v) be dependent variables.

- Consider the third order system of PDEs with two independent and dependent variables i.e.

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$$E_1(t,x,u,v,u_t,v_t,...)=0, \quad E_2(t,x,u,v,u_t,v_t,...)=0. \tag{1}$$

Let $u \to U_x$ and $v \to V_x$ then the given third order system becomes fourth order system in U, V variables, i.e.

$$G_1(t,x,U,V,U_t,V_t,...)=0, \quad G_2(t,x,U,V,U_t,V_t,...)=0. \tag{2}$$

- The Euler operator is:

$$\frac{\delta}{\delta U}=\frac{\partial}{\partial U}-D_t\frac{\partial}{\partial U_t}-D_x\frac{\partial}{\partial U_x}+D_t^2\frac{\partial}{\partial U_{tt}}+D_x^2\frac{\partial}{\partial U_{xx}}+...,\tag{3}$$

$$\frac{\delta}{\delta U}=\frac{\partial}{\partial U}-D_t\frac{\partial}{\partial U_t}-D_x\frac{\partial}{\partial U_x}+D_t^2\frac{\partial}{\partial U_{tt}}+D_x^2\frac{\partial}{\partial U_{xx}}+...,\tag{4}$$

where,

$$D_t=\frac{\partial}{\partial t}+U_t\frac{\partial}{\partial U}+V_t\frac{\partial}{\partial V}+U_{tt}\frac{\partial}{\partial U_t}+V_{tt}\frac{\partial}{\partial V_t}+U_{tx}\frac{\partial}{\partial U_x}+...\tag{5}$$

$$D_x=\frac{\partial}{\partial x}+U_x\frac{\partial}{\partial U}+V_x\frac{\partial}{\partial V}+U_{xx}\frac{\partial}{\partial U_x}+V_{xx}\frac{\partial}{\partial V_x}+U_{tx}\frac{\partial}{\partial U_t}+...\tag{6}$$

are known as the total derivative operators.

- The generalized operator is defined by:

$$X=\tau\frac{\partial}{\partial t}+\xi\frac{\partial}{\partial x}+\phi\frac{\partial}{\partial U}+\eta\frac{\partial}{\partial V}+\phi^x\frac{\partial}{\partial U_x}+\eta^x\frac{\partial}{\partial V_x}+\phi^t\frac{\partial}{\partial U_t}+...\tag{7}$$

- A standard Lagrangian $L = L(t, x, U, V, U_t, V_t...)$ A (space of differential functions) and satisfies:

$$\frac{\delta L}{\delta U}=0 \text{ and } \frac{\delta L}{\delta V}=0.$$

- The generalized operator is known as Noether operator associated with a standard Lagrangian L if it satisfies:

$$X^{[2]}L+L(\tau_t+\tau_U U_t+\tau_V V_t+\xi_x+\xi_U U_x+\xi_V V_x)=B_t^1+B_U^1U_t+B_V^1V_t+B_x^2+B_U^2U_x+B_V^2V_x.\tag{8}$$

In Eq.(8), B^i's are known as the gauge terms, while $X^{[2]}$ is the second prolongation of the generator X.

- The equation:

$$(T^1)_t+(T^2)_x=0,\tag{9}$$

evaluated on the solution space given by (1) is known as the conservation laws for Eq. (1). The vector $T = (T^1, T^2)$ is a conserved vector where T^1, T^2 are its components.

The conserved vectors of the system (1) associated with a Noether operator X can be determined from the formula:

$$T^i=B^i-N^i(L) \quad i=1,2,\tag{10}$$

where,

$$N^1 = \tau + (\phi - \tau U_t - \xi U_x)\frac{\delta}{\delta U_t} + (\eta - \tau V_t - \xi V_x)\frac{\delta}{\delta V_t} + D_t(\phi - \tau U_t - \xi U_x)\frac{\delta}{\delta U_{tt}} + D_t(\eta - \tau V_t - \xi V_x)\frac{\delta}{\delta V_{tt}} + D_x(\phi - \tau U_t - \xi U_x)\frac{\delta}{\delta U_{tx}} + \ldots, \quad (11)$$

$$N^2 = \xi + (\phi - \tau U_t - \xi U_x)\frac{\delta}{\delta U_x} + (\eta - \tau V_t - \xi V_x)\frac{\delta}{\delta V_x} + D_x(\phi - \tau U_t - \xi U_x)\frac{\delta}{\delta U_{xx}} + D_x(\eta - \tau V_t - \xi V_x)\frac{\delta}{\delta V_{xx}} + D_t(\phi - \tau U_t - \xi U_x)\frac{\delta}{\delta U_{tx}} + \ldots \quad (12)$$

Noether Operators and Conservation Laws: In this section, we will calculate the conserved quantities of the bi-Hamiltonian Boussinesq system and the system of dispersive wave equations respectively.

Bi-Hamiltonian Boussinesq System: The bi-Hamiltonian Boussinesq system is:

$$u_t - v_x = 0,\ v_t - \frac{1}{3}u_{xxx} - \frac{8}{3}(uu_x) = 0. \quad (13)$$

It should be noted that system (13) belongs to the class of non-variational problems and hence does not possess a standard Lagrangian. In order to make it variational let us take $u = U_x$ and $v = V_x$. The system (13) becomes:

$$U_{tx} - V_{xx} = 0, V_{tx} - \frac{1}{3}U_{xxxx} - \frac{8}{3}[U_x U_{xx}] = 0. \quad (14)$$

The standard Lagrangian for system (14) is:

$$L = \frac{1}{6}U_{xx}^2 + \frac{1}{2}V_x^2 + \frac{4}{9}U_x^3 - \frac{1}{2}U_t V_x - \frac{1}{2}U_x V_t \quad (15)$$

Using Eq. (15) in Eq.(8) and separating with respect to derivatives of U and V one gets:

$$(i): \xi_x = 0,\quad (ii): \xi_U = 0,\quad (iii): \xi_V = 0,\ (iv): \eta_U = 0, \quad (16)$$

$$(i): \tau_V = 0, (ii): \tau_U = 0, (iii): \tau_x = 0, (iv): \tau_t = 0, (v): \eta_V = 0, \quad (17)$$

$$(i): \phi_U = 0,\ (ii): \phi_V = 0,\ (iii): \phi_x = 0,\quad (iv): B_V^1 = 0, \quad (18)$$

$$(i): B_U^1 = -\frac{1}{2}\eta_x, (ii): B_U^2 = -\frac{1}{2}\eta_t,\ (iii): B_V^2 = -\frac{1}{2}\phi_t + \eta_x \quad (19)$$

and

$$B_t^1 + B_x^2 = 0. \quad (20)$$

The solution of Eqs. (16)-(20) is:

$$\tau = c_1,\ \xi = c_2,\ \phi = a,\ \eta = b + c_3 x, \quad (21)$$

$$B^1 = -\frac{1}{2}c_3 U + \alpha(t,x),\ B^2 = -\frac{1}{2}b_t U - \frac{1}{2}a_t V + c_3 V + \beta(t,x). \quad (22)$$

where,

$$\alpha_t + \beta_x = 0, \tag{23}$$

while a and b are any arbitrary functions of t. Without loss of generality one can take $\alpha = 0 = \beta$. Thus the Noether operators for system (14) will be:

$$X_1 = \frac{\partial}{\partial t},\quad X_2 = \frac{\partial}{\partial x},\quad X_3 = x\frac{\partial}{\partial V}, X_{(a,b)} = a\frac{\partial}{\partial U} + b\frac{\partial}{\partial V}. \tag{24}$$

Using (10) and after applying the inverse transformation, i.e. $U \to \int u dx$ and $V \to \int v dx$ we get the following corresponding conserved vectors for system (13).

- For X_1, we have the following components

$$T_1^1 = -\frac{1}{2}v^2 - \frac{1}{6}u_x^2 - \frac{4}{9}u_x^3,\ T_1^2 = \frac{1}{3}u_t u_x + \frac{1}{3}u_{xx}\int u_t dx + \frac{4}{3}u^2\int u_t dx + v\int v_t dx - \int u_t dx\int v_t dx. \tag{25}$$

- X_2 gives the following components of T_2

$$T_2^1 = -uv,\quad T_2^2 = \frac{1}{2}v^2 - \frac{1}{6}u_x^2 + \frac{8}{9}u^3 - \frac{1}{3}uu_{xx}. \tag{26}$$

- The components of the conserved vector for X_3 are

$$T_3^1 = \frac{1}{2}u + \frac{1}{2}\int u dx,\quad T_3^2 = -xv + \int v dx + \frac{x}{2}\int u_t dx. \tag{27}$$

- $X_{(a,\,b)}$ yields the following components of the conserved vector:

$$\begin{aligned} T_{(a,b)}^1 &= av + bu - D_x\left(\int (av+bu)dx\right), \\ T_{(a,b)}^2 &= -\frac{1}{2}a_t - \frac{1}{2}b_t u - a\left(-\frac{1}{3}u_{xxx} + \frac{4}{3}u_x^2 - \frac{1}{2}v_t^2\right) - b\left(v_x - \frac{1}{2}u_t\right). \end{aligned} \tag{28}$$

System of Dispersive Wave Equations: Let us consider the system of dispersive wave equations, i.e.

$$v_t + uv_x + vu_x = 0,\quad u_t + uu_x + v_x + kv_{xxx} = 0. \tag{29}$$

Working on the same line as did in the last section the following variational fourth order system is obtained from (29)

$$V_{tx} + U_{xx}V_x + U_x V_{xx} = 0,\ \ U_{tx} + U_x U_{xx} + V_{xx} + kV_{xxxx} = 0. \tag{30}$$

The standard Lagrangian for (30) is:

$$L = \frac{1}{2}\left[kV_{xx}^2 - U_x V_t - U_t V_x - V_x U_x^2 - V_x^2\right]. \tag{31}$$

Substituting Eq. (31) in Eq. (8) and after some manipulations one gets:

$$(i):\xi_x = 0, (ii):\xi_U = 0, (iii):\xi_V = 0, (iv):\tau_V = 0, \tag{32}$$

$$(i):\tau_t = 0, (ii):\tau_x = 0, (iii):\eta_x = 0, (iv):\eta_V = 0, (v):\eta_U = 0, \tag{33}$$

$$(i):\phi_U = 0,\ (ii):\phi_V = 0,\ (iii):\xi_t - \phi_x = 0, \tag{34}$$

$$(i): B_U^1 = 0, (ii): B_U^2 = -\frac{1}{2}\eta_t,\ (iii): B_V^1 = -\frac{1}{2}\phi_x, \tag{35}$$

$$(i): B_V^1 = -\frac{1}{2}\phi_x, \quad (ii): B_V^2 = -\frac{1}{2}\phi_t \tag{36}$$

where,

$$B_t^1 + B_x^2 = 0. \tag{37}$$

The solution of Eqs (32) – (37) will be

$$\tau = c_1,\ \xi = c_2 + c_3 t,\ \phi = c_3 x + a,\ \eta = b, \tag{38}$$

$$B^1 = -\frac{1}{2}c_2 V + \alpha(t,x),\ B^2 = -\frac{1}{2}b_t U - \frac{1}{2}a_t V + \beta(t,x), \tag{39}$$

where a and b are any arbitrary function of t and

$$\alpha_t + \beta_x = 0. \tag{40}$$

After assuming

$$\alpha = 0 = \beta,$$

the Noether operators for system (30) will be:

$$X_1 = \frac{\partial}{\partial t},\ \ X_2 = \frac{\partial}{\partial x},\ X_3 = t\frac{\partial}{\partial x} + x\frac{\partial}{\partial U},\ X_{(a,b)} = a\frac{\partial}{\partial U} + b\frac{\partial}{\partial V}. \tag{41}$$

After doing the routine calculation we have the following components of conserved vectors corresponding to X_1, X_2, X_3 and $X_{(a,\,b)}$ for system (29) are given below respectively:

(i) $$T_1^1 = -\frac{k}{2}v_x^2 + \frac{1}{2}v^2 + \frac{1}{2}u^2 v,\ T_1^2 = kv_t v_x - uv\int u_t dx - \left(kv_{xx} + v + \frac{u^2}{2}\right)\int v_t dx - \int u_t dx \int v_t dx. \tag{42}$$

(ii) $$T_2^1 = -uv, \quad T_2^2 = \frac{k}{2}v_x^2 - \frac{1}{2}v^2 - u^2 v - kvv_{xx}. \tag{43}$$

(iii)
$$T_3^1 = -tuv + xv - D_x\left(\frac{x}{2}\int vdx\right),$$
$$T_3^2 = -ktvv_{xx} + \frac{1}{2}ktv_x^2 - tu^2 v + xuv - \frac{1}{2}tv^2 + \frac{x}{2}D_t\left(\int vdx\right). \tag{44}$$

(iv)
$$T_{(a,b)}^1 = av + bu - D_x\left(\int \frac{1}{2}(av + bu)dx\right),$$
$$T^2(a,b) = auv + kbv_{xx} + bv + b\frac{u^2}{2} - \frac{1}{2}a_t\int vdx - \frac{1}{2}b_t\int udx + \frac{a}{2}\int v_t dx + \frac{b}{2}\int u_t dx. \tag{45}$$

CONCLUSION

In this paper, Noether approach was applied to the non-variational third order systems of PDEs. In order to convert the considered systems into variational problems the transformation $u = U_x$ and $v = V_x$ was applied. Then a standard Lagrangian was reported for each system. Further conserved quantities for fourth order systems in U , V variables were found by using Noether approach. Moreover, an inverse transformation was used to obtain the conserved quantities for the main problems.

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